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APPLICATION OF A THEOREM OF M. G. KREIN TO SINGULAR INTEGRALS

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ABSTRACT. We give Hölder and L^2 estimates for singular integrals on homogeneous spaces in the sense of Coifman and Weiss. The fundamental tool which allows us to pass from Hölder to L^2 estimates, is a theorem of M. G. Krein.

1. Introduction. In this paper we study L^2 and Hölder estimates for singular integrals on homogeneous spaces in the sense of Coifman, Weiss [**2**, **3**]. The connection between both types of estimates is established by

M. G. KREIN'S THEOREM (cf. Gohberg, Krupnik [**7**, p. 183] for a proof). *Let H be a real or complex Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|_H$. Let further $B \subset H$ be a Banach space dense in H with norm $\|\cdot\|_B$ such that $\|x\|_H \leq C_{B,H}\|x\|_B$ $(x \in B)$. Then for any two linear operators $T_1, T_2\colon B \to B$ satisfying $\|T_i x\|_B \leq C_i\|x\|_B$ $(x \in B,\ i = 1,2)$, $\langle T_1 x, y\rangle = \langle x, T_2 y\rangle$ $(x, y \in B)$, we have*

$$\|T_i x\|_H \leq (C_1 C_2)^{1/2}\|x\|_H \qquad (x \in B,\ i = 1,2).$$

In particular, T_1, T_2 can be extended to bounded operators on H.

Usually Hölder estimates are much easier to prove than L^2 estimates for which only two tools are known: the Fourier transform and Cotlar's Lemma [**4**, **8**]. Using Cotlar's Lemma, G. David and J. L. Journé [**5**] characterized L^2 boundedness of elliptic singular integrals on $\mathbf{R}^n$ but their method is not applicable to general homogeneous spaces.

It is convenient to discuss our results first for ordinary singular integrals. Obviously, $L^2(\mathbf{R}^n)$ will be the Hilbert space H in Krein's theorem. Because of the preceding remarks the Hölder space $\Lambda^\alpha(\mathbf{R}^n)$ $(0 < \alpha < 1)$, which consists of all functions $f\colon \mathbf{R}^n \to \mathbf{C}$ satisfying

$$\|f\|_\infty := \sup\{|f(x)|\colon x \in \mathbf{R}^n\} < \infty,$$
$$|f|_\alpha := \sup\{|f(x) - f(y)|\,|x-y|^{-\alpha}\colon x, y \in \mathbf{R}^n, x \neq y\} < \infty,$$

would be a good candidate. However $\Lambda^\alpha(\mathbf{R}^n)$ has two disadvantages:

1. $\Lambda^\alpha(\mathbf{R}^n)$ is not contained in $L^2(\mathbf{R}^n)$.
2. Singular integrals behave only well on functions $f \in \Lambda^\alpha(\mathbf{R}^n)$ with "small" carrier. For instance, even for the Hilbert transform we have

$$\lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{\varepsilon\leq|x-y|<\varepsilon^{-1}}\frac{1}{x-y}f(y)\,dy = -\infty \qquad (x \in \mathbf{R})$$

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where

$$f(y) := \begin{cases} 0 & \text{if } y \le 0, \\ y & \text{if } 0 \le y \le 1, \\ 1 & \text{if } y \ge 1. \end{cases}$$

These disadvantages disappear if we work on compact spaces as, for instance, the n-dimensional torus. On the other hand, it is possible to adapt Krein's theorem also to the noncompact situation:

LEMMA (CF. LEMMA 2.5). *Let* $0 < \alpha < 1$ *and*

$$\Lambda_0^\alpha(\mathbf{R}^n) := \{f \in \Lambda^\alpha(\mathbf{R}^n) \colon \lambda^n(\{f \neq 0\}) < \infty\},$$

where λ^n *is the* n*-dimensional Lebesgue measure. Let further* $S, T \colon \Lambda_0^\alpha(\mathbf{R}^n) \to \Lambda^\alpha(\mathbf{R}^n)$ *be two linear operators satisfying*

(i) $\int \bar{g} Sf \, d\lambda = \int f \overline{Tg} \, d\lambda$ $(f, g \in \Lambda_0^\alpha(\mathbf{R}^n))$,
(ii) $|Sf|_\alpha,\ |Tf|_\alpha \le C_{S,1} |f|_\alpha$,
(iii) $\|Sf\|_\infty,\ \|Tf\|_\infty \le C_{S,2} |f|_\alpha \lambda^n(\{f \neq 0\})^{\alpha/n}$.

Then there exists a constant C_S *depending only on* $C_{S,1}, C_{S,2}$ *and* n *such that*

$$\|Sf\|_{L^2}, \|Tf\|_{L^2} \le C_S \|f\|_{L^2} \qquad (f \in \Lambda_0^\alpha(\mathbf{R}^n)).$$

In particular, S, T *extend to bounded linear operators on* $L^2(\mathbf{R}^n)$.

Unfortunately a precise estimate like (iii) cannot be found in the literature. We prove estimates (ii), (iii) for very general singular integrals on homogeneous spaces. On $\mathbf{R}^n$ this theorem, which is of independent interest, reads as follows.

THEOREM A (CF. THEOREM 2.1). *Let* $0 < \alpha < \alpha_0 \le 1$, $W \subset \mathbf{R}^n$, $\lambda^n(\mathbf{R}^n \setminus W) = 0$ *and* $k \colon \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{C}$ *a measurable (singular integral) kernel such that*

(i)
$$|k(x,y)| \le C_1 |x-y|^{-n},$$

(ii)
$$|k(x,y) - k(x',y)| \le C_2 |x-x'|^{\alpha_0} |x-y|^{-n-\alpha_0} \qquad (x, x', y \in \mathbf{R}^n, |y-x| \ge 2|x-y'|),$$

(iii)
$$\left| \int_{r \le |y-x| < s} k(x,y)\, dy \right| \le C_3 \qquad (0 < r \le s < \infty),$$

(iv)
$$\lim_{\varepsilon_2 \to 0} \sup_{0 < \varepsilon_1 \le \varepsilon_2} \left| \int_{\varepsilon_1 \le |x-y| < \varepsilon_2} k(x,y)\, dy \right| = 0 \qquad (x \in W),$$

(v)
$$h(x) := \lim_{\varepsilon \to 0} \int_{\varepsilon \le |y-x| < \varepsilon^{-1}} k(x,y)\, dy$$

exists for any $x \in W$. *Let the linear operators* $T_{k,\varepsilon}$, $\hat{T}_{k,\varepsilon}$ *be defined by*

$$T_{k,\varepsilon} f(x) := \int_{\varepsilon \le |x-y| < \varepsilon^{-1}} k(x,y) f(y)\, dy,$$

$$\hat{T}_{k,\varepsilon} f(x) := \int_{\varepsilon \le |x-y| < \varepsilon^{-1}} k(x,y)(f(y) - f(x))\, dy.$$

Then there exist linear operators $T_k, \hat{T}_k \colon \Lambda_0^\alpha(\mathbf{R}^n) \to L^\infty(\mathbf{R}^n)$ *such that*

(a) $\lim_{\varepsilon \to 0} T_{k,\varepsilon} f(x) = T_k f(x)$, $\lim_{\varepsilon \to 0} \hat{T}_{k,\varepsilon} f(x) = \hat{T}_k f(x)$ $(x \in W)$,

(b) $|\hat{T}_k f|_\alpha \le C_{S,1}|f|_\alpha$,
(c) $\|T_k f\|_\infty$, $\|\hat{T}_k f\|_\infty \le C_{S,2}|f|_{C^\alpha}\lambda^n(\{f \ne 0\})^{\alpha/n}$.
If $h \in C^\alpha(W)$ then we have also
(d) $|T_k f|_\alpha \le C'_{S,1}|f|_\alpha$.
The constants $C_{S,1}, C_{S,2}$ depend only on $C_1, C_2, C_3, \alpha, \alpha_0$ and $C'_{S,1}$ depends also on $|h|_\alpha$.

From the last theorem we see that the operators $\hat{T}_{k,\varepsilon}$ and $\hat{T}_k$ often behave better than the operators $T_{k,\varepsilon}$ and T_k. This is the reason why we can work with Krein's theorem even in situations when T_k does not preserve Hölder continuity. The way to L^2-estimates is now fairly easy.

THEOREM B (THEOREM 2.6 AND COROLLARY 2.7). *Let $0 < \alpha_0 \le 1$ and $k\colon \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{C}$ a measurable kernel such that k and the adjoint kernel $k^*(x,y) := \overline{k(y,x)}$ satisfy estimates* (i)–(iii) *of Theorem* A. *Then there exists a constant C_S depending only on $C_1, C_2, C_3, n, \alpha_0$ such that*
(a) $|||T_{k,\varepsilon}|||_{L^2,L^2}$, $|||\hat{T}_{k,\varepsilon}|||_{L^2,L^2} \le C_S$.
If $W \subset \mathbf{R}^n$, $\lambda^n(\mathbf{R}^n \setminus W) = 0$ and k satisfies condition (iv) *of Theorem* A *then*
(b) $|||T_k|||_{L^2,L^2}$, $|||\hat{T}_k|||_{L^2,L^2} \le C_S$,
(c) $\lim_{\varepsilon\to 0}\|T_{k,\varepsilon}f - T_k f\|_{L^2} = 0$ $(f \in L^2(\mathbf{R}^n))$.

