PATTERNS OF COMMUTATIVITY: THE COMMUTANT OF THE FULL PATTERN*

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Abstract. Identified are a number of conditions on square patterns that are closely related to allowing commutativity with the full pattern. Implications and examples that show non-implications are given, along with a graph that summarizes the provided information. A complete description of commutativity with the full pattern is given in both the irreducible case and the reducible case in which there are two irreducible components.

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1. Introduction, problem statement and notation. By a pattern $\mathcal{P}$ (respectively, sign pattern $\mathcal{S}$) we mean an array of $*$'s and 0's ($+$'s, $-$'s and 0's) in which a $*$ ($+$ or $-$) indicates a nonzero (positive or negative) entry. A real matrix $A = (a_{i,j})$ belongs to pattern $\mathcal{P}$ (sign pattern $\mathcal{S}$) if its dimensions agree with those of $\mathcal{P}$ ($\mathcal{S}$) and $a_{i,j} \neq 0$ if and only if the i,j entry of $\mathcal{P}$ is a $*$ ($a_{i,j} > 0$, $a_{i,j} < 0$ iff the i,j entry of $\mathcal{S}$ is, respectively, $+$ or $-$). We say that two n-by-n patterns $\mathcal{P}$ and $\mathcal{Q}$ (a pattern $\mathcal{P}$ and a sign pattern $\mathcal{S}$) *commute* (or allow commutativity) if there exist matrices $A \in \mathcal{P}$, $B \in \mathcal{Q}$ ($\in \mathcal{S}$) that commute, i.e., $AB = BA$. In general we say that a pattern *allows* a given property if there exists a matrix of the pattern with that property (e. g. we are considering pairs of patterns that allow commutativity); a pattern *requires* a given property if every matrix of the pattern has that property. The *commutant* of a pattern $\mathcal{P}$ (sign pattern $\mathcal{S}$) is simply the set of all patterns $\mathcal{Q}$ that commute with $\mathcal{P}$ ($\mathcal{S}$). Let $C(\mathcal{P})$ ($C(\mathcal{S})$) denote the commutant of $\mathcal{P}$ ($\mathcal{S}$).

Our interest here lies in determining the commutant of the full (all $*$'s) pattern $\mathcal{F}$ and of the all $+$ sign pattern $\mathcal{J}$. Of course $C(\mathcal{J}) \subseteq C(\mathcal{F})$, but, as we will see later, the opposite inclusion is not true.

We begin with a discussion of (new) conditions that are necessary for a pattern to be in $C(\mathcal{F})$, then identify several conditions (some familiar) that are sufficient and identify implications (and non-implications) among these. We also discuss necessary and sufficient conditions in terms of the number of components in the Frobenius normal form of $\mathcal{P}$.

Many matrix concepts and notation, such as irreducibility, submatrices, and multiplication by a permutation or diagonal matrix, extend in an unambiguous way to patterns, and we use them in the context of patterns without comment.

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2. The irreducible case and general necessary conditions. First note:

THEOREM 2.1. *Each irreducible pattern lies in* $C(\mathcal{J})$.

Proof. If $\mathcal{P}$ is an n-by-n irreducible pattern, consider $A \in \mathcal{P}$ such that $A \geq 0$ (entry-wise). As A is also irreducible, it follows that $(I + A)^{n-1} > 0$, i.e., $(I + A)^{n-1} \in \mathcal{J}$. As this matrix is a polynomial in A, it commutes with A. $\square$

We can conclude from Theorem 2.1 that $C(\mathcal{F})$ contains all irreducible patterns. There exist, however, reducible patterns that do not lie in $C(\mathcal{F})$. We first state a simple necessary condition for a pattern to belong to $C(\mathcal{F})$.

THEOREM 2.2. *If* $\mathcal{P}$ *is a pattern whose* i^{th} *row (column) is all zeros and whose* j^{th} *column (row) has exactly one* $*$, *then* $\mathcal{P} \notin C(\mathcal{F})$.

