## Learning Objectives

This lesson is designed to transition students from an understanding of right triangle trigonometry to an understanding of the unit circle and trig ratios for angles greater than $90°$ or less than $0°$.  They should develop a solid conceptual connection between right triangles and the unit circle, as well as understand the expanded definitions for sine, cosine, and tangent.

### *Math Objectives*

- The student will understand the connections between the right triangle trigonometry and coordinates of a point on a unit circle.
- The student will understand and be able to use the expanded definitions of sine, cosine, and tangent for angles greater than $90°$ or less than $0°$.

### *Technology Objectives*

- The student will become familiar with use of the various tools in the Geometry Expressions program, including drawing, constraining, animating, and constructing a locus

## Math Prerequisites

- The student must understand and be proficient in right-triangle trigonometry.
- The student must understand a four-quadrant Cartesian coordinate system.

## Technology Prerequisites

- Students should be able to complete this lesson with little to no background with Geometry Expressions.

## Materials

- Computer with Geometry Expressions for each student or pair of students.

## Overview for the Teacher

In this lesson, students will transition from an understanding of trigonometry which is limited to right triangles, to an understanding of the unit circle. This is done in a way that strongly emphasizes the logical connection and partial equivalence of the two aspects of trigonometry. Most students should be able to work through this lesson independently – it is not designed to require any direct instruction from the teacher. However, some common issues to look for and be aware of are outlined below.

This lesson involves a minimal amount of writing; most of the work will be done on the computer. You may want to institute a set of checkpoints, where you confirm the work has been done on the screen, and initial student papers. We suggest steps 8, 14, and 16 would be appropriate. Furthermore, the independent practice is not included, since exercises of this type are readily available. Following this lesson, students will need to do some practice (in worksheet form or otherwise) with…

- identifying coordinates of points on a unit circle given an angle of rotation
- identifying the sign of sine and cosine in the four quadrants
- correlating coordinates of points on a unit circle to their respective trig values.

Steps 1-5: In early versions of GX, some constraint contradictions tended to happen if one constructed the triangle by constraining point C to the axis directly or through its coordinates. If the location of C is established using a slope of 0 or direction of $0°$, the problem is avoided. This shouldn't be a problem in later versions of the software.

By step 8, student diagrams should resemble the following. (Numbers will be different depending on the angle measure chosen and the length of c.)
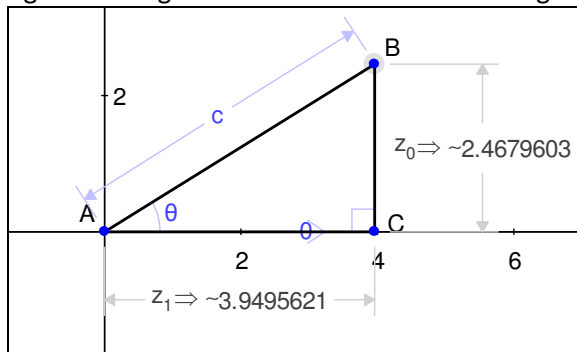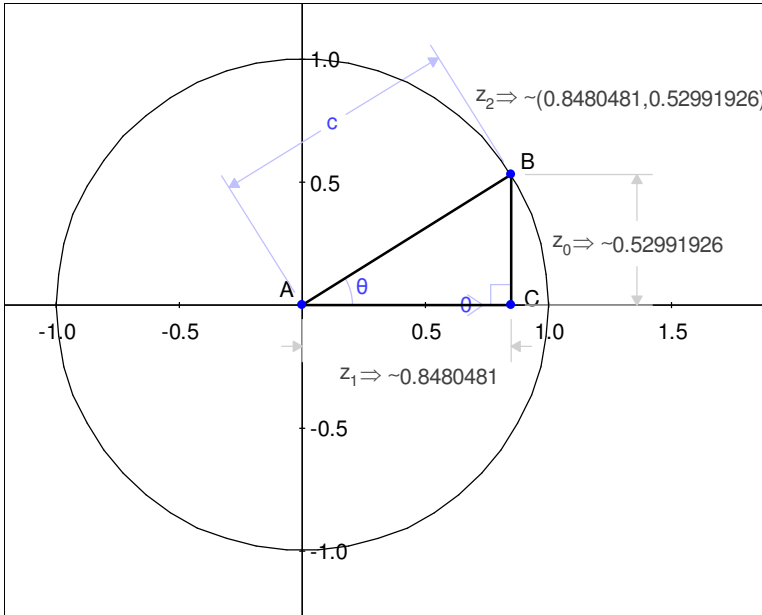


Step 14: Students may want to delete the side length measurements after they figure out the equivalence with the coordinates of point B. This is fine, and cleans up the diagram.