The antisymmetric case, which we will state below, is obviously a special case of the theorem of G. David and J. L. Journé. In general, it doesn't seem so easy to show that the above assumptions imply the assumptions of the David-Journé theorem.

COROLLARY (CF. COROLLARY 2.8). *Let $0 < \alpha_0 \le 1$ and $k\colon \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{C}$ a measurable antisymmetric kernel, i.e. $k(y,x) = -k(x,y)$, satisfying the conditions* (i)–(iii) *of Theorem* A. *Then there exists a linear operator T_k on L^2 such that*
(a) *$T_{k,\varepsilon}f$ converges to $T_k f$ in the weak topology on $L^2(\mathbf{R}^n)$ for any $f \in L^2(\mu)$,*
(b) $|||T_k|||_{L^2,L^2} \le C_S$.

In order to give concrete applications we consider the following two forms of the Cauchy kernel:

$$\begin{aligned} k_a(x,y) &:= \frac{1}{(y-x) + i(a(y) - a(x))}, \\ \tilde{k}_a(x,y) &:= \frac{1 + ia'(y)}{(y-x) + i(a(y)-a(x))} \qquad (x, y \in \mathbf{R}, x \ne y), \\ k_a(x,y) &:= 0 =: \tilde{k}_a(x,y) \qquad (x,y \in \mathbf{R}), \end{aligned}$$

where $a\colon \mathbf{R} \to \mathbf{R}$ is a Lipschitz continuous function and a' is the almost everywhere existing derivative of a. It is easy to see that k_a, k_a^* and $\tilde{k}_a$ satisfy the conditions (i) and (ii) of Theorem A. It is a quite simple property of the Cauchy integral that $\tilde{k}_a$ satisfies condition (iii), and that, if a behaves well at infinity, also condition (iv) and (v) hold for $\tilde{k}_a$, provided we choose for W the set of points for which a' exist. Since the behavior at infinity is not essential we get Hölder estimates for $T_{\tilde{k}_a}$. Unfortunately the kernel $\tilde{k}_a^*$ is very bad if a is only Lipschitz continuous. However, if $a' \in \Lambda^\alpha(\mathbf{R})$ then we can pass from $T_{\tilde{k}_a}$ to T_{k_a} and get Hölder estimates also for this operator. Since the kernel k_a is antisymmetric we have then also the L^2 boundedness of T_{k_a}.

A very different application of our results to singular integrals studied by Knapp, Stein [**8**] is mentioned in Remark 2.9(b). Actually the method of Knapp and Stein can be used to prove Theorem B and Theorem 2.6 when $C_3 = 0$ but in general this assumption is quite articial.

2. L^2 and Hölder estimates for singular integrals. In the sequel (X, d, μ) will be a *homogeneous space* in the sense of Coifman and Weiss [**3**]. That means

(i) $d\colon X \times X \to \mathbf{R}_+$ is a *pseudo-distance*, i.e.

$$\begin{aligned} &d(x,y) = d(y,x) && (x,y \in X),\\ &d(x,y) = 0 \Leftrightarrow x = y && (x,y \in X),\\ &d(x,z) \le C_d(d(x,y) + d(y,z)) && (x,y,z \in X). \end{aligned}$$

(ii) X is a topological space such that for any $x \in X$, the sets $B(x,r) := \{y \in X\colon d(x,y) \le r\}$ form a neighborhood base of x.

(iii) $B(x,r)$ is Borel measurable for any $x \in X$, $r > 0$ and μ is a measure on the Borel sets of X such that

$$\mu(B(x,2r)) \le C_\mu \mu(B(x,r)) < \infty \qquad (x \in X, r > 0).$$

Next we introduce the Hölder spaces $C^\alpha(W)$ ($W \subset X$, $0 < \alpha \le 1$) which consist of all functions $f\colon W \to \mathbf{C}$ satisfying

$$\begin{aligned} &\|f\|_\infty := \sup\{|f(x)|\colon x \in W\} < \infty,\\ &|f|_{C^\alpha} := \sup\{|f(x) - f(y)|\mu(B(x, d(x,y)))^{-\alpha}\colon x,y \in W\} < \infty. \end{aligned}$$

If we endow $C^\alpha(W)$ with the norm

$$\|\cdot\|_{C^\alpha} := \|\cdot\|_\infty + |\cdot|_{C^\alpha}$$

then $C^\alpha(W)$ becomes a Banach space.

Obviously, for any $0 < \alpha \le 1$,

$$C_0^\alpha(X) := \{f \in C^\alpha(X)\colon \mu(\{f \ne 0\}) < \infty\}$$

is a subspace of $L^p(\mu)$ $(1 \le p \le \infty)$. However, it is not clear whether $C_0^\alpha(X)$ is dense in $L^p(\mu)$ $(1 \le p < \infty)$. Already in classical situations it happens for large α that $C^\alpha(X)$ consists only of constant functions. We will come back later to this question, when we derive L^2 estimates from the Hölder estimate which we now state.

2.1 THEOREM. *Let $k\colon X \times X \to \mathbf{C}$ measurable, $0 < \alpha < \gamma \le 1$, $\beta > 1$ and $W \subset X$ be such that*

(i) $$|k(x,y)| \le C_1 \mu(B(x, d(x,y)))^{-1} \qquad (x,y \in X),$$

(i) $$|k(x,y) - k(x',y)| \le C_2 \mu(B(x, d(x,x'))^\gamma \mu(B(x, d(x,y))))^{-1-\gamma}$$
$$(x, x', y \in X, d(x,y) > \beta d(x,x')),$$

(iii) $$\left|\int_{r\le d(x,y)<s} k(x,y)\,\mu(dy)\right| \le C_3 \qquad (x\in X,\ 0<r\le s<\infty),$$

(iv) $$\lim_{\varepsilon_2\to 0}\ \sup_{0<\varepsilon_1\le\varepsilon_2}\left|\int_{\varepsilon_1\le d(x,y)<\varepsilon_2} k(x,y)\,\mu(dy)\right| = 0 \qquad (x\in W),$$

(v) $$h(x) := \lim_{\varepsilon\to 0}\int_{\varepsilon\le d(x,y)<\varepsilon^{-1}} k(x,y)\,\mu(dy)$$

exists for any $x\in W$. *Assume that either*

(vi) $$\liminf_{r\to\infty}\mu(B(x,r))^{-1}\mu(B(x,r)\setminus B(x',r)) = 0 \qquad (x,x'\in X)$$

or

(vii) *for any* $x\in X$ *there exists* $r>0$ *such that* $k(x,y)=0$ $(y\in X,\ d(x,y)\ge r)$.

Then, for any $f\in C_0^\alpha(X)$ *and* $x\in W$,

$$T_k f(x) := \lim_{\varepsilon\to 0}\int_{\varepsilon\le d(x,y)} k(x,y)f(y)\,\mu(dy),$$
$$\hat{T}_k f(x) := \lim_{\varepsilon\to 0}\int_{\varepsilon\le d(x,y)<\varepsilon^{-1}} k(x,y)(f(y)-f(x))\,\mu(dy)$$

exist and we have

(a)
$$T_k f = \hat{T}_k f + f_{|W}\cdot h,$$

(b)
$$\hat{T}_k f\in C^\alpha(W) \quad \textit{and} \quad |\hat{T}_k f|_{C^\alpha}\le C_{S,1}|f|_{C^\alpha},$$

(c)
$$\|\hat{T}_k f\|_\infty \le C_{S,2}|f|_{C^\alpha}\mu(\{f\ne 0\})^\alpha,$$
$$\|T_k f\|_\infty \le C_{S,2}|f|_{C^\alpha}\mu(\{f\ne 0\})^\alpha + C_3\|f\|_\infty.$$

If $\mu(X)=\infty$ *then we have*

(d)
$$\|T_k f\|_\infty \le C'_{S,2}|f|_{C^\alpha}\mu(\{f\ne 0\})^\alpha.$$

If $h\in C^\alpha(W)$ *then we have*

(e)
$$|T_k f|_{C^\alpha} \le (C_{S,1}+C_3)|f|_{C^\alpha} + |h|_{C^\alpha}\|f\|_\infty.$$

If $h\in C^\alpha(W)$ *and* $\mu(X)=\infty$ *then we have also*

(f)
$$|T_k f|_{C^\alpha} \le C'_{S,1}|f|_{C^\alpha}.$$

The constants $C_{S,1}$, $C_{S,2}$, $C'_{S,2}$ *depend only on* $C_1, C_2, C_3, C_d, C_\mu, \alpha, \beta, \gamma$ *and* $C'_{S,1}$ *depends also on* $\|h\|_{C^\alpha}$.

REMARKS. (a) Assumption (vi) is always satisfied if $\mu(X)<\infty$. Also for all "natural" homogeneous spaces (vi) is satisfied.