Proof. For any $A \in \mathcal{P}$ and any $B \in \mathcal{F}$, $(AB)_{i,j} = 0$ $((AB)_{i,j} \neq 0)$ and $(BA)_{i,j} \neq 0$ $((BA)_{i,j} = 0)$. $\square$

The necessary condition stated in Theorem 2.2 is not sufficient:

EXAMPLE 2.3. The reducible pattern $\begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \notin C(\mathcal{F})$.

Another necessary condition for a pattern to belong to $C(\mathcal{F})$ is what we call the *swath conditions*. Any pattern $\mathcal{P}$ is permutation similar to an ("irreducible") Frobenius normal form

$$\mathcal{P}' = \begin{bmatrix} \mathcal{P}_{1,1} & \mathcal{P}_{1,2} & \cdots & \mathcal{P}_{1,k} \\ 0 & \mathcal{P}_{2,2} & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \mathcal{P}_{k,k} \end{bmatrix}$$

in which each $\mathcal{P}_{i,i}$ is a square and irreducible pattern (note that this includes the possibility that $\mathcal{P}_{i,i}$ is 1-by-1 and either $[0]$ or $[*]$, cases that will be important later). Such a form is not necessarily unique. Since $\mathcal{F}$ is permutation similarity invariant, $\mathcal{P} \in C(\mathcal{F})$ if and only if $\mathcal{P}' \in C(\mathcal{F})$. We refer to the diagonal blocks of $\mathcal{P}'$, or their index sets, as the irreducible components of $\mathcal{P}$, or $\mathcal{P}'$.

THEOREM 2.4. *If* $\mathcal{P}' \in C(\mathcal{F})$, *then for each* j, $j = 1, \ldots, k$, *there are either* 0 *or* 2 *or more* $*$'s *among the subpatterns:*

$$\mathcal{P}_{1,j}, \ldots, \mathcal{P}_{j-1,j}, \mathcal{P}_{j,j+1}, \ldots, \mathcal{P}_{j,k}.$$

We refer to the k conditions of Theorem 2.4 as the *swath conditions* for $\mathcal{P}'$ (or for the original $\mathcal{P}$). Note that the first *swath* is $\mathcal{P}_{1,2}, \ldots, \mathcal{P}_{1,k}$, as, for $j = 1$, no blocks occur above $\mathcal{P}_{1,1}$, and the last swath is $\mathcal{P}_{1,k}, \ldots, \mathcal{P}_{k-1,k}$ as, for $j = k$, there are no blocks beside $\mathcal{P}_{k,k}$. All other swaths are "L" shaped.

Proof. Let $A \in \mathcal{P}'$ and $B \in \mathcal{F}$ be matrices such that $AB = BA$. Partition each of A and B conformally with $\mathcal{P}'$, then equate the j, j block of AB with that of BA. This gives

$$(2.1) \qquad \begin{aligned} A_{j,j}B_{j,j} + A_{j,j+1}B_{j+1,j} + \cdots + A_{j,k}B_{k,j} = \\ B_{j,1}A_{1,j} + B_{j,2}A_{2,j} + \cdots + B_{j,j}A_{j,j}. \end{aligned}$$

Since $tr\,(A_{j,j}B_{j,j}) = tr\,(B_{j,j}A_{j,j})$, when we equate the traces of the two sides of (2.1), we have

$$(2.2) \qquad \begin{aligned} tr\,(A_{j,j+1}B_{j+1,j}) + \cdots + tr\,(A_{j,k}B_{k,j}) = \\ tr\,(B_{j,1}A_{1,j}) + \cdots + tr\,(B_{j,j-1}A_{j-1,j}). \end{aligned}$$

If it were the case that there were only 1 nonzero entry among the A blocks appearing in (2.2), it would follow, as $B \in \mathcal{F}$, that precisely 1 of the above traces is nonzero, a contradiction to equality (2.1). $\square$

The swath conditions are not sufficient conditions for commutativity with the full pattern:

EXAMPLE 2.5. The pattern $\mathcal{P} = \begin{bmatrix} * & * & * & * \\ * & * & 0 & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}$ satisfies the swath conditions, but, according to Theorem 2.2, it does not belong to $C(\mathcal{F})$.