Step 16: The default variable for constructing a locus is c, but $\theta$ is needed. If students get a straight line instead of a circle, have them make sure they chose variable $\theta$.

Final diagram should look like:

# Intro to Unit Circle Trigonometry

Before you get started:

Open a new Geometry Expressions file. Make sure you are in degree mode and display your axes. (Edit/Preferences/Math/Math/Angle Mode -> Degrees; toggle the axis/grid button on

the top toolbar ▦)

Right Triangle:

1) Pick a number between 10 and 80. Write it here: _____

2) You are going to create a triangle in the first quadrant. Use **draw line segment** to create triangle ABC, with A snapped to the origin, and B and C both somewhere in the first quadrant.

3) Highlight line segment AC and use **constrain slope** to set the slope to zero. This makes the segment line up with the x-axis.

4) Highlight line segments BC and AC at the same time by holding down the shift key

and clicking on each. Use **constrain perpendicular** to form a right angle.

5) Highlight line segments AB and AC at the same time and use **constrain angle** to constrain the angle measure to $\theta$.

You have just created a right triangle in **standard position**. A right triangle in standard position has the vertex with the angle you are concerned about ($\theta$) at the origin, one leg of the triangle on the positive x-axis, and the third vertex in the first quadrant. Note that any right triangle could be positioned this way by rotating, translating, and/or reflecting it.
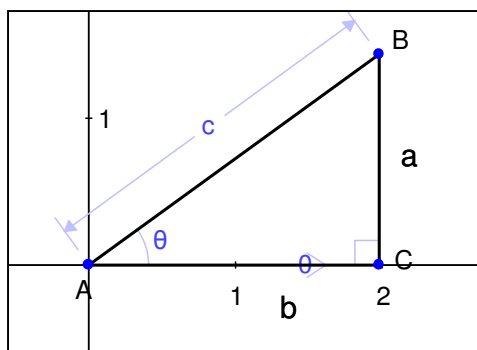
6) Highlight segment AB and use **constrain distance** to determine its length to be the variable $c$.

7) Now make the triangle your own. Use the Variables tool panel, highlight $\theta$, then below that box, type the number you chose in part 1. Lock $\theta$ by clicking on the icon next to the number.

8)      Use **calculate real distance** 🔲 to calculate lengths *a* and *b* of your triangle; length *c* is displayed in the Variables window.  Use those lengths to determine your sine, cosine, and tangent ratios.

| Side | Length |
|------|--------|
| a    |        |
| b    |        |
| c    |        |

|              | Ratio | Decimal |
|--------------|-------|---------|
| Sine ($\theta$)    | /     |         |
| Cosine ($\theta$)  | /     |         |
| Tangent ($\theta$) | /     |         |

9)      Use the trig functions on your calculator to check the ratios.  Make sure your calculator is in degree mode.

10)     What should happen to the triangle if we increase or decrease length *c*?   Test your idea on GX by highlighting *c* in the Variables tool panel and changing the value.  You can increase or decrease *c* by either typing in new numbers, or using the scroll bar underneath.

11)     What will happen to the trig ratios as you do this?

12)     For the rest of the activity, set *c* equal to 1 and lock it in the Variables tool panel.

        You will probably need to scale up ( 🔍 button on the top tool bar – like zooming in) so the diagram takes up much of your screen.

13)     Highlight point B and use **calculate real coordinates** to find the location of B. Compare this to your results in step 8.  What do you notice?

14) Now adjust your diagram for some other angle measures by unlocking $\theta$ (click on the icon again), then either using the Variables tool panel, or clicking and dragging point B in your diagram. Test whether the trig ratios for two different angle measures correspond to coordinates for point B.

| $\theta$ | Coordinates of B | Cos($\theta$) | Sin($\theta$) |
|---|---|---|---|
| | ( , ) | | |
| | ( , ) | | |

Why does this work? If a right triangle is in standard position, the opposite side length is equal to the y-coordinate of point B, and the adjacent side length is equal to the x-coordinate of B.