If $x,x'\in X$, $r\ge d(x,x')$, $\tilde{n} := \inf\{n\in\mathbf{N}\colon 2C_d\le 2^n\}$ then

$$\mu(B(x',r)) \le \mu(B(x,2C_d r)) \le C_\mu^{\tilde{n}}\mu(B(x,r))$$

together with (vi) implies

$$\liminf_{r\to\infty} \mu(B(x,r))^{-1}\mu(B(x',r)\setminus B(x,r)) = 0.$$

Thus (vi) is equivalent to

$$\text{(vi}'\text{)} \qquad \liminf_{r\to\infty} \mu(B(x,r))^{-1}\mu((B(x,r)\setminus B(x',r))\cup(B(x',r)\setminus B(x,r))) = 0 \qquad (x,x'\in X).$$

(b) The general idea behind the proof of Theorem 2.1 is very old and seems to appear first in A. Korn [**9**]. For translation invariant singular integrals acting on periodic functions special cases of Theorem 2.1 were proved by A. P. Calderón and A. Zygmund [**1**] and by M. H. Taibleson [**13**]. We should also mention that Zygmund [**14**] has characterized all translation invariant bounded operators acting on Hölder spaces of periodic functions. For general homogeneous spaces, singular integrals on somewhat different Hölder spaces were studied by R. Macias and C. Segovia [**11**]. A theorem of N. G. Meyers [**12**], which was generalized to homogeneous spaces of R. Macias, C. Segovia [**10**], shows that both types of Hölder spaces coincide. The method of Macias and Segovia is quite different from ours. They require the L^2 boundedness of the kernel and assume also that h is constant. On the other hand they need not assume (iii) and (iv) and use a weaker form of (v) which would have been also sufficient for us. The principal advantage of our method is that via Krein's theorem it allows to prove L^2-estimates, a problem, which was considered to be much harder than the Hölder estimates (cf. C. Fefferman [**6**, p. 102]).

For the proof of the theorem we need the following

2.2 LEMMA. *Let $f \in C^\alpha(X)$ $(0 < \alpha \le 1)$ such that $\{f = 0\} \neq \emptyset$ and $\mu(\{f\neq 0\}) < \infty$. Then we have*

$$\|f\|_\infty \le C_\mu |f|_{C^\alpha}\mu(\{f\neq 0\})^\alpha.$$

The above assumption is fulfilled for any $f \in C_0^\alpha(X)$ if $\mu(X) = \infty$.

PROOF. Let $x\in X$ with $f(x)\neq 0$. We choose $0 < r_x < \infty$, $x'\in X$ such that $B(x,r_x)\subset\{f\neq 0\}$, $x'\in B(x,2r_x)$, $f(x')=0$ and we conclude

$$\begin{aligned}|f(x)| = |f(x)-f(x')| &\le |f|_{C^\alpha}\mu(B(x,d(x,x'))) \le |f|_{C^\alpha}\mu(B(x,2r_x))\\ &\le C_\mu|f|_{C^\alpha}\mu(B(x,r_x)) \le C_\mu |f|_{C^\alpha}\mu(\{f\neq 0\}).\end{aligned}$$

PROOF OF THE THEOREM. (a) is trivial. For any $x\in X$, $0\le r\le\infty$ denote $h_x(r) := \mu(B(x,r))$. We will use frequently

$$(1) \qquad \int_{r\le d(x,y)<s} f\circ h_x(d(x,y))\,\mu(dy) \le \int_{h_x(r)}^{h_x(s)} f(t)\,dt$$

$$(f\colon \mathbf{R}_+\to\mathbf{R}_+ \text{ decreasing and continuous}).$$

We even have equality for arbitrary measurable functions, but we do not need this fact. Now (1) is a special case of

$$(2) \qquad \int_{r\le d(x,y)} 1_A (f\circ h_x)(d(x,y))\,\mu(dy) \le \int_{h_x(r)}^{h_x(r)+\mu(A)} f(t)\,dt$$

$$(f\colon \mathbf{R}_+\to\mathbf{R}_+ \text{ decreasing and continuous}, A\subset X \text{ measurable}).$$

Formula (2) is an immediate consequence of the distribution function inequality

$$\begin{aligned}&\mu(\{x\in A\colon f\circ h_x(d(x,y))>s, d(x,y)\geq r\})\\&\qquad\leq \lambda(\{h_x(r)\leq t\leq h_x(r)+\mu(A)\colon f(t)>s\})\quad (s>0)\end{aligned}$$

where λ is one-dimensional Lebesgue measure.

If $f(t)\leq s$ for all $t\geq h_x(r)$ then there is nothing to prove. If this is not the case, we define

$$\begin{aligned}s'&:=\sup\{t\in\mathbf{R}_+\colon f(t)>s\}>h_x(r),\\ s''&:=\sup\{t\in\mathbf{R}_+\colon h_x(t)<s'\}.\end{aligned}$$

If $s''<\infty$ then

$$\mu(B(x,s''))\geq s',\qquad f(s')=s$$

and we can conclude

$$\begin{aligned}\mu(\{x\in A\colon f\circ h_x(d(x,y))>s, d(x,y)\geq r\}) &= \mu(A\cap\{y\in X\colon r\leq d(x,y)<s''\})\\ &\leq \min(\mu(A), h_x(s'')-h_x(r))\\ &=\lambda([h_x(r),h_x(r)+\mu(A)]\cap[h_x(r),h_x(s'')])\\ &\leq\lambda([h_x(r),h_x(r)+\mu(A)]\cap[h_x(r),s'])\\ &=\lambda(\{h_x(r)\leq t\leq h_x(r)+\mu(A)\colon f(t)>s\}).\end{aligned}$$

The case $s''=\infty$ is trivial.

Let now $0<\alpha\leq\gamma$, $f\in C_0^\alpha(X)$ and $x\in W$. Then we have for any $0<\varepsilon_1\leq\varepsilon_2<\infty$

$$\begin{aligned}&\left|\int_{\varepsilon_1\leq d(x,y)}k(x,y)f(y)\,\mu(dy)-\int_{\varepsilon_2\leq d(x,y)}k(x,y)f(y)\,\mu(dy)\right|\\ &\quad\leq\left|\int_{\varepsilon_1\leq d(x,y)<\varepsilon_2}k(x,y)(f(y)-f(x))\,\mu(dy)\right|+\left|\int_{\varepsilon_1\leq d(x,y)<\varepsilon_2}f(x)k(x,y)\,\mu(dy)\right|\\ &\quad=: J(\varepsilon_1,\varepsilon_2)+J'(\varepsilon_1,\varepsilon_2).\end{aligned}$$

By (iv), $J'(\varepsilon_1,\varepsilon_2)$ tends to 0 if ε_2 tends to 0. In order to show that $T_kf(x)$, $\hat{T}_kf(x)$ exist we have to show the same for $J(\varepsilon_1,\varepsilon_2)$. This will be accomplished by $\lim_{\varepsilon_2\to 0}h_x(\varepsilon_2)-h_x(\varepsilon_1)=0$ and by the following useful estimate for $J(\varepsilon_1,\varepsilon_2)$:

$$\begin{aligned}(3)\qquad J(\varepsilon_1,\varepsilon_2)&\leq\int_{\varepsilon_1\leq d(x,y)<\varepsilon_2}C_1h_x^{-1}(d(x,y))|f|_{C^\alpha}h_x^\alpha(d(x,y))\,\mu(dy)\\ &\leq C_1|f|_{C^\alpha}\int_{h_x(\varepsilon_1)}^{h_x(\varepsilon_2)}t^{-1+\alpha}\,dt=\frac{C_1}{\alpha}|f|_{C^\alpha}(h_x^\alpha(\varepsilon_2)-h_x^\alpha(\varepsilon_1)).\end{aligned}$$

Let now $x\in X$ with $f(x)=0$. Denoting $A:=\{f\neq 0\}$ and using (2) we get, for any $\varepsilon>0$,

$$\begin{aligned}&\left|\int_{\varepsilon\leq d(x,y)}k(x,y)f(y)\,\mu(dy)\right|=\left|\int_{\varepsilon\leq d(x,y)}k(x,y)(f(y)-f(x))\,\mu(dy)\right|\\ &\quad\leq C_1|f|_{C^\alpha}\int_A h_x^{-1+\alpha}(d(x,y))\,\mu(dy)\leq C_1|f|_{C^\alpha}\int_0^{\mu(A)}t^{-1+\alpha}\,dt\\ &\quad=\frac{C_1}{\alpha}|f|_{C^\alpha}\mu(A)^\alpha.\end{aligned}$$

Thus we have proved

(4) $$|T_k f(x)| = |\hat{T}_k f(x)| \le (C_1/\alpha)|f|_{C^\alpha}\mu(\{f \neq 0\})^\alpha \qquad (x \in W,\ f(x) = 0).$$