For a pattern $\mathcal{P}$ to be in $C(\mathcal{F})$ there is an important consistency condition on the first and last diagonal blocks of the Frobenius normal form. Each irreducible component may be classified as follows: type 1 means that it allows a nonzero eigenvalue and type 2 means that it allows the eigenvalue 0. Of course a pattern may be both type 1 and type 2. The only such pattern that is not type 1 is the type 2 pattern $[0]$. A pattern that is type 1 and not type 2 must require nonsingularity. The patterns that require nonsingularity are precisely those that are permutation equivalent ($P\mathcal{A}Q$, P and Q independent permutation matrices) to a triangular pattern with all diagonal entries nonzero. It then follows from Theorem 2.2 that:

COROLLARY 2.6. *Let $\mathcal{P}$ be an n-by-n pattern. If $\mathcal{P} \in C(\mathcal{F})$, then the first and last blocks of any Frobenius normal form of $\mathcal{P}$ must share the same type.*

Example 2.5 also shows that it is possible for the swath conditions, as well as the first and last block conditions, to be met, without $\mathcal{P} \in C(\mathcal{F})$.

3. Sufficient conditions. In [1] a portion of $C(\mathcal{S})$ has been determined, namely those patterns $\mathcal{Q}$ that allow constant line sums and, thus, commutativity with the all 1's matrix J in $\mathcal{J}$. The proof of the following theorem may be deduced from [1]. For reference later we begin to identify particular properties of a pattern with subscripted roman capital P's. The strongest conditions on $\mathcal{P}$, sufficient for $\mathcal{P} \in C(\mathcal{F})$ are $P_1 - P_4$:

THEOREM 3.1. *Let $\mathcal{P}$ be an n-by-n pattern. The following properties are equivalent:*

P_1 : *$\mathcal{P}$ allows constant line sums;*

P_2 : *$\mathcal{P}$ allows commutativity with J;*

P_3 : *$\mathcal{P}$ allows right and left constant eigenvectors associated with the same eigenvalue;*

P_4 : *$\mathcal{P}$ satisfies the J-S "single $*$" condition found in* [1, Corollary 10].

It is clear that any of the conditions in Theorem 3.1 is sufficient for commutativity with $\mathcal{J}$, but the following example shows that they are not necessary:

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EXAMPLE 3.2. The pattern $\mathcal{Q} = \begin{bmatrix} * & * \\ * & 0 \end{bmatrix}$ is irreducible, so it belongs to $C(\mathcal{J})$, but, clearly, it does not allow constant line sums.

Intermediate sufficient conditions are collected in the following:

THEOREM 3.3. *Let $\mathcal{P}$ be an n-by-n pattern. The following properties are equivalent:*

P_5 : *$\mathcal{P}$ allows positive right and left eigenvectors associated with the same eigenvalue;*

P_6 : *$\mathcal{P}$ commutes with a positive rank 1 matrix;*

P_7 : *$\mathcal{P} \in C(\mathcal{J})$.*

Proof. To show that P_5 implies P_6, let $A \in \mathcal{P}$ be such that there are positive vectors x and y satisfying $Ax = \lambda x$ and $y^T A = \lambda y^T$. Clearly xy^T is a full positive rank 1 matrix, and $A\left(xy^T\right) = (Ax)\,y^T = (\lambda x)\,y^T = x\left(\lambda y^T\right) = \left(xy^T\right) A$.

It is obvious that P_6 implies P_7.

To show that P_7 implies P_5, suppose $\mathcal{P} \in C(\mathcal{J})$ and A is a matrix of pattern $\mathcal{P}$ that commutes with a positive matrix B. Then $B \in \mathcal{J}$ and $AB = BA$. Let λ be the Perron eigenvalue of B and x and y be positive vectors such that $Bx = \lambda x$ and $y^T B = \lambda y^T$.

