Introduction to Patterns Part 1

Before students can create or recognize a pattern in a sequence of numbers, they must be able to tell how far apart the successive terms in the sequence are. Students can count up on their fingers, if necessary, to find the gap between two numbers.

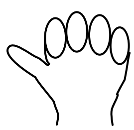
Here is a foolproof method for identifying the gaps between numbers:

EXAMPLE: How far apart are 8 and 11?

<u>Step 1</u>: Say the lower number (8) with your fist closed.

8

<u>Step 2</u>: Count up by ones, raising your thumb first then one finger at a time, until you reach the higher number (11).



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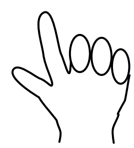

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11

<u>Step 3</u>: The number of fingers you have up when you reach the higher number is the answer. In this case, you have three fingers up, so three is the difference between 8 and 11.

Even the weakest student can find the difference between two numbers using the above method, which you can teach in one lesson. Make sure students say the first number with their fists closed! (Some students will want to put their thumbs up to start.)

Eventually, you should wean students off using their fingers to find the gap between a pair of numbers. The exercises in the Mental Math section of this manual will help with this.

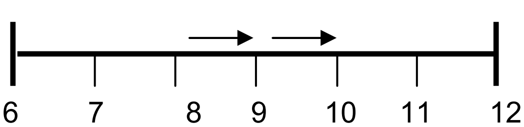
Here is one approach you can use to help students find larger gaps between larger numbers:

1. Have students memorize the gap between the number 10 and each of the numbers from 1 to 9. EXAMPLE: the gap between 8 and 10 is 2 (you need to add 2 to 8 to get 10).

You could make flash cards to help your student learn these facts.

8 + ? = 10		**8 + 2 = 10**
Front of card		Back of card

You could also draw a picture of a number line to help your students visualize the gaps.



2. Have students memorize the gap between 10 and each of the numbers from 11 to 19. Again, you might use flashcards for this:

10 + ? = 17	**10 + 7 = 17**
Front of card	Back of card

Point out that the gap between 10 and any number from 11 to 19 is merely the ones digit of the larger number. EXAMPLE: 16 minus 10 is 6, but 6 is just the ones digit of 16. Once students know this, they will have no trouble recognizing the gap between 10 and any number from 11 to 19.

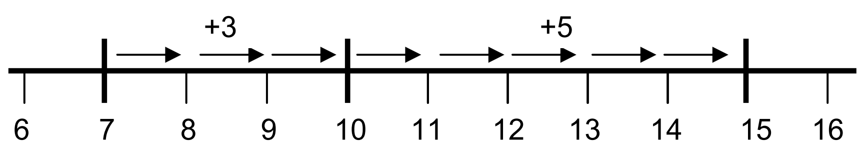
3. Students can now find the gap between a number from 1 to 9 and a number from 11 to 19—say, between 7 and 15—as follows:

Step 1: Find the gap between 7 and 10 (by now, your students will know this is 3).

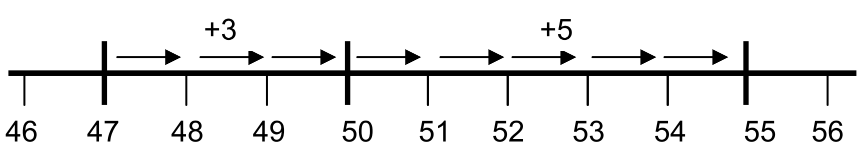
Step 2: Find the gap between 10 and 15 (your students will know this is 5).

Step 3: Add the two numbers you found in steps 1 and 2: 3 + 5 = 8. So the gap between 7 and 15 is 8.

Show students why this works with a picture:



4. Students can use the method introduced in part 3 to find the gap between any pair of two-digit numbers whose leading digits differ by 1. EXAMPLE: The gap between 47 and 55 is 8—start at 47, add 3 to get to 50, and then add 5 to get to 55.



This method can ultimately be used to find the gap between **any pair** of two digit numbers. EXAMPLE: To go from 36 to 72 on the number line, you add 4 to reach 40, then add 30 to reach 70, then add 2 to reach 72; the gap between 36 and 72 is 4 + 30 + 2 = 36. (NOTE: Before students can attempt questions of this sort, they must be able to find the gap between pairs of numbers that have zeros in their ones place. They can find those gaps by mentally subtracting the tens digits of the numbers. EXAMPLE: the gap between 80 and 30 is 50, since 8 − 3 = 5.)

Do not discourage students from counting on their fingers until they can add and subtract readily in their heads. You should expect students to answer all of the questions in this unit, even if they have to rely on their fingers for help.