Introducing

$$g(s) := \inf\{n \in \mathbf{N} \colon s \le 2^n\} \qquad (s > 0)$$

we have

$$\mu(B(x, sr)) \le C_\mu^{g(s)} \mu(B(x, r)) \qquad (x \in X,\ r, s > 0).$$

Let now $x, x' \in X$, $\delta := d(x, x')$ and let $\beta' := \max(\beta, 2C_d)$. Then we have, for any $0 < \varepsilon \le \delta^{-1}$,

$$\begin{aligned}
&\left| \int_{\varepsilon \le d(x,y) < \varepsilon^{-1}} k(x,y)(f(y) - f(x))\,\mu(dy) \right. \\
&\qquad\qquad \left. - \int_{\varepsilon \le d(x',y) < \varepsilon^{-1}} k(x',y)(f(y) - f(x'))\,\mu(dy) \right| \\
&\quad \le \left| \int_{\varepsilon \le d(x,y) < \beta'\delta} k(x,y)(f(y) - f(x))\,\mu(dy) \right| \\
&\quad + \left| \int_{\varepsilon \le d(x',y) < \beta'\delta} k(x',y)(f(y) - f(x'))\,\mu(dy) \right| \\
&\quad + \left| \int_{\beta'\delta \le d(x,y) < \varepsilon^{-1}} (k(x,y) - k(x',y))(f(y) - f(x))\,\mu(dy) \right| \\
&\quad + \left| \int_{\beta'\delta \le d(x,y) < \varepsilon^{-1}} k(x',y)(f(y) - f(x))\,\mu(dy) \right. \\
&\qquad\qquad \left. - \int_{\beta'\delta \le d(x',y) < \varepsilon^{-1}} k(x',y)(f(y) - f(x))\,\mu(dy) \right| \\
&\quad + \left| \int_{\beta'\delta \le d(x',y) < \varepsilon^{-1}} k(x',y)(f(x) - f(x'))\,\mu(dy) \right| \\
&\quad =: I_1(\varepsilon) + I_2(\varepsilon) + I_3(\varepsilon) + I_4(\varepsilon) + I_5(\varepsilon).
\end{aligned}$$

From (3) we get

$$\begin{aligned}
I_1(\varepsilon) = J(\varepsilon, \beta'\delta) &\le (C_1/\alpha)|f|_{C^\alpha}\mu(B(x, \beta'\delta))^\alpha \\
&\le (C_1/\alpha) C_\mu^{\alpha g(\beta')} |f|_{C^\alpha} \mu(B(x,\delta))^\alpha
\end{aligned}$$

and similarily

$$\begin{aligned}
I_2(\varepsilon) &\le (C_1/\alpha)|f|_{C^\alpha}\mu(B(x', \beta'\delta))^\alpha \le (C_1/\alpha)|f|_{C^\alpha}\mu(B(x, C_d(\delta + \beta'\delta)))^\alpha \\
&\le (C_1/\alpha) C_\mu^{\alpha g(C_d(1+\beta'))} |f|_{C^\alpha} \mu(B(x,\delta))^\alpha.
\end{aligned}$$

Using (ii) and (1) we get

$$\begin{aligned} I_3(\varepsilon) &\le \int_{\beta'\delta \le d(x,y)} C_2 h_x^\gamma(\delta) h_x^{-1-\gamma}(d(x,y)) |f|_{C^\alpha} h_x^\alpha(d(x,y))\, \mu(dy) \\ &\le C_2 |f|_{C^\alpha} h_x^\gamma(\delta) \int_{h_x(\beta'\delta)}^\infty t^{\alpha-1-\gamma}\, dt = \frac{C_2}{\gamma-\alpha} |f|_{C^\alpha} h_x^\gamma(\delta) h_x^{\alpha-\gamma}(\beta'\delta) \\ &\le \frac{C_2}{\gamma-\alpha} |f|_{C^\alpha} \mu(B(x,\delta))^\alpha. \end{aligned}$$

Obviously

$$I_5(\varepsilon) \le C_3 |f|_{C^\alpha} \mu(B(x,\delta))^\alpha.$$

Setting $\beta'' := C_d(1+\beta')$ and $D_\varepsilon := (B(x,\varepsilon^{-1})\backslash B(x',\varepsilon^{-1})) \cup (B(x',\varepsilon^{-1})\backslash B(x,\varepsilon^{-1}))$ we have

$$\begin{aligned} I_4(\varepsilon) \le& \int_{\delta \le d(x',y) < \beta''\delta} |k(x',y)|\, |f(y) - f(x')|\, \mu(dy) \\ &+ \int_{\delta \le d(x',y) < \beta''\delta} |k(x',y)|\, |f(x') - f(x)|\, \mu(dy) \\ &+ \int_{(2C_d)^{-1}\varepsilon^{-1} \le d(x',y)} 1_{D_\varepsilon} |k(x',y)|\, |f(y) - f(x)|\, \mu(dy) \\ &=: I_{4,1}(\varepsilon) + I_{4,2}(\varepsilon) + I_{4,3}(\varepsilon). \end{aligned}$$

Using (3) we can estimate $I_{4,1}(\varepsilon)$ similarly as $I_2(\varepsilon)$:

$$I_{4,1}(\varepsilon) \le (C_1/\alpha) |f|_{C^\alpha} \mu(B(x',\beta''\delta))^\alpha \le (C_1/\alpha) C_\mu^{\alpha g(C_d(1+\beta''))} |f|_{C^\alpha} \mu(B(x,\delta))^\alpha.$$

Obviously we have

$$I_{4,2}(\varepsilon) \le C_3 |f|_{C^\alpha} \mu(B(x,\delta))^\alpha.$$

If (vii) is satisfied, then $I_{4,3}(\varepsilon) = 0$ provided ε is so small such that $k(x',y) = 0$ when $d(x',y) \ge (2C_d)^{-1}\varepsilon^{-1}$. If (vi) is satisfied, then

$$\begin{aligned} I_{4,3}(\varepsilon) &\le \int 1_{D_\varepsilon} C_1 \mu(B(x',(2C_d\varepsilon)^{-1}))^{-1} 2\|f\|_\infty\, \mu(dy) \\ &\le C2 \|f\|_\infty \mu(B(x',\varepsilon^{-1}))^{-1} \mu(D_\varepsilon) \end{aligned}$$

together with (vi′) implies

$$\liminf_{\varepsilon\to 0} I_{4,3}(\varepsilon) = 0.$$

Collecting all these estimates and letting ε tend to 0, assertion (b) follows.

Let now $x \in A = \{f \ne 0\}$ and assume that $\{f = 0\} \ne \emptyset$. Then there exist $0 < r_x < \infty$, $x' \in X$ such that $B(x,r_x) \subset A$, $x' \in B(x,2r_x)$, $f(x') = 0$. Observing first that

$$\mu(B(x,2r_x)) \le C_\mu \mu(B(x,r_x)) \le C_\mu \mu(A)$$

and using (b) and (4) we get

$$\begin{aligned} |\hat{T}_k f(x)| &\le |\hat{T}_k f(x')| + |\hat{T}_k f(x) - \hat{T}_k f(x')| \\ &\le (C_1/\alpha) |f|_{C^\alpha} \mu(A)^\alpha + C_{S,1} |f|_{C^\alpha} \mu(B(x,d(x,x')))^\alpha \\ &\le ((C_1/\alpha) + C_{S,1} C_\mu) |f|_{C^\alpha} \mu(A)^\alpha. \end{aligned}$$

If $\{f=0\}=\emptyset$ then $\mu(X)<\infty$ and hence $1\in C_0^\alpha(X)$. Choosing an arbitrary point $x_0\in X$, the function $f'(x):=f(x)-f(x_0)$ satisfies $\{f'=0\}\neq\emptyset$ and therefore

$$|\hat{T}_k f(x)| = |\hat{T}_k f'(x)| \le (C_1/\alpha + C_{S,1})|f'|_{C^\alpha}\mu(A) = (C_1/\alpha + C_{S,1})|f|_{C^\alpha}\mu(A).$$

Together with Lemma 2.2 the assertions are now immediate.

The following rather technical lemma, which is based on results of R. Macias, C. Segovia [**10**], is obvious for all classical homogeneous spaces and is therefore not necessary in order to understand the proof of Proposition 2.5.

2.3 LEMMA. (a) *There exists* $\tilde{\alpha}>0$ *such that* $C_0^\alpha(X)$ *is dense in* $L^p(\mu)$ $(0<\alpha\le\tilde{\alpha},\ 1\le p<\infty)$.

(b) *There exists a family* $(\varphi_r)_{0<r<\infty}$ *of functions on* X *such that*

(i) $0\le\varphi_r\le 1$, $\varphi_r\le\varphi_s$ $(0<r\le s<\infty)$,

(ii) $\{\varphi_r\neq 0\}\subset\{\varphi_{2r}=1\}$,

(iii) $\bigcup_{r>0}\{\varphi_r=1\}=X$,

(iv) $\mu(\{\varphi_r\neq 0\})\le C_{\varphi,1}r$,

(v) $\varphi_r\in C_0^\alpha(X)$, $|\varphi_r|_{C^\alpha}\le C_{\varphi,2}r^{-\alpha}$ $(0<\alpha\le\tilde{\alpha},\ r>0)$.

$\tilde{\alpha}$ *and the constants* $C_{\varphi,1}, C_{\varphi,2}$ *depend only on* C_d *and* C_μ.

PROOF. The proof will be divided into three steps, from which the first is the essential one.