As $B(Ax) = (BA)\,x = (AB)\,x = A(Bx) = A(\lambda x) = \lambda(Ax)$, if $Ax \neq 0$, Ax is also a right eigenvector of B associated with λ and, as the Perron eigenspace is 1-dimensional, there is a ρ such that $Ax = \rho x$. Thus x is a right eigenvector of A associated with ρ.

Similarly, $\left(y^T A\right) B = y^T (AB) = y^T (BA) = \left(y^T B\right) A = \left(\lambda y^T\right) A = \lambda \left(y^T A\right)$, so that, if $y^T A \neq 0$, $y^T A$ is also a left eigenvector of B associated with λ. Thus there is a μ such that $y^T A = \mu y^T$ and y^T is a left eigenvector of A associated with μ.

It is enough now to conclude that $\rho = \mu$. As $\rho y^T x = y^T (Ax) = \left(y^T A\right) x = \mu y^T x$ and $y^T x \neq 0$, we conclude $\rho = \mu$.

Using the above calculation and the fact that Ax and $y^T A$ are uniformly signed when they are nonzero, it follows that if $Ax = 0$ $(y^T A = 0)$ then $y^T A\,(Ax)$ is also 0 and A has positive right and left eigenvectors associated with 0. This completes the proof. $\square$

As we noted in the introduction, $C(\mathcal{J}) \subseteq C(\mathcal{F})$, but using Theorem 3.3, we can see that the opposite inclusion is not true:

EXAMPLE 3.4. The pattern $\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ commutes with $\mathcal{F}$ (the matrices

$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & -1 & 2 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} -1 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & -1 & -1 & -1 \end{bmatrix}$ commute) but it does not allow any

positive right eigenvector.

The following theorem gives another set of sufficient conditions for a pattern to belong to $C(\mathcal{F})$, but Example 3.4 also shows that they are not necessary.

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THEOREM 3.5. *Let $\mathcal{P}$ be an n-by-n pattern. The following properties are equivalent:*

P_8 : *$\mathcal{P}$ allows totally nonzero right and left eigenvectors for the same eigenvalue;*

P_9 : *$\mathcal{P}$ commutes with a totally nonzero, rank 1 matrix.*

Proof. The proof that P_8 implies P_9 is similar to the proof that P_5 implies P_6, replacing "positive" by "totally nonzero". To show that P_9 implies P_8, let $A \in \mathcal{P}$ and let B be a rank 1 full matrix such that $AB = BA$. There exist totally nonzero vectors x and y such that $B = xy^T$. As A and B commute, $A\left(xy^T\right) = \left(xy^T\right) A$. If $A\left(xy^T\right) = \left(xy^T\right) A = 0$, then $(Ax) y^T = x\left(y^T A\right) = 0$ and, as x and y are totally nonzero vectors, $Ax = y^T A = 0$ and we conclude that x is a right eigenvector of A for the eigenvalue 0 and that y is a left eigenvector for the eigenvalue 0. Suppose now that $AB \neq 0$. If $y^T = \left[\begin{array}{ccc} y_1 & \cdots & y_n \end{array}\right]$ then

$$A\left(xy^T\right) = (Ax) y^T = \left[\begin{array}{cccc} y_1 Ax & y_2 Ax & \cdots & y_n Ax \end{array}\right] \text{ and}$$
$$\left(xy^T\right) A = \left[\begin{array}{cccc} \left(y^T A\right)_1 x & \left(y^T A\right)_2 x & \cdots & \left(y^T A\right)_n x \end{array}\right].$$