PA7-1: Extending Patterns

Goal: Students will use the gaps between terms to extend patterns

Prior Knowledge Required:
Can add, subtract, and count up to subtract

Vocabulary:
increasing sequence
decreasing sequence

Curriculum Expectations:
review

Materials: 5 strips of paper (see details below) for each student
a pair of scissors for each student,
tape for each student

Introduce patterns. Use one or both of the Möbius strip activities below. They can be done independently of each other. Then ASK: Why are patterns useful? Explain that patterns allow you to make predictions about things that may be difficult to check by hand. Do you want to try turning the paper in either Activity 100 times? And yet, from the pattern, we can see what will happen without checking.

Extending sequences that were made by adding or subtracting the same number to each term. See Questions 1 and 2.

EXAMPLE: Extend the pattern 3, 6, 9, ... up to six terms.

<u>Step 1</u>: Identify the gap between successive pairs of numbers in the sequence. (Students may count on their fingers, if necessary – see the Introduction.) The gap in this example is three. Check that the gap between successive terms in the sequence is always the same, otherwise you cannot continue the pattern by adding a fixed number. Write the gap between each pair of successive terms above the pairs.

$$\begin{array}{ccccccccccc} & 3 & & 3 & & & & & & & \\ 3 & , & 6 & , & 9 & , & \_\_\_\_ & , & \_\_\_\_ & , & \_\_\_\_ \end{array}$$

<u>Step 2</u>: Say the last number in the sequence with your fist closed. Count by ones until you have raised the same number of fingers as the gap, in this case, three. The number you say when you have raised your third finger is the next number in the sequence.

$$\begin{array}{ccccccccccc} & 3 & & 3 & & & & & & & \\ 3 & , & 6 & , & 9 & , & \underline{12} & , & \_\_\_\_ & , & \_\_\_\_ \end{array}$$

<u>Step 3</u>: Repeat Step 2. Continue adding terms to extend the sequence.

3 , 6 , 9 , _*12*_ , _*15*_ , _*18*_

<u>Extra Practice for Question 1:</u> Extend the pattern.

a) 6, 9, 12, 15, _____, _____ b) 5, 11, 17, 23, ____, ____ c) 2, 10, 18, 26, ____, ____

Bonus: a) 99, 101, 103, ____, ____ b) 654, 657, 660, ____, ____

Extra Practice for Question 2: Extend the pattern.
a) 21, 19, 17, 15, ____, ____ b) 34, 31, 28, _____, ____, ____ c) 48, 41, 34, 27, ____, ____

Bonus:
a) 141, 139, 137, 135, _____ b) 548, 541, 534, 527, _____ c) 234, 221, 208, ____

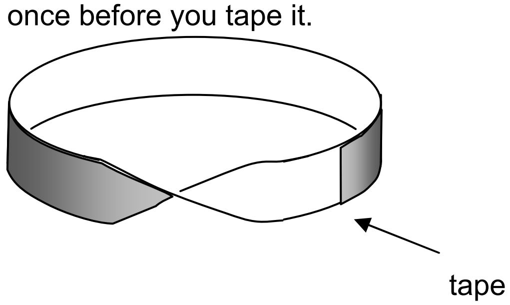
Extend sequences by extending the pattern in the gaps. PE – Looking for a pattern See Question 3. In parts a), b), and d), the gaps form sequences similar to those in Questions 1 and 2. In parts c) and e), the gaps form the same sequence as the original.

Extra practice for Question 3: Extend the pattern. 1, 3, 6, 10, 15, ____, ____, ____
Bonus: 1, 4, 10, 20, 35, ____, ____, ____
Extra Bonus: 1, 5, 15, 35, 70, ____, ____, _____

Activity 1 PE – Looking for a pattern
Each student will need 3 strips of paper (11″ × 1″), 2 longer strips of paper (say, 22″ × 1″), a pair of scissors, and tape (staples do not work for this activity).

Show your students a sheet of paper and ask how many sides it has. (2) Repeat with an 11″ × 1″ strip of paper with the ends taped so that it looks like a ring. Trace each side of the ring with your finger, naming them “inside” and “outside.” Point out that you could colour one side and leave the other side blank. Have students create their own rings and colour one side. ASK: If an ant is walking along the coloured side (and never goes over the edge), will it always stay on the coloured side? (yes)

Take another strip and tape the ends together as though to make a ring, but this time turn one of the ends once before you tape it.



Have students do the same. Have students put a finger somewhere on the strip and ASK: Is your finger on the inside or on the outside of the strip? Ask students to slide their fingers along the strip until everyone has their finger on the outside. Have students continue sliding their fingers along the strip until everyone has their finger on the inside. ASK: How did you slide your finger from the outside to the inside without going over the edge? Could you have done that with the original ring? (no)

Show your own strip and explain that you think it has two sides (point to two opposite “sides” at the same point). Suggest to students that if there were two sides, they should be able to colour only one side. Challenge them to do so. Students will see that if they colour a whole side, they have to colour every part of the paper, even what was originally the other side of the strip!