1. *Step.* We assume that (X,d,μ) is a *normalized homogeneous space*, i.e. there exists a constant $C_N>0$ with

$$(1)\qquad C_N^{-1}r\le\mu(B(x,r))\le C_N r\qquad(x\in X,\ r>0).$$

Then $C^\alpha(X)$ coincides with the "metric" Hölder space $H^\alpha(X)$ which consists of all functions $f\colon X\to\mathbf{C}$ satisfying

$$\|f\|_\infty<\infty, |f|_{H^\alpha}:=\sup\{|f(x)-f(y)|d(x,y)^{-\alpha}\colon x,y\in X,\ x\neq y\}.$$

Moreover, we have

$$(2)\qquad C_N^{-\alpha}|f|_{H^\alpha}\le|f|_{C^\alpha}\le C_N^\alpha|f|_{H^\alpha}.$$

Now we employ [**10**], Theorem 2 to find a pseudo-distance d' on X and $0<\tilde{\alpha}<1$ satisfying

$$(3)\qquad C_1^{-1}d'(x,y)\le d(x,y)\le C_1 d'(x,y)\qquad(x,y\in X),$$

$$(4)\quad |d'(x,z)-d'(y,z)|\le C_2\max(d'(x,z),d'(y,z))^{1-\tilde{\alpha}}d'(x,y)^{\tilde{\alpha}}\qquad(x,y,z\in X),$$

where $\tilde{\alpha}:=(\log 3C_d^2)^{-1}\log 2$, $C_1:=4^{1/\tilde{\alpha}}$, $C_2:=\tilde{\alpha}^{-1}$. For any $z\in X$, $r>0$ we can now define the Urysohn function

$$\varphi_{r,z}(x):=\sup(0,\inf(1,2-r^{-1}d'(x,z))).$$

Obviously, we have

$$(5)\qquad 0\le\varphi_{r,z}\le 1,\qquad\{\varphi_{r,z}\neq 0\}\subset B'(z,2r),\qquad B'(z,r)\subset\{\varphi_{r,z}=1\},$$

$$(6)\qquad \varphi_{r,z}\le\varphi_{s,z}\qquad(0<r\le s<\infty),$$

where $B'(z,r):=\{y\in X\colon d'(x,y)\le r\}$. From (4) we get for any $x,y\in B'(z,3C_d r)$

$$|\varphi_{r,z}(x)-\varphi_{r,z}(y)|\le r^{-1}C_2(3C_d r)^{1-\tilde{\alpha}}d'(x,y)^{\tilde{\alpha}}\le C_2 3C_d r^{-\tilde{\alpha}}C_1^{\tilde{\alpha}}d(x,y)^{\tilde{\alpha}}.$$

Since $d(x,y)^{\tilde\alpha} \le (6C_d^2 r)^{\tilde\alpha-\alpha} d(x,y)^\alpha \le 6C_d^2 r^{\tilde\alpha-\alpha} d(x,y)^\alpha$ we get
$|\varphi_{r,z}(x)-\varphi_{r,z}(y)| \le C_3 r^{-\alpha} d(x,y)^\alpha \quad (0<\alpha\le\tilde\alpha,\ r>0,\ z\in X,\ x,y\in B(z,3C_d r))$.
For any $x\in X\setminus B'(z,3C_d r)$, $y\in B'(z,2r)$ we have $d'(x,y)\ge r$ and therefore

$$|\varphi_{r,z}(x)-\varphi_{r,z}(y)| = |\varphi_{r,z}(y)| \le 1 \le (r^{-1}d'(x,y))^\alpha \le r^{-\alpha}C_1^\alpha d(x,y)^\alpha.$$

For any $x\in X\setminus B'(z,3C_d r)$, $y\in X\setminus B'(z,2r)$ we have

$$|\varphi_{r,z}(x)-\varphi_{r,z}(y)| = 0.$$

Altogether we have

$$|\varphi_{r,z}|_{H^\alpha} \le C'_{\varphi,2} r^{-\alpha} \qquad (0<\alpha\le\tilde\alpha,\ r>0,\ z\in X). \tag{7}$$

Thus if we choose an arbitrary point $z_0\in X$ and let $\varphi_r := \varphi_{r,z_0}$ assertion (b) follows from (7), (5), (6), (1), (2).

To prove (a), we note first that the balls $B'(x,r)$ $(x\in X,\ r>0)$ are relatively compact in the completion of X with respect to d' (cf. [**2**, p. 67]). Since, by (7), there exist so many Urysohn functions in $C_0^\alpha(X)$ $(0<\alpha\le\tilde\alpha)$ (a) is now immediate.

2. *Step.* We assume now only that $\{y\in X\colon d(x,y)<r\}$ is open for any $r>0$. By [**10**], Theorem 3 (X,d'',μ) is a normalized homogeneous space, if we define

$$d''(x,y) := \begin{cases} 0 & \text{if } x=y,\\ \inf\{\mu(B(z,r))\colon z\in X,\ r>0,\ x,y\in B(z,r)\}, \end{cases}$$

Moreover, denoting by H''^α the metric Hölder space with respect to d'' we have obviously $C^\alpha(X)\subset H''^\alpha(X)$, $|f|_{H''^\alpha}\le |f|_{C^\alpha}$. Since the constants C''_d, C''_μ, C''_N of the normalized homogeneous space (X,d'',μ) depend only on C_d, C_μ the assertion follows if we apply the first step to (X,d'',μ).

3. *Step.* If (X,d,μ) is an arbitrary homogeneous space, then we apply [**10**, Theorem 2] once more to find a pseudo-distance d' such that (3) and (4) hold. It follows from (3) that (X,d',μ) is a homogeneous space with $C'_d := C_1^2 C_d$, $C_\mu := C_\mu^{\tilde n}$, $\tilde n := \inf\{n\colon 2C_1^2 \le 2^n\}$ and that

$$C'^\alpha(X) = C^\alpha(X), \quad C_r^{-1}|f|_{C'^\alpha} \le |f|_{C^\alpha} \le C_4 |f|_{C'^\alpha},$$

where

$$B'(x,d'(x,y)) \subset B'(x,C_1 d(x,y)) \subset B(x,C_1^2 d(x,y)) \subset B(x,2^{\tilde n-1}d(x,y)),$$
$$B(x,d(x,y)) \subset B'(x,C_1^2 d'(x,y)) \subset B'(x,2^{\tilde n-1}d'(x,y))$$

shows that we may choose $C_4 := (C'_\mu)^{\tilde n-1}$. Since by (4) the balls $B(x,r)$ are open the assertion follows from the second step.

The next lemma adapts Krein's Theorem to the present situation.

2.4 LEMMA. *Let $0<\alpha\le 1$ be such that $C_0^\alpha(X)$ is dense in $L^2(\mu)$ and such that there exists a family $(\varphi_r)_{r>0}$ satisfying* 2.3(i)–(iv). *Let further $S,T\colon C_0^\alpha(X)\to C^\alpha(X)$ be two linear operators satisfying*
(i) $\int f\overline{Sg}\,d\mu = \int \overline{g}Tf\,du$ $(f,g\in C_0^\alpha(X))$,
(ii) $\|Sf\|_{C^\alpha}, \|Tf\|_{C^\alpha} \le C_{S,1}|f|_{C^\alpha}$,
(iii) $\|Sf\|_\infty, \|Tf\|_\infty \le C_{S,2}|f|_{C^\alpha}\mu(\{f\ne 0\})^\alpha$.
Then we have

$$\|Sf\|_{L^2}, \|Tf\|_{L^2} \le (C_{S,1} + 2C_{\varphi,1}^\alpha C_{\varphi,2}C_{S,2})\|f\|_{L^2} \qquad (f\in C_0^\alpha(X)).$$

In particular, S,T extend to bounded linear operators on $L^2(\mu)$.

REMARK. Note that we did not assume that $Sf, Tf \in L^2(\mu)$ $(f \in C_0^\alpha(X))$. Nevertheless the integrals in (i) exist always.

PROOF. For any $r > 0$ we define

$$\begin{aligned}
&B_r := \{\varphi_r f\colon f \in C^\alpha(X)\} \subset C_0^\alpha(X),\\
&\|f\|_{B_r} := C_{\varphi,2}\|f\|_\infty + r^\alpha |f|_{C^\alpha} \qquad (f \in B_r),\\
&S_r, T_r\colon B_r \to B_r\\
&S_r f := \varphi_r S(\varphi_r f), \qquad T_r f := \varphi_r T(\varphi_r f).
\end{aligned}$$

Finally let H_r be the closure of B_r in $L^2(\mu)$ endowed with the inner product $\langle \cdot, \cdot \rangle_{H_r}$ induced by $L^2(\mu)$. From

$$\begin{aligned}
&\|\varphi_r f\|_\infty \le \|f\|_\infty,\\
&|\varphi_r f|_{C^\alpha} \le |\varphi_r|_{C^\alpha}\|f\|_\infty + \|\varphi_r\|_\infty |f|_{C^\alpha} \le C_{\varphi,2} r^{-\alpha}\|f\|_\infty + |f|_{C^\alpha}
\end{aligned}$$

we get, for any $f \in B_r$,

$$\begin{aligned}
\|S_r f\|_\infty &\le \|S(\varphi_r f)\|_\infty \le C_{S,2}(C_{\varphi,2} r^{-\alpha}\|f\|_\infty + |f|_{C^\alpha})\mu(\{\varphi_r \ne 0\})^\alpha\\
&\le C_{S,2} C_{\varphi,1}^\alpha (C_{\varphi,2}\|f\|_\infty + r^\alpha |f|_{C^\alpha}) = C_{S,2} C_{\varphi,1}^\alpha \|f\|_{B_r},\\
|S_r f|_{C^\alpha} &\le C_{\varphi,2} r^{-\alpha}\|S(\varphi_r f)\|_\infty + |S(\varphi_r f)|_{C^\alpha}\\
&\le C_{\varphi,2} C_{\varphi,1}^\alpha C_{S,2} r^{-\alpha}\|f\|_{B_r} + C_{S,1}|\varphi_r f|_{C^\alpha}\\
&\le C_{\varphi,2} C_{\varphi,1}^\alpha C_{S,2} r^{-\alpha}\|f\|_{B_r} + C_{S,1} r^{-\alpha}\|f\|_{B_r},\\
\|S_r f\|_{B_r} &\le C_{\varphi,2} C_{S,2} C_{\varphi,1}^\alpha \|f\|_{B_r} + C_{\varphi,2} C_{\varphi,1}^\alpha C_{S,2}\|f\|_{B_r} + C_{S,1}\|f\|_{B_r}
\end{aligned}$$

and therefore

$$|||S_r|||_{B_r,B_r} \le C_{S,1} + 2C_{\varphi,1}^\alpha C_{\varphi,2} C_{S,2} =: C_0.$$