We can assume, without loss of generality, $y_1 Ax \neq 0$ and so, as $y_1 Ax = \left(y^T A\right)_1 x$, then $Ax = \left(\frac{\left(y^T A\right)_1}{y_1}\right) x$ and x is a right eigenvector of A for an eigenvalue $\lambda \neq 0$. On the other hand, if $x^T = \left[\begin{array}{ccc} x_1 & \cdots & x_n \end{array}\right]$, then

$$\left(xy^T\right) A = \left[\begin{array}{c} x_1 y^T A \\ x_2 y^T A \\ \vdots \\ x_n y^T A \end{array}\right] \text{ and } A\left(xy^T\right) = \left[\begin{array}{c} (Ax)_1 y^T \\ (Ax)_2 y^T \\ \vdots \\ (Ax)_n y^T \end{array}\right].$$

Again we can assume that $x_1 y^T A \neq 0$ and so, as $x_1 y^T A = (Ax)_1 y^T$ and $y^T A = \left(\frac{(Ax)_1}{x_1}\right) y^T$ and y is a left eigenvector of A for an eigenvalue $\mu \neq 0$. As $Axy^T = \lambda xy^T$, $xy^T A = \mu xy^T$ and $Axy^T = xy^T A$, we conclude that $\lambda = \mu$, which completes the proof. □

It is interesting to compare Theorem 3.5 with Theorem 3.3. Of course P_5 implies P_8, but we do not know if a pattern satisfying P_8 must also satisfy P_5. If the eigenvectors in question have positive Hadamard product, then a signature (diagonal, orthogonal) similarity will convert them to both positive and not change the pattern. Thus, in this event, P_8 implies P_5. Also, if the eigenvalue is 0, the sign pattern of one eigenvector may be converted to that of the other (via multiplication by a signature matrix) without altering the problem; so that, again for the eigenvalue 0, P_8 is equivalent to P_5.

4. The graph of implications. In what follows we say that a pattern satisfies condition P_{12} if it does not have a zero row (column) together with a column (row) with exactly one $*$ (Theorem 2.2) and that a pattern satisfies condition P_{11} if it satisfies the swath conditions (Theorem 2.4). The conditions P_1 to P_9 are the ones in Theorems 3.1, 3.3 and 3.5. We label the condition that $\mathcal{P} \in C\left(\mathcal{F}\right)$ as P_{10}. In the

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following graph an arrow means an implication in the indicated direction and the horizontal lines mean equivalence. The general results and examples (for arbitrarily many blocks in the Frobenius normal form) thus far, may be summarized as follows:
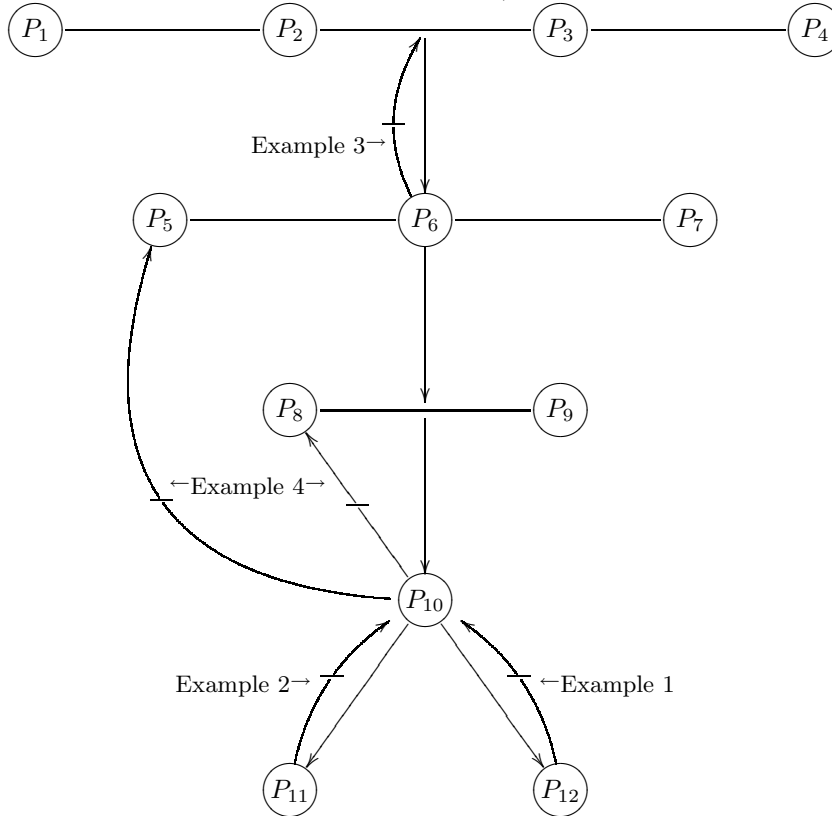
$P_1 - \mathcal{P}$ allows constant line sums.