Explain to the students that when they taped the two ends together after turning one of the ends, they created only one side—they glued the “inside” to the “outside.” Explain that this surface is called a Möbius strip. Show an 11” × 1” strip of paper with one side coloured. Then demonstrate turning it into a Möbius strip and show how the coloured side becomes the white side.

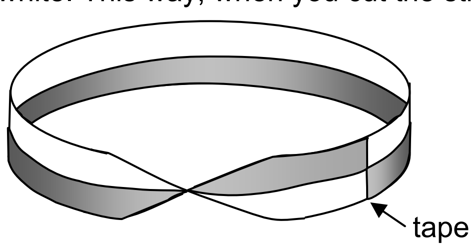
Then ask students what they think will happen if they make two turns instead of one before taping the ends together. Do they end up with one side or two sides? (two sides) Have them predict and then check their prediction. Repeat with three turns (one side), and four turns (two sides), this time using the longer strips of paper. Have students predict what will happen with five turns and six turns. What about 99 turns? (one side) 100 turns? (two sides)

Activity 2
Each student will need 3 strips of paper (11″ × 1″) and 2 longer strips of paper (say, 22″ × 1″) with a line drawn lengthwise in the middle of each strip, a pair of scissors, and tape (staples do not work for this activity). Draw the lines with a marker that bleeds through the paper, so that the lines are visible on the other side of the strip as well.

Ask students to tape the ends of one of the strips of paper to make a ring, so that the ends of the line in the middle meet. SAY: I want to cut this ring along the line (hold up your own ring to illustrate what you mean). What will I get? (two thinner rings). Have students check their predictions by cutting their rings.

Take another strip of paper and tape the ends together, this time turning one of the ends once before you tape them. Make sure the ends of the line meet, as before. Have students do the same. Ask students to predict what will happen when they cut the strip along the line. (Students who have never seen a Möbius strip before will likely predict that there will be two rings. Ask students who have seen or done this before not to reveal the right answer.) Then have students cut their strips to check their prediction. (There will be only one ring!)

Explain that the surface students made before cutting it in half is called a Möbius strip. Then ASK: What will happen to the surface when I cut it if I turn one end two times before taping it to the other end? Have students check their prediction. (There will be two rings linked together.) Repeat with three turns (one ring) and four turns (two rings), this time using the longer strips. Have students predict what will happen with five turns and six turns. What about 99 turns? 100 turns? (For even numbers of turns, there will be two rings. For odd numbers of turns, there will be a single ring.) To explain why this happens, colour half of the strip along the middle line (e.g. colour the bottom half of both sides of the strip). When you turn the end once, the coloured half is glued to the white half, and the resulting ring is half coloured and half white. When you turn the end twice, the coloured half is glued back to the coloured half, and the white to the white. This way, when you cut the strip, you separate the coloured ring from the white ring.



Extensions
1. Find the incorrect number in each pattern and correct it.
   a) 2, 5, 7, 11 add 3
   b) 7, 12, 17, 21 add 5
   c) 6, 8, 14, 18 add 4
   d) 29, 27, 26, 23 subtract 2
   e) 40, 34, 30, 22 subtract 6

2. PA – [R, C, PS, 7m1, 7m7]Each pattern was made by adding or subtracting the same number each time. Find the missing number(s) in each pattern and explain the strategy you used.

a) 2, 4, ___, 8	b) 9, 7, ___, 3	c) 7, 10, ___, 16	d) 16, ___, 8, 4
e) 3, ___, 11, 15	f) 15, 18, ___, 24, ___, 30	g) 14, ___, ___, 20	h) 57, ___, ___, 48

SOLUTION: In parts a)–f), you can find the gap directly, since two consecutive terms are given, then use the gap to find the missing terms. Parts g) and h) require more work. Here are two possible strategies students can use:

- [redacted] **Guess, check, and revise.** For example, for part g), you know you have to add because 20 is more than 14. Try adding 1 each time; this only gets you to 17: 14, 15, 16, 17. Try adding 2 each time; this gets you to 20: 14, 16, 18, 20.
- [redacted] **Find the gap by determining the number of steps needed to get from one given term to the next.** For example, in part g), you have to increase 14 by 6 in 3 equal steps, so each step must be an increase of 2. Similarly, for h), you need to decrease 57 by 9 in 3 equal steps, so each step must be a decrease of 3.