Analogously we have $|||T_r|||_{B_r,B_r} \le C_0$. Since we have, by (i),

$$\langle S_r f, g\rangle_{H_r} = \langle f, T_r g\rangle_{H_r} \qquad (f, g \in B_r)$$

we may now apply Krein's Theorem to conclude that

$$\|S_r f\|_{H_r}, \|T_r f\|_{H_r} \le C_0 \|f\|_{H_r} \qquad (f \in B_r).$$

By 2.4(i), (ii), we have $\varphi_r f = f$ $(f \in B_s, 0 < s < 2s \le r)$ and therefore $B_s \subset B_r$, $S_r f = \varphi_r S f$ $(f \in B_s,\ 0 < s < 2s \le r)$ together with (1) and $|\varphi_r Sf| \uparrow |Sf|$ implies

(2) $$\|Sf\|_{L^2} = \lim_{r\to\infty} \|S_r f\|_{L^2} \le C_0 \|f\|_{L^2} \qquad (f \in B_s, s > 0).$$

Since for any $f \in C_0^\alpha(X)$

$$\lim_{r\to\infty} \|f - \varphi_r f\|_\infty = 0, \qquad \lim_{r\to\infty} \mu(\{(f - \varphi_r f) \ne 0\}) = 0,$$

by 2.3(i), (iii) and Lemma 2.2, we get from 2.4(iii) and (2)

$$\lim_{r\to\infty} \|Sf - S(\varphi_r f)\|_\infty = 0, \qquad \lim_{r,s\to\infty} \|S(\varphi_r f) - S(\varphi_s f)\|_{L^2} = 0$$

whence

$$\|Sf\|_{L^2} = \lim_{r\to\infty} \|S(\varphi_r f)\|_{L^2} \le C_0 \|f\|_{L^2} \qquad (f \in C_0^\alpha(X)).$$

Since the same holds for T, the proof is complete.

2.5 PROPOSITION. *Let* $0 < \gamma \leq 1$, $W \subset X$, $\mu(X \setminus W) = 0$ *and* $k\colon X \times X \to \mathbf{C}$ *a measurable kernel such that the assumptions of Theorem 2.1 are fulfilled with respect to k and with respect to the adjoint kernel* k^* *defined by* $k^*(x,y) := \overline{k(y,x)}$. *Then there exists a constant* C'_S *depending only on* $C_1, C_2, C_3, C_d, C_\mu$ *and* γ *and two bounded linear operators* $T_k, \hat{T}_k\colon L^2(\mu) \to L^2(\mu)$ *such that*

$$T_k f(x) = \lim_{\varepsilon \to 0} \int_{\varepsilon \leq d(x,y)} k(x,y) f(y)\, \mu(dy),$$

(a)
$$\hat{T}_k f(x) = \lim_{\varepsilon \to 0} \int_{\varepsilon \leq d(x,y) < \varepsilon^{-1}} k(x,y)(f(y) - f(x))\, \mu(dy)$$
$$(x \in W,\ f \in C_0^\alpha(X),\ 0 < \alpha < \gamma),$$

(b)
$$|||T_k|||_{L^2,L^2}, |||\hat{T}_k|||_{L^2,L^2} \leq C'_S.$$

PROOF. We cannot apply Lemma 2.4 directly to Theorem 2.1 and must therefore decompose k into

$$k_1 := \tfrac{1}{2}(k + k^*), \quad k_2 := \tfrac{1}{2i}(k - k^*),$$
$$k_{1,1} := \operatorname{Re} k_1, \quad k_{1,2} := \operatorname{Im} k_1, \quad k_{2,1} := \operatorname{Re} k_1, \quad k_{2,2} := \operatorname{Im} k_2.$$

By our assumptions on k and k^*, the kernels $k_{i,j}$ satisfy the assumptions of Theorem 2.1. In particular, for any $0 < \alpha < \gamma$, $1 \leq i,j \leq 2$, there exists the linear operator $\hat{T}_{k_{i,j},\alpha}\colon C_0^\alpha(X) \to C^\alpha(X)$ satisfying

(1)
$$\hat{T}_{k_{i,j},\alpha} f(x) := \lim_{\varepsilon \to 0} \int_{\varepsilon \leq d(x,y) < \varepsilon^{-1}} k_{i,j}(x,y)(f(y) - f(x))\, \mu(dy)$$
$$(x \in W,\ f \in C_0^\alpha(X)),$$

(2)
$$|\hat{T}_{i_{i,j},\alpha} f|_{C^\alpha} \leq C_{S,1} |f|_{C^\alpha} \qquad (f \in C_0^\alpha(X)),$$

(3)
$$\|\hat{T}_{k_{i,j},\alpha} f\|_\infty \leq C_{S,2} |f|_{C^\alpha} \mu(\{f \neq 0\})^\alpha \qquad (f \in C_0^\alpha(X)).$$

Since $k_{i,j}$ is symmetric or antisymmetric and real valued we have also

(4)
$$\int g \overline{\hat{T}_{k_{i,j},\alpha} f}\, d\mu = \pm \int \overline{f} \hat{T}_{k_{i,j},\alpha} g\, d\mu \qquad (f, g \in C_0^\alpha(X)).$$

Taking $\tilde{\alpha}$ as in Lemma 2.3, we choose $\alpha := \min(\tilde{\alpha}, \gamma/2)$. Then by (1)–(4) and Lemma 2.3 the assumptions of Lemma 2.4 are fulfilled and we get

$$\|\hat{T}_{k_{i,j},\alpha} f\|_{L^2} \leq (C_{S,1} + 2C_{\varphi,1}^\alpha C_{\varphi,2} C_{S,2}) \|f\|_{L^2} \qquad (f \in C_0^\alpha(X)).$$

Since

$$\hat{T}_{k,\alpha} = \hat{T}_{k_1,\alpha} + i\hat{T}_{k_2,\alpha} = (\hat{T}_{k_{1,1},\alpha} + i\hat{T}_{k_{1,2},\alpha}) + i(\hat{T}_{k_{2,1},\alpha} + i\hat{T}_{k_{2,2},\alpha}),$$
$$|||T_{k,\alpha} - \hat{T}_{k,\alpha}|||_{L^2,L^2} \leq C_3,$$

we get

$$|||T_{k,\alpha}|||_{L^2,L^2}, |||\hat{T}_{k,\alpha}|||_{L^2,L^2} \leq C'_S := C_3 + 8(C_{S,1} + C_{\varphi,1}^\alpha C_{\varphi,2} C_{S,2}).$$

Since $C_{\varphi,1}, C_{\varphi,2}$ depend only on C_d, C_μ and α depends only on C_d, C_μ and γ, the assertion follows.

2.6 THEOREM. *Assume that (X, d, μ) is a normalized homogeneous space, i.e.*

$$C_N^{-1} r \le \mu(B(x,r)) \le C_N r \qquad (x \in X,\ r > 0).$$

Let $0 < \gamma \le 1$ and $k\colon X \times X \to \mathbf{C}$ be a measurable kernel such that k and k^* satisfy 2.1(i)–(iii). Then there exists a constant C_S depending only on $C_1, C_2, C_3, C_d, C_\mu, C_N, \beta, \gamma$ such that the operators $T_{k,\varepsilon}, \hat{T}_{k,\varepsilon}\colon L^2(\mu) \to L^2(\mu)$ defined by

$$T_{k,\varepsilon} f(x) := \int_{\varepsilon \le d(x,y) < \varepsilon^{-1}} k(x,y) f(y)\, \mu(dy),$$

$$\hat{T}_{k,\varepsilon} f(y) := \int_{\varepsilon \le d(x,y) < \varepsilon^{-1}} k(x,y)(f(y) - f(x))\, \mu(dy)$$

satisfy $|||T_{k,\varepsilon}|||_{L^2,L^2},\ |||\hat{T}_{k,\varepsilon}|||_{L^2,L^2} \le C_S$.

PROOF. We return to Step 1 of the proof of Lemma 2.3. There we have found $\tilde{\alpha} > 0$ and we have constructed functions $\varphi_{r,z}$ satisfying

(1) $\varphi_{r,z} \in C_0^\alpha(X)$, $|\varphi_{r,z}|_{C^\alpha} \le C_{\varphi,2} r^{-\alpha}$ $(0 < \alpha \le \tilde{\alpha}, r > 0, z \in X)$,

(2) $0 \le \varphi_{r,z} \le \varphi_{s,z} \le 1$ $(0 < r \le s, z \in X)$,

(3) $B(z, C_{\varphi,3}^{-1} r) \subset \{\varphi_{r,z} = 1\} \subset B(z, C_{\varphi,3} r)$, $\{\varphi_{r,z} \ne 0\} \subset B(z, 2C_{\varphi,3} r)$ $(r > 0, z \in X)$, where $C_{\varphi,3}$ is identical with the constant C_1 of the proof of Lemma 2.3.