Also, notice that we scaled the triangles back so that the hypotenuse is always 1. This can be done to any right triangle without changing the ratios of the sides.
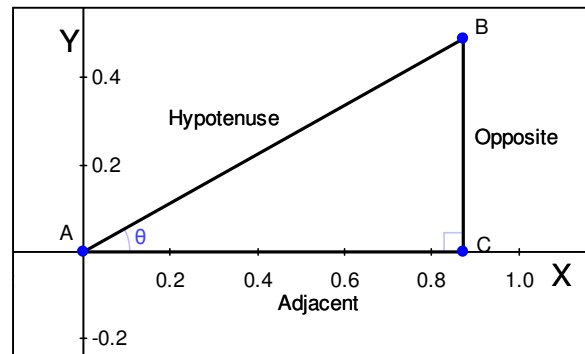
Therefore,

$$\sin\theta = \frac{opposite}{hypotenuse} = \frac{y}{1} = y$$

$$\cos\theta = \frac{adjacent}{hypotenuse} = \frac{x}{1} = x$$

$$\tan\theta = \frac{opposite}{adjacent} = \frac{y}{x} = \frac{Sin\theta}{Cos\theta}$$

These facts can be used to expand the definitions of sine, cosine, and tangent to angle measures greater than $90°$ and less than $0°$, as we will see shortly.

15) We want to look at all the possible values of $\theta$. To do this, highlight $\theta$ in the Variables tool panel, and then look at the animation tools at the bottom of that section.

These tools allow you to move continuously through a range of values for a given variable. The bottom left and right boxes indicate the minimum and maximum values for $\theta$. We want to see between $0°$ and $360°$. You can manually run through those values by clicking and dragging the scroll bar, or have the computer do it automatically by clicking on the play button. Notice the path that point B takes.
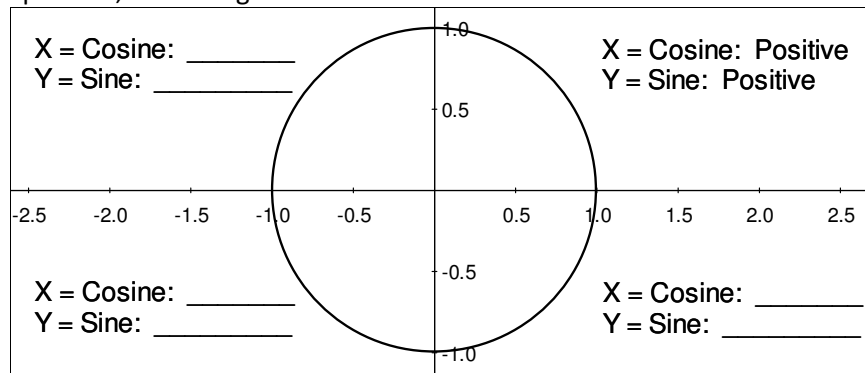
16) To see the pathway more clearly, highlight point B, and use the **construct locus** tool. You'll see a pop-up menu; make sure it is set to variable $\theta$, start value 0, and end value 360. Click OK. Then repeat the animation from step 15.

When you consider the values of $0°$ through $90°$ - all the possibilities for a right triangle - the point B makes an arc. As you continue the pattern beyond the triangle, you create a circle with radius one, centered at the origin, which is called a **unit circle**. The coordinates of B, as we saw earlier, are ($\cos\theta$, $\sin\theta$), where $\theta$ is the angle made with the positive x axis. The trick here is to think of $\theta$ a little differently. Instead of just being an acute angle of a right triangle, think of $\theta$ as the number of degrees that point B has rotated through since leaving the x-axis in the animation. For measures between $0°$ and $90°$, these two definitions of $\theta$ are exactly the same. However, the second definition allows us to use angle measures greater than $90°$.

17) Similarly, angle measure of less than $0°$ indicate a rotation in the opposite direction. Adjust your beginning value in the variables menu to −360, and use the scroll bar to investigate.

In short, the expanded definitions of sine and cosine are the coordinates of a point on a unit circle, after the point has rotated $\theta$ degrees counterclockwise from the positive x-axis. The coordinates are ($\cos\theta$, $\sin\theta$). Similarly, tangent of $\theta$ becomes $\dfrac{y}{x}$.

18) Pause the animation at various angle measures in the different quadrants and determine from the coordinates where sine is positive and negative, and where cosine is positive and negative. Complete the diagram below with the words "positive," and "negative."

X = Cosine: _____    X = Cosine: Positive
Y = Sine: _____    Y = Sine: Positive

X = Cosine: _____    X = Cosine: _____
Y = Sine: _____    Y = Sine: _____

(Graph with unit circle, x-axis marked -2.5, -2.0, -1.5, -1.0, -0.5, 0.5, 1.0, 1.5, 2.0, 2.5 and y-axis marked 1.0, 0.5, -0.5, -1.0)

19) Based on what you have learned so far, or by observing the points on the unit circle, what is the greatest and least value you could get for sine of some angle $\theta$? This is called the range of sine.

20) What is the greatest and least value you could get for cosine of some angle $\theta$? This is the range of cosine.