Bonus: 15, ___, ___, 24, ___ 59, ___, ___, ___, 71 100, ___, ___, ___, ___, ___, 850

PA – [R, PS, C, 7m1, 7m7] Question 4

PA7-2: Describing Patterns

Goal: Students will describe increasing, decreasing, and repeating patterns by writing a rule.

Vocabulary:
increases
decreases
term

Prior Knowledge Required:
Can add, subtract, multiply, and divide
Can count up to subtract

Curriculum Expectations:
Ontario: 5m63, 5m66

Match sequences to descriptions. See Question 2.

<u>Extra Practice for Question 2:</u>

Match the description to the pattern. Each description may fit more than one pattern.

17, 16, 14, 12, 11	3, 5, 7, 3, 5, 7, 3	A: Increases by the same amount
10, 14, 18, 22, 26	4, 8, 11, 15, 18,	B: Decreases by the same amount
54, 47, 40, 33, 26	4, 5, 3, 2, 6, 2, 4	C: Increases by different amounts
4, 8, 12, 8, 6, 4, 6	11, 19, 27, 35, 43	D: Decreases by different amounts
74, 69, 64, 59, 54	12, 15, 18, 23, 29	E: Repeating pattern
98, 95, 92, 86, 83	67, 71, 75, 79, 83	F: Increases and decreases

Match the description to the pattern.

97, 96, 94, 92, 91	A: Increases by the same amount
210, 214, 218, 222	B: Decreases by the same amount
654, 647, 640, 633	C: Increases by different amounts
3, 5, 8, 9, 3, 5, 8, 9	D: Decreases by different amounts
444, 448, 451, 456	E: Repeating pattern
741, 751, 731, 721	F: Increases and decreases by different amounts

Identify terms in patterns. Write these three patterns:

1, 5, 10, 1, 5, 10, 1, 5, 10, …

red, blue, green, yellow, red, blue, green, yellow, red, blue, green, yellow, …

do re mi fa so la ti do re mi fa so la ….

ASK: What is the same about all these patterns? (they are all repeating patterns) What is different? (the length of the core—the part that repeats—is different; the patterns consist of different types of things—numbers, colours, and musical notes)

Explain that each thing in a pattern—whether it's a number, a colour, a musical note, a shape, or anything else—is called a **term**. Have volunteers identify the third term in each sequence above. (10, green, and mi)

<u>Extra Practice:</u>

a) What is the third term of the sequence 2, 4, 6, 8?

b) What is the fourth term of the sequence 17, 14, 11, 8?
c) Extend each sequence to find the sixth term.
i) 5, 10, 15, 20 ii) 8, 12, 16, 20 iii) 131, 125, 119, 113, 107

Sequences made by multiplying and dividing each term by the same number. See Questions 4 and 5.

<u>Extra Practice for Questions 4 and 5:</u>

What operation was performed on each term in the sequence to make the next term?
a) 2, 4, 8, 16, … (multiply each term by 2)
b) 10 000, 1 000, 100, 10, … (divide each term by 10)
c) 10 000, 5 000, 2 500, 1 250, … (divide each term by 2)
d) 5, 15, 45, 135, … (multiply each term by 3)

Introduce rules of the form "Start at ___, add/subtract/multiply by/divide by ___."
<u>Extra Practice for Question 6:</u>
1. Write the rule for each pattern.
a) 12, 15, 18, 21, … b) 19, 17, 15, 13, … c) 132, 136, 140, 144, …
d) 1, 3, 9, 27, …. e) 224, 112, 56, 28, … f) 25, 75, 225, 675, …

2. Use the description of each sequence to find the 4^{th} term of the sequence.
a) Start at 5 and add 3.
b) Start at 40 and subtract 7.
c) Start at 320 and divide by 2.
d) Start at 5 and multiply by 4.

Write rules for repeating patterns. See Question 8.

Extensions

1. One of these sequences was not made by adding or subtracting the same number each time. Find the sequence and state the rules for the other two sequences.

A 25, 20, 15, 10
B 6, 8, 10, 11
C 9, 12, 15, 18

2. The first term of a sequence of numbers is 2. Each term after the first is obtained by multiplying the preceding term by 5 then subtracting 6. What is the 5^{th} term of the sequence?

3. Match each pattern to its description.
1, 4, 13, 40, 121 A. Multiply by 5 and subtract 1.
1, 4, 7, 10, 13 B. Multiply by 3 and add 1.
1, 4, 19, 94, 469 C. Add 5 and subtract 2.

PA7-3: T-tables

Goal: Students will use T-tables to solve word problems.