(4) $\varphi_{r,z}(y) = \varphi_{r,y}(z)$ $(r > 0, y, z \in X)$,

(5) $(x, y) \to \varphi_{r,x}(y)$ is measurable on X $(r > 0)$.

We want to show that the kernels

$$g_r(x,y) := \varphi_{r,x}(y) k(x,y)$$

satisfy 2.1(i), (ii) with suitable constants C_1', C_2' and $\gamma' := \min(\gamma, \tilde{\alpha})$, $\beta' := \max(\beta, 2C_d)$. Obviously 2.1(i) holds with $C_1' := C_1$. To show 2.1(ii), let $x, x' \in X$, $\beta' d(x,x') \le d(x,y)$ and consider

$$\begin{aligned} &|g_r(x,y) - g_r(x',y)| \\ &\quad \le |\varphi_{r,x'}(y)|\,|k(x,y) - k(x',y)| + |k(x,y)|\,|\varphi_{r,x}(y) - \varphi_{r,x'}(y)| =: I_1 + I_2. \end{aligned}$$

Then we have

$$\begin{aligned} I_1 &\le C_2 \mu(B(x, d(x,x')))^\gamma \mu(B(x, d(x,y)))^{-1-\gamma} \\ &\le C_2 \mu(B(x, d(x,x')))^{\gamma'} \mu(B(x, d(x,y)))^{-1-\gamma'}, \end{aligned}$$

and if $d(x,y) \ge 6 C_d C_{\varphi,3} r$ then

$$d(x',y) \ge C_d^{-1}(d(x,y) - C_d d(x,x')) \ge (2C_d)^{-1} d(x,y) \ge 3C_{\varphi,3r}$$

together with (3) implies

$$I_2 \le |k(x,y)|\,|0 - 0| = 0.$$

If $d(x,y) \le 6 C_d C_{\varphi,3} r$ then we get from (1), (4) that

$$\begin{aligned} I_2 &\le |k(x,y)|\,|\varphi_{r,y}(x) - \varphi_{r,y}(x')| \\ &\le C_1 \mu(B(x, d(x,y)))^{-1} C_{\varphi,2} r^{-\gamma'} \mu(B(x, d(x,x')))^{\gamma'} \\ &\le C_1 C_{\varphi,2} \mu(B(x, d(x,y)))^{-1} (6 C_d C_{\varphi,3})^{\gamma'} (6 C_d C_{\varphi,3} r)^{-\gamma'} \mu(B(x, d(x,x')))^{\gamma'} \\ &\le C_1 C_{\varphi,2} (6 C_d C_{\varphi,3})^{\gamma'} \mu(B(x, d(x,y)))^{-1} \\ &\quad \times C_N^{-\gamma'} \mu(B(x, d(x,y)))^{-\gamma'} \mu(B(x, d(x,x')))^{\gamma'}. \end{aligned}$$

Thus we may choose

$$C_2' := C_2 + C_1 C_{\varphi,2}(2C_d C_{\varphi,3})^{\gamma'} C_N^{-\gamma'}.$$

Now the kernels

$$g_{r,s} := g_s - g_r \qquad (0 < r \le s < \infty)$$

satisfy 2.1(i), (ii) with constants $C_1'' := 2C_1'$, $C_2'' := 2C_2'$. Next we compare the kernels $g_{r,s}$ with the kernels $k_{r,s}$ defined by

$$k_{r,s}(x,y) := \begin{cases} 0 & \text{if } d(x,y) < r \text{ or } d(x,y) \ge s, \\ k(x,y) & \text{if } r \le d(x,y) < s, \end{cases}$$

and obtain from (3) that

$$(6) \qquad |k_{r,s}(x,y) - g_{r,s}(x,y)| \le \begin{cases} 0 & \text{if } d(x,y) \le C_{\varphi,3}^{-1} r, \\ |k(x,y)| & \text{if } C_{\varphi,3}^{-1} r \le d(x,y) < 2C_{\varphi,3} r, \\ 0 & \text{if } 2C_{\varphi,3} r \le d(x,y) < C_{\varphi,3}^{-1} s, \\ |k(x,y)| & \text{if } C_{\varphi,3}^{-1} s \le d(x,y) < 2C_{\varphi,3} s, \\ 0 & \text{if } 2C_{\varphi,3} s \le d(x,y). \end{cases}$$

Denoting

$$T_{k_{r,s}} f(x) := \int k_{r,s}(x,y) f(y)\, \mu(dy)$$

and analogously $T_{g_{r,s}} f(x)$ we see that $T_{k_{r,s}}, T_{g_{r,s}}$ are bounded operators on L^p $(1 \le p \le \infty)$. From (6) we get for any $f \in L^\infty(\mu)$

$$\begin{aligned} |T_{k_{r,s}} f(x) - T_{g_{r,s}} f(x)| &\le \int_{C_{\varphi,3}^{-1} r \le d(x,y) < 2C_{\varphi,3} r} C_1 \mu(B(x, d(x,y)))^{-1} \|f\|_\infty \,\mu(dy) \\ &\quad + \int_{C_{\varphi,3}^{-1} s \le d(x,y) < 2C_{\varphi,3} s} C_1 \mu(B(x, d(x,y)))^{-1} \|f\|_\infty \,\mu(dy) \\ (7) \qquad &\le C_1 \|f\|_\infty (\log(\mu(B(x, 2C_{\varphi,3} r))) - \log(\mu(B(x, C_{\varphi,3}^{-1} r))) \\ &\quad + \log(\mu(B(x, 2C_{\varphi,3} s))) - \log(\mu(B(x, C_{\varphi,3}^{-1} s)))) \\ &\le C_1 \|f\|_\infty \cdot 2\tilde{n} \log(C_\mu). \end{aligned}$$

Here we have used inequality (1) of the proof of Theorem 2.1 and $\tilde{n} := \inf\{n \in \mathbf{N} \colon 2C_{\varphi,3}^2 \le 2^n\}$. Since the same holds for the adjoint operators $T_{k_{r,s}}^*, T_{g_{r,s}}^*$ we get by duality also

$$\|T_{k_{r,s}} f - T_{g_{r,s}} f\|_{L^1} \le 2C_1 \tilde{n} \log(C_\mu) \|f\|_{L^1} \qquad (f \in L^1(\mu)).$$

Applying the Riesz-Thorin theorem we get for any $1 \le p \le \infty$,

$$(8) \qquad \|T_{k_{r,s}} f - T_{g_{r,s}} f\|_{L^p} \le 2C_1 \tilde{n} \log(C_\mu) \|f\|_{L^p} \qquad (0 < r \le s,\ f \in L^p(\mu)).$$

In particular $(p = \infty)$,

$$\left| \int_{r' \le d(x,y) < s'} g_{r,s}(x,y)\, \mu(dy) \right| \le C_3 + 2C_1 \tilde{n} \log(C_\mu) =: C_3'' \qquad (0 < r' \le s').$$

Thus the kernels $g_{r,s}$, $\gamma' = \min(\tilde{\alpha}, \gamma)$, $\beta' = \max(\beta, 2C_d)$ satisfy 2.1(i)–(iii) with constants C_1'', C_2'', C_3'' depending only on $C_1, C_2, C_3, C_d, C_\mu, C_n, \gamma, \beta$. Because of

(4) the same is true for the adjoint kernels $g^*_{r,s}$. Since 2.1(iv), (v), (vii) are also obvious for $g_{r,s}$ and $g^*_{r,s}$ we may apply Proposition 2.5 to get

$$|||T_{g_{r,s}}|||_{L^2,L^2} \le C'_S \qquad (0 < r \le s < \infty).$$

Together with (8) the assertion follows from

$$T_{k,\varepsilon} = T_{k_{r,s}}, \qquad r := \varepsilon,\ s := \varepsilon^{-1}.$$

REMARK. Setting $r := \varepsilon$, $s := \varepsilon^{-1}$ we get from 2.1(c) and the inequality (7) of the last proof that $(0 < \alpha < \min(\tilde{\alpha}, \gamma))$

$$\begin{aligned}\|T_{k,\varepsilon} f\|_\infty &\le \|T_{k_{r,s}} f - T_{g_{r,s}} f\|_\infty + \|T_{g_{r,s}} f\|_\infty \\ &\le C''_3 \|f\|_\infty + C''_{S,2} |f|_{C^\alpha} \mu(\{f \ne 0\})^\alpha \qquad (f \in C^\alpha_0(X)).\end{aligned}$$

If $\mu(X) = \infty$ then we get from Lemma 2.2 that $(0 < \alpha < \min(\tilde{\alpha}\gamma))$

$$\|T_{k,\varepsilon} f\|_\infty \le C''_{S,2} |f|_{C^\alpha} \mu(\{f \ne 0\})^\alpha \qquad (f \in C^\alpha_0(X))$$

where $C''_{S,2}, C'''_{S,2}$ depend only on $C_1, C_2, C_3, C_d, C_\mu, C_N, \beta, \gamma, \alpha$.