$P_2 - \mathcal{P}$ allows commutativity with J.

$P_3 - \mathcal{P}$ allows right and left constant eigenvectors associated with the same eigenvalue.

$P_4 - \mathcal{P}$ satisfies the J-S "single $*$" condition found in [1, Corollary 10].

$P_5 - \mathcal{P}$ allows positive right and left eigenvectors associated with the same eigenvalue.

$P_6 - \mathcal{P}$ commutes with a positive rank 1 matrix.

$P_7 - \mathcal{P} \in C\left(\mathcal{J}\right).$

$P_8 - \mathcal{P}$ allows totally nonzero right and left eigenvectors for the same eigenvalue.

$P_9 - \mathcal{P}$ commutes with a totally nonzero, rank 1 matrix.

$P_{10} - \mathcal{P} \in C\left(\mathcal{F}\right).$

$P_{11} - \mathcal{P}$ satisfies the swath conditions.

$P_{12} - \mathcal{P}$ doesn't have an all zeros row (column) together with a column (row) with a single $*$.

5. The 2-by-2 block case. We next examine in detail the two-component case of the Frobenius normal form. It is possible to satisfy the swath condition (very simple in this case) and not the consistency of the first and last block (or the $0,*$ condition

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of Theorem 2.2). Thus, we assume such a condition, beyond the swath condition, when relevant. The main result will be that the swath condition (P_{11}) together with condition P_{12} is necessary and sufficient in this case.

A useful observation that underlies the following key fact (that may be of independent interest) is that a vector lies in the column space or right hand range of a matrix if and only if it is orthogonal to every vector in the left null space of the matrix. Let $gm_B(\mu)$ denote the geometric multiplicity of μ as an eigenvalue of the matrix B.

THEOREM 5.1. *Let* $A = \begin{bmatrix} A_{1,1} & A_{1,2} \\ 0 & A_{2,2} \end{bmatrix}, A \in M_n, A_{1,1} \in M_k, A_{2,2} \in M_{n-k}$. *Then, for any scalar* λ, $gm_A(\lambda) \leq gm_{A_{1,1}}(\lambda) + gm_{A_{2,2}}(\lambda)$, *with equality if and only if* $y_1^T A_{1,2} x_2 = 0$ *whenever* $y_1^T A_{1,1} = \lambda y_1^T$ *and* $A_{2,2} x_2 = \lambda x_2$. *Moreover, in case of equality, there is a basis for the left (right) eigenspace of A consisting of a basis of the left (right) eigenspace of* $A_{2,2}$ $(A_{1,1})$ *extended by $0's$ on the left (downward) together with a basis of the left (right) eigenspace of* $A_{1,1}$ $(A_{2,2})$ *extended to the right (upward).*

Proof. The inequality is known and follows from a simple analysis of the (right) null space of $A - \lambda I$. For the case of equality, suppose $gm_{A_{1,1}}(\lambda) = p$ and $gm_{A_{2,2}}(\lambda) = q$. For sufficiency we want to find $p+q$ linearly independent vectors in the $RNS(A - \lambda I)$, the right null space of $A - \lambda I$. There are p of the form $\begin{bmatrix} x_1 \\ 0 \end{bmatrix}$, $x_1 \in RNS(A_{1,1} - \lambda I)$. Thus, we need q of the form $\begin{bmatrix} u_1 \\ x_2 \end{bmatrix}$ with $0 \neq x_2 \in RNS(A_{2,2} - \lambda I)$. But

$$(A - \lambda I) \begin{bmatrix} u_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (A_{1,1} - \lambda I) u_1 + A_{1,2} x_2 \\ (A_{2,2} - \lambda I) x_2 \end{bmatrix} = \begin{bmatrix} (A_{1,1} - \lambda I) u_1 + A_{1,2} x_2 \\ 0 \end{bmatrix}.$$