Vocabulary:
T-table
rule

Prior Knowledge Required:
Can find the gaps between numbers
Can extend patterns obtained by doing one operation successively

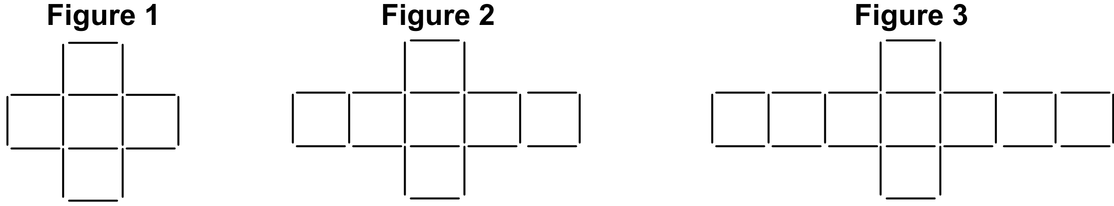
Curriculum Expectations:
Ontario: 5m63, 5m64, 5m65, 7m1, 7m6
WNCP: [R, PS]

Introduce T-tables using the example on the worksheet. PE – Organizing data Work through the example at the top of the worksheet together. You could point out that this type of chart is called a T-table because the central part of the chart looks like a T.

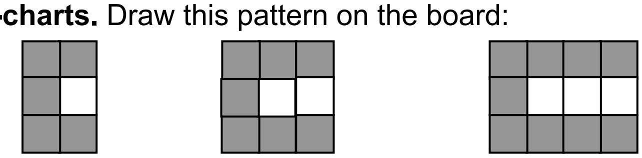
Teach students how to use and create T-tables by following the progression in Questions 1–4: start by identifying the rules for patterns from completed T-tables (Question 1); then use T-tables to extend patterns (Questions 2 and 3); then have students create their own T-tables to extend patterns (Question 4).

Extra Practice: PE – Looking for a pattern
Count the number of toothpicks in each figure. Then use a T-table to determine how many toothpicks make up Figure 5.

Figure 1 **Figure 2** **Figure 3**



Double T-charts. Draw this pattern on the board:



Make a double T-chart—a T-chart with 3 columns—with headings Number of Unshaded Squares, Number of Shaded Squares, and Number of Squares. Have students copy the blank chart in their notebooks and fill it in independently. Then have students use the chart to answer these questions:
a) How many shaded squares will be needed for a figure with 7 unshaded squares?
b) How many squares will be needed for a figure with 15 shaded squares?

Show the following double T-chart and ask students to answer the questions below.

Time (min)	Fuel (L)	Distance from airport (km)
0	1200	525
5	1150	450
10	1100	375

a) How much fuel will be left in the airplane after 25 minutes?
b) How far from the airport will the plane be after 30 minutes?
c) How much fuel will be left in the airplane when it reaches the airport?

Students will need to make double T-charts to solve Question 10 on the worksheet.

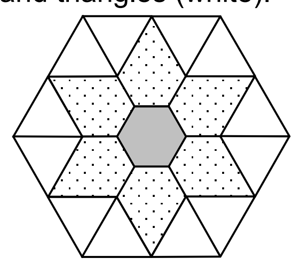
Extra Practice with Word Problems: PE – Problem-Solving

1. The snow is 17 cm deep at 5 p.m. Four centimetres of snow falls each hour.
   How deep is the snow at 9 p.m.? (33 cm)

2. Philip has $42 in savings at the end of July. Each month he saves $9.
   How much will he have by the end of October? ($69)

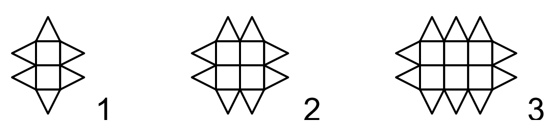
3. Carol's plant is 3 cm high and grows 5 cm per week.
   Ron's plant is 9 cm high and grows 3 cm per week.
   How many weeks will it take for the plants to reach the same height? (3 weeks)

4. Rita made an ornament using a hexagon (shaded), pentagons (dotted), and triangles (white).
   a) How many pentagons does Rita need to make 7 ornaments? (42)
   b) Rita used 12 hexagons to make ornaments.
      How many triangles did she use? (144)
   c) Rita used 12 pentagons to make ornaments.
      How many triangles did she use? (24)



5. A store rents snowboards at $7 for the first hour and $5 for every hour after that.
   How much does it cost to rent a snowboard for 6 hours? ($32)

6. a) How many triangles would Ann need to make a figure with 10 squares? (14)
   b) Ann says that she needs 15 triangles to make the sixth figure. Is she correct? (No, to make the sixth figure she needs 16 triangles.)



7. Merle saves $55 in August. She saves $6 each month after that.
   Alex saves $42 in August. He saves $7 each month after that.
   Who has saved the most money by the end of January? (Merle has $85, whereas Alex only has $77, so Merle has saved the most money by the end of January.)