2.7 COROLLARY. *Assume that (X, d, μ) is a normalized homogeneous space. Let $0 < \gamma \le 1$, $W \subset X$, $\mu(X \setminus W) = 0$ and $k\colon X \times X \to \mathbf{C}$ a measurable kernel satisfying* 2.1(i)–(iv). *Assume further that k^* satisfies* 2.1(i)–(iii). *Then there exists a bounded linear operator $T_k\colon L^2(\mu) \to L^2(\mu)$ such that*

(a) $|||T_k|||_{L^2,L^2} \le C_S$,

(b) $\lim_{\varepsilon \to 0} \|T_k f - T_{k,\varepsilon} f\|_{L^2} = 0$ $(f \in L^2(\mu))$.

Here C_S is the same constant as in Theorem 2.6.

PROOF. From Step 1 in the proof of Lemma 2.3 we see that, for sufficiently small $\alpha > 0$, the space $C^\alpha_{00}(X)$ of all function $f \in C^\alpha(X)$ with $\sup\{d(x,y)\colon x, y \in X, f(x) \ne 0 \ne f(y)\} < \infty$ is dense in $L^2(\mu)$. Because of the preceding theorem, it is therefore sufficient to show that $T_{k,\varepsilon} f$ is convergent in $L^2(\mu)$ for any $f \in C^\alpha_{00}(X)$. We may assume $\alpha < \gamma$ and hence the last remark yields

$$C(k,f) := \sup_{\varepsilon > 0} \|T_{k,\varepsilon} f\|_\infty < \infty \qquad (f \in C^\alpha_{00}(X)). \tag{1}$$

Let now $f \in C^\alpha_{00}(X)$ and $x_0 \in X$, $r_0 > 0$ with $\{f \ne 0\} \subset B(x_0, r_0)$. For any $x \in X$, $d(x, x_0) \ge 2C_d r_0$ we have

$$\inf\{d(x,y)\colon f(y) \ne 0\} \ge C_d^{-1} d(x, x_0) - r_0 \ge \tfrac{1}{2} C_d^{-1} d(x, x_0) =: e(x),$$

and therefore, by 2.1(i),

$$\begin{aligned}|T_{k,\varepsilon} f(x)| &\le \int_{B(x_0,r_0)} |k(x,y)|\, \|f\|_\infty\, \mu(dy) \\ &\le C_1 \mu(B(x, e(x)))^{-1} \|f\|_\infty \mu(B(x_0, r_0)) \\ &\le C_1 C_N e(x)^{-1} \|f\|_\infty \mu(B(x_0, r_0)) =: C_f d(x, x_0)^{-1}.\end{aligned}$$

Hence, for any $\varepsilon > 0$ and $x \in X$,

$$|T_{k,\varepsilon} f(x)| \le g(x) := \begin{cases} C(k,f) & \text{if } x \in B(x_0, 2C_d r_0), \\ C_f d(x, x_0)^{-1} & \text{otherwise.} \end{cases}$$

If we can show that $g \in L^2(\mu)$ then the assertion follows from the dominated convergence theorem, since $T_{k,\varepsilon}f$ converges pointwise on W by the proof of Theorem 2.1. Now on one hand we have

$$\int_{d(x,x_0)<2C_d r_0} |g(x)|^2 \, \mu(dx) \le C(k,f)^2 \mu(B(x_0, 2C_d r_0) < \infty$$

and on the other hand we have

$$\begin{aligned}\int_{2C_d r_0 \le d(x,x_0)} &|g(x)|^2 \, \mu(dx) \\ &\le \int_{2C_d r_0 \le d(x,x_0)} C_f d(x,x_0)^{-2} \, \mu(dx) \\ &\le \int_{2C_d r_0 \le d(x,x_0)} C_f C_N^2 \mu(B(x_0, d(x_0,x)))^{-2} \, \mu(dx) \\ &\le \int_{h_{x_0}(2C_d r_0)} C_f C_N^2 t^{-2} \, dt < \infty.\end{aligned}$$

Here we have used inequality (1) of the proof of Theorem 2.1.

2.8 COROLLARY. *Assume that (X, d, μ) is a normalized homogeneous space. Let $0 < \gamma \le 1$ and $k\colon X \times X \to \mathbf{C}$ a measurable antisymmetric kernel, i.e. $k(y,x) = -k(x,y)$, satisfying* 2.1(i)–(iii). *Then there exists a bounded linear operator T_k on $L^2(\mu)$ such that*

(a) $|||T_k|||_{L^2,L^2} \le C_S$,

(b) *$T_{k,\varepsilon}f$ converges ($\varepsilon \to 0$) to $T_k f$ in the weak topology of the Hilbert space $L^2(\mu)$ for any $f \in L^2(\mu)$.*

Again C_S is the same constant as in Theorem 3.6.

PROOF. As in Corollary 2.7 it is enough to show (b) for all $f \in C_{00}^\alpha(X)$ and $\alpha > 0$ sufficiently small. Moreover, since $\{T_{k,\varepsilon}f \colon \varepsilon > 0\}$ is relatively compact with respect to the weak topology, it is enough to prove that $\int g T_{k,\varepsilon} f \, d\mu$ converges for any $f, g \in C_0^\alpha(X)$. This will be accomplished by means of the following formula for antisymmetric kernels:

$$\begin{aligned}&\left| \iint_{\varepsilon_1 \le d(x,y) < \varepsilon_2} f(x) k(x,y) f(y) \, \mu(dy) \, \mu(dx) \right| \\ &\quad = \left| \frac{1}{2} \iint_{\varepsilon_1 \le d(x,y) < \varepsilon_2} k(x,y)(g(x)f(y) - g(y)f(x)) \, \mu(dy) \, \mu(dx) \right| \\ &\quad \le \frac{1}{2} \iint_{\varepsilon_1 \le d(x,y) < \varepsilon_2} |k(x,y)| \, |f(y) - f(x)| \, |g(x)| \, \mu(dy) \, \mu(dx) \\ &\qquad + \frac{1}{2} \iint_{\varepsilon_1 \le d(x,y) < \varepsilon_2} |k(x,y)| \, |g(x) - g(y)| \, |f(x)| \, \mu(dy) \, \mu(dx) \\ &\quad =: I_1(\varepsilon_1, \varepsilon_2) + I_2(\varepsilon_1, \varepsilon_2).\end{aligned}$$

Using the estimate (3) from the proof of Theorem 2.1 we obtain

$$I_1(\varepsilon_1, \varepsilon_2) \le \frac{1}{2} \int |g(x)| \frac{C_1}{\alpha} (\mu(B(x, \varepsilon_2))^\alpha - \mu(B(x, \varepsilon_1))^\alpha) \, \mu(dx)$$

and therefore

$$\lim_{\varepsilon_2\to 0}\sup_{0<\varepsilon_1\le\varepsilon_2} I_1(\varepsilon_1,\varepsilon_2)=0.$$

Of course the same is true for $I_2(\varepsilon_1,\varepsilon_2)$. Because $\delta(f,g):=\sup\{d(x,y)\colon f(x)\neq 0, g(y)\neq 0\}<\infty$, we have

$$\iint_{\varepsilon_2^{-1}\le d(x,y)<\varepsilon_1^{-1}} g(x)k(x,y)f(y)\,\mu(dy)\,\mu(dx)=0 \qquad (0<\varepsilon_1\le\varepsilon_2<\delta(f,g)^{-1}).$$

Having done this we can conclude $(0<\varepsilon_1\le\varepsilon_2<\delta(f,g)^{-1})$ from

$$\begin{aligned}\left|\int gT_{k_{\varepsilon_2}}f\,d\mu-\int gT_{k_{\varepsilon_1}}f\,d\mu\right| &\le \left|\iint_{\varepsilon_1\le d(x,y)<\varepsilon_2} f(x)k(x,y)f(y)\,\mu(dy)\,\mu(dx)\right|\\ &\quad+\left|\iint_{\varepsilon_2^{-1}\le d(x,y)<\varepsilon_1^{-1}} g(x)k(x,y)f(y)\,\mu(dy)\,\mu(dx)\right|\\ &\le I_1(\varepsilon_1,\varepsilon_2)+I_2(\varepsilon_1,\varepsilon_2)+0\end{aligned}$$

that

$$\lim_{\varepsilon_2\to 0}\sup_{0<\varepsilon_1\le\varepsilon_2}\left|\int gT_{k_{\varepsilon_2}}f\,d\mu-\int gT_{k_{\varepsilon_1}}f\,d\mu\right|=0.$$

2.9 EXAMPLES. (a) If we set $d(x,y):=|x-y|^n$ then $(\mathbf{R}^n,d,\lambda^n)$ is a normalized homogeneous space and the results of the introduction follow from the corresponding results in this section. Note that $\Lambda^\alpha=C^{\alpha/n}$. We may choose $\tilde{\alpha}:=\alpha/n$.

(b) The special case of Theorem 1 in A. W. Knapp, E. M. Stein [**8**], which they proved first, is an obvious consequence of Corollary 2.7. The general case follows also from Corollary 2.7 if we use the simple Lemma 1.2 of [**8**]. Moreover, using Theorem 2.1, we get also Hölder estimates for the singular integrals of Knapp and Stein.

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