Activity
Give each student a set of blocks and ask them to build a sequence of figures that grows in a regular way (according to some pattern rule) and that could be a model for a given T-table. Here are some sample T-tables you can use for this activity:

Figure	# of Blocks
1	4
2	6
3	8

Figure	# of Blocks
1	3
2	7
3	11

Figure	# of Blocks
1	1
2	5
3	9

Extensions
1. PA – [PS, R, V, 7m1, 7m6]
a) How many 11s would there be in this sequence? 1 3 3 5 5 5 … (ANSWER: 6)
b) How many 7s would there be in this sequence? 1 2 2 2 2 3 3 3 3 3 3 3 … (ANSWER: 19)
HINT: Make a T-chart with the headings Number and Number of Times it Occurs

2. **Magic Squares** Show students the following 3 × 3 array of numbers:

4	7	4
5	5	5
6	3	6

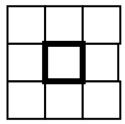
Explain that this is a **magic square** because all the numbers in each row, column, and diagonal add to the same number, in this case 15. Verify this together. (4 + 7 + 4 = 15, 5 + 5 + 5 = 15, and so on)

A **pure 3 × 3 magic square** places each of the numbers from 1 to 9 exactly once in a 3 × 3 grid in such a way that each row, column, and diagonal adds to the same number. Follow the steps below to make a pure 3 × 3 magic square.

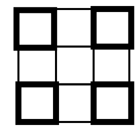
a) By pairing numbers that add to 10, find 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9. (ANSWER: 45)
b) Your answer to part a) tells you what all 3 rows add to. What does **each row** add to? (ANSWER: 45 ÷ 3 = 15) This is called the magic sum.
c) PE – Organizing data List all possible ways of adding **3 different numbers** from 1 to 9 to total 15 (EXAMPLE: 2 + 4 + 9 works, but 3 + 3 + 9 and 6 + 9 do not)

ANSWER:	1 + 5 + 9	1 + 6 + 8	2 + 4 + 9	2 + 5 + 8
	2 + 6 + 7	3 + 4 + 8	3 + 5 + 7	4 + 5 + 6

d) PE – Using logical reasoning Look at a 3 × 3 grid. How many sets of numbers that add to 15 must the number in the middle square be a part of? ANSWER: 4—the middle row, the middle column, and both diagonals.



Look at your list from part c) to determine which number must be in the middle.
ANSWER: Only the number 5 occurs four times, so 5 must be in the middle.



e) PE – Looking for a pattern Which numbers must be corner numbers? Why?
ANSWER: The corner numbers are each part of three sums. This happens for 2, 4, 6, and 8. (The numbers 1, 3, 7, and 9 only occur in two sums, so these must be in the remaining four squares.)

f) Write the numbers in the grid to make a pure 3 × 3 magic square!

PE – Connecting Compare your magic square with those of other people. What transformations (e.g., rotations or reflections) can you do to a magic square to get another magic square? (SAMPLE ANSWERS: rotate 90° clockwise; reflect vertically using middle column as a mirror line)

Now make a magic square with the numbers 2, 3, 4, 5, 6, 7, 8, 9, 10.
What will the new magic sum be? What if you use the numbers 3, 4, 5, 6, 7, 8, 9, 10, 11?

Make a T-table of magic sums:

Numbers Used in the Magic Square	**Magic Sum**
1–9	15
2–10	18
3–11	21
4–12	24

Notice that the second number in the first column isn't necessary—students could just make this T-table:

Least Number Used	**Magic Sum**
1	15
2	18
3	21
4	24

Predict the magic sum for a magic square made with the numbers 8, 9, 10, 11, 12, 13, 14, 15, 16.

PA7-4: Patterns (Advanced)

Goal: Students will investigate patterns in geometrical sequences, the Fibonacci sequence, and Pascal's triangle.

Prior Knowledge Required:
Can extend patterns
Can use charts and T-tables to display sequences

Vocabulary:
even
odd

Curriculum Expectations:
Ontario: 7m1, 7m3, 7m5, 7m6, 7m7
WNCP: [R, ME, C, CN]

Finding patterns within patterns. PE – Looking for a pattern, Organizing data Have students extend this pattern: 1, 4, 7, 10, …. Then tell students that you would like to look for a pattern within this pattern. Review the terms even and odd, then have students identify each term in the pattern as even or odd and record their answers in a table like this one:

Term Number	1	2	3	4	5	6	7	8	9	10
Term	1	4	7	10						
Even or Odd	O	E								

Have students predict whether the 100th term will be even or odd and explain their prediction. (The odd-even pattern is "O, E, then repeat." The 100th term will be even because every even-numbered term is even.)

Now have students extend this pattern: 2, 4, 8, 16, ….
Tell students that you would like to know if there is a pattern in the ones digits of this sequence. How about in the tens digits in this sequence? (The ones digits form a repeating pattern: 2, 4, 8, 6, repeat; the tens digits form no easily discernible pattern.)

Finally, have students extend the pattern 1, 4, 9, 16, … and look for an odd-even pattern. (O, E, repeat)
Bonus: Look for a pattern in the ones digits. (1, 4, 9, 6, 5, 6, 9, 4, 1, 0, repeat; notice the symmetry in the core of this pattern)

Pascal's triangle. PE – Looking for a pattern. See Question 3.

<u>Extra Practice:</u> Describe the pattern in the numbers along the 2nd diagonal of Pascal's triangle.

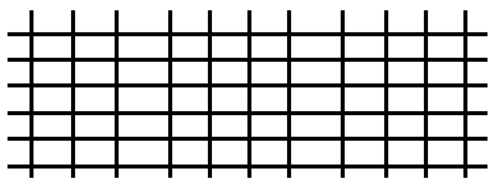
Bonus 1: Add the numbers in each row of Pascal's triangle. For example, the numbers in the third row add to 1 + 2 + 1 = 4. Use a T-table to find the sum of each of the first five rows, then predict the sum of the 8^{th} row.

Bonus 2: Describe the pattern in the numbers along the 4^{th} diagonal of Pascal's triangle.

Estimating and thinking before solving a problem. PE – Mental math and estimation Before assigning the Investigation, show students the diagram on the worksheet. If possible, have it reproduced on an overhead transparency so that you can show it for a brief time and remove it quickly (to prevent students from counting the lines).

Have students guess how many lines there are without counting them. Take all answers.

PE- Reflecting on what made the problem easy or hard Discuss what makes this problem hard. If students say "Because the lines intersect in so many places", show them the following set of lines to count:



Challenge students to articulate why this set of lines is easier to count.

Solving the problem in the Investigation. Discuss strategies for solving the problem. Guide the students to do a simpler problem first. Suggest that if they find the answer in easier cases first, they might find a pattern. ASK: How can we make this problem easier? PROMPT: There are 8 dots with all possible lines joining them. What problem can we solve that would be easier? Listen to students' suggestions and, if they don't bring it up, point out that if they solve the same problem with 1 dot, 2 dots, 3 dots, and 4 dots, they might find a pattern.

Reflecting on other ways to solve the problem. After students finish the Investigation, explain that you noticed another pattern within the pattern of the number of lines for each given number of dots:

Number of Dots	1	2	3	4	5	6	7
Number of Lines	$0=\frac{0\times1}{2}$	$1=\frac{1\times2}{2}$	$3=\frac{2\times3}{2}$	$6=\frac{3\times4}{2}$	$10=\frac{4\times5}{2}$	$15=\frac{5\times6}{2}$	$21=\frac{6\times7}{2}$

Challenge students to predict the expression for the number of lines for 8 dots. Does their expression give the right answer, 28? Then challenge students to explain why the expression works: What does 7 × 8 tell you? Why do we divide by 2? (There are 8 dots and 7 lines extending from each dot, so 7 × 8 tells you the number of **endpoints** altogether. Since each line has two endpoints, the total number of lines is 7 × 8 ÷ 2 = 28.)

Activity PE – Reflecting on what made a problem easy or hard, Organizing data, Connecting
Have students get into groups of 2 and shake hands with everyone else in their group. How many handshakes were there? (1) Repeat with groups of 3, groups of 4, and groups of 5. (Ensure that different students are left out, when necessary, of each round.) What do students notice about their answers to this problem and their answers to the Investigation on the worksheet? (They are the same!) Discuss why this happened. Students could arrange themselves in a circle, so that each student represents a point and each handshake represents the line between the two points. Have students arrange themselves into groups of 7 or 8 and to count the handshakes directly. Encourage students to be organized in their counting. Was counting handshakes easier or harder than counting lines between points? Why? (Students might find it easier to keep track of who they have already shaken hands with than which lines they have already counted.)

Extensions

1. Sudoku is a popular mathematical game that is a regular feature in many newspapers. See Extra Worksheets and Blackline Masters (p XXX) for Sudoku suitable for children with step-by-step instructions. Once students master this easier form of Sudoku they can try the real thing.
2. Pick one number from each row in the grid below; each number must be in a different column. Add the numbers. Now repeat with a different set of selections. What do you notice about the two sums? (ANSWER: They are the same.) Will this always happen? (yes) Can you explain why it happens?

1	**2**	**3**	**4**	**5**
6	**7**	**8**	**9**	**10**
11	**12**	**13**	**14**	**15**
16	**17**	**18**	**19**	**20**
21	**22**	**23**	**24**	**25**

EXPLANATION: Let's label each row according to the first number in the row: the 1 row, the 6 row, the 11 row, the 16 row, the 21 row. One number is selected from each row. If you select a number in, say, the 16 row, you can either pick 16 + 0, 16 + 1, 16 + 2, 16 + 3, or 16 + 4. No matter which row you pick from, you are either adding 0, 1, 2, 3, or 4 to the first number in that row. Since you pick one number from each column, you add 0 once, 1 once, 2 once, 3 once, and 4 once, so the sum is 1 + 6 + 11 + 16 + 21 + 0 + 1 + 2 + 3 + 4 = 65.

To help students discover this explanation, ask them to answer the same questions for one or more of the arrays below (or others like them). On the first card, how often are the numbers from the first part of each sum (1, 6, 11, 16, and 21) selected? How often are the numbers 0, 1, 2, 3, and 4 from the second part selected? (once each)

1 + 0	1 + 1	1 + 2	1 + 3	1 + 4
6 + 0	6 + 1	6 + 2	6 + 3	6 + 4
11 + 0	11 + 1	11 + 2	11 + 3	11 + 4
16 + 0	16 + 1	16 + 2	16 + 3	16 + 4
21 + 0	21 + 1	21 + 2	21 + 3	21 + 4

1 + 0	1 + 1	1 + 2	1 + 3	1 + 4
11 + 0	11 + 1	11 + 2	11 + 3	11 + 4
21 + 0	21 + 1	21 + 2	21 + 3	21 + 4
31 + 0	31 + 1	31 + 2	31 + 3	31 + 4
41 + 0	41 + 1	41 + 2	41 + 3	41 + 4

1 + 0	1 + 1	1 + 3	1 + 5	1 + 7
6 + 0	6 + 1	6 + 3	6 + 5	6 + 7
11 + 0	11 + 1	11 + 3	11 + 5	11 + 7
16 + 0	16 + 1	16 + 3	16 + 5	16 + 7
21 + 0	21 + 1	21 + 3	21 + 5	21 + 7

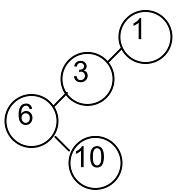
1 + 0	1 + 2	1 + 4	1 + 5	1 + 8
11 + 0	11 + 2	11 + 4	11 + 5	11 + 8
21 + 0	21 + 2	21 + 4	21 + 5	21 + 8
31 + 0	31 + 2	31 + 4	31 + 5	31 + 8
41 + 0	41 + 2	41 + 4	41 + 5	41 + 8

Students can make up their own such 5 × 5 grid and present it as a magic trick to a younger class. One way to present this as a magic trick would be as follows: Each student gives a grid to a younger buddy. The student tells the younger buddy to pick 5 numbers, one from each row and column. (Your student may need to explain to the buddy what a row is and what a column is.) The buddy then adds the numbers but doesn't tell your student the sum. Your student asks questions that may seem relevant, but really are not. (EXAMPLES: Is the number in the third column bigger or smaller than the number in the second column? How far apart are the two biggest numbers?) Students will have to be careful to ask questions that are compatible with their buddies' level. Students then give the correct answer, to their buddies' surprise.

3. PA – [PS, R, 1m1, 1m2] Look at Pascal's triangle.

a) Start at the top right of any (right to left) diagonal and move along the diagonal, adding the numbers you encounter. Stop at any point you wish. Where will you find the sum? (ANSWER: Just below and to the right of the last number you added.)

EXAMPLE:



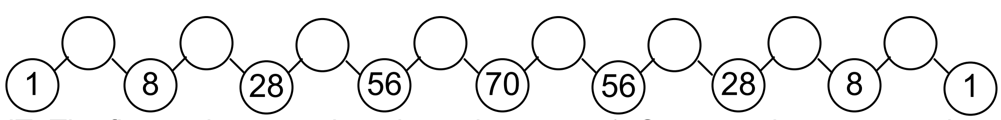
1 + 3 + 6 = 10

b) How can you find the sum 1 + 2 + 3 + 4 + 5 quickly using Pascal's triangle?

HINT: Use the pattern you found in part a).

c) Extend Pascal's triangle to 15 rows and shade the even numbers. What patterns do you see?

d) Without extending Pascal's triangle, can you find the missing numbers in the 8th row?



8th row
9th row

HINT: The first and last numbers in each row are 1. Some students may notice that since the rows are symmetrical, they can reduce their work by half.

PA – [7m1, 7m2, PS, R] Investigation