So, we want $(A_{1,1} - \lambda I) u_1 + A_{1,2} x_2 = 0$ to have q linearly independent solutions; so, consider the linear systems $(A_{1,1} - \lambda I) u_1 = -A_{1,2} x_2$ as x_2 runs through the q-dimensional set $RNS(A_{2,2} - \lambda I)$. Since $y_1^T A_{1,2} x_2 = 0$ whenever $y_1 \in LNS(A_{1,1} - \lambda I)$, each such $-A_{1,2} x_2 \in Range(A_{1,1} - \lambda I)$ and there is a solution u_1 to $(A_{1,1} - \lambda I) u_1 = -A_{1,2} x_2$ for each such x_2. For necessity, if $gm_A(\lambda) = p + q$, we must have $p + q$ linearly independent left (resp; right) null vectors for $(A - \lambda I)$. There are p right ones of the form $\begin{bmatrix} x_1 \\ 0 \end{bmatrix}$, $x_1 \in RNS(A_{1,1} - \lambda I)$ and q left ones of the form $\begin{bmatrix} 0 \\ y_2 \end{bmatrix}^T$, $y_2^T \in LNS(A_{2,2} - \lambda I)$. Others must be of the form $\begin{bmatrix} u_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ v_2 \end{bmatrix}^T$, resp., with $x_2 \in RNS(A_{2,2} - \lambda I)$, $y_1^T \in LNS(A_{1,1} - \lambda I)$. But $0 = \begin{bmatrix} y_1^T & v_2^T \end{bmatrix} (A - \lambda I) \begin{bmatrix} u_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1^T & v_2^T \end{bmatrix} \begin{bmatrix} (A_{1,1} - \lambda I) u_1 + A_{1,2} x_2 \\ (A_{2,2} - \lambda I) x_2 \end{bmatrix} = \begin{bmatrix} y_1^T & v_2^T \end{bmatrix} \begin{bmatrix} (A_{1,1} - \lambda I) u_1 + A_{1,2} x_2 \\ 0 \end{bmatrix} = y_1^T (A_{1,1} - \lambda I) u_1 + y_1^T A_{1,2} x_2$.

As $y_1^T (A_{1,1} - \lambda I) = 0$, $y_1^T A_{1,2} x_2 = 0$, whenever $x_2 \in RNS(A_{2,2} - \lambda I)$, and $y_1^T \in LNS(A_{1,1} - \lambda I)$. □

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We may now apply Theorem 5.1 to the two-component case. First:

COROLLARY 5.2. *Suppose that* $\mathcal{P} = \begin{bmatrix} \mathcal{P}_{1,1} & \mathcal{P}_{1,2} \\ 0 & \mathcal{P}_{2,2} \end{bmatrix}$ *is a pattern and that* $A_{1,1} \in$ $\mathcal{P}_{1,1}$ *and* $A_{2,2} \in \mathcal{P}_{2,2}$ *may be chosen, so that, for a common eigenvalue* $\lambda \in \sigma(A_{1,1}) \cap$ $\sigma(A_{2,2})$ *with geometric multiplicity 1 in each,* $A_{1,1}$ *has a positive left and right eigenvector associated with* λ, *as does* $A_{2,2}$. *Then, if* $\mathcal{P}_{1,2}$ *has 0 or 2 or more nonzeros,* $A \in \mathcal{P}$ *may be chosen so that it has positive left and right eigenvectors associated with the same eigenvalue.*

Proof. To apply Theorem 5.1, we let the upper left block of A be $A_{1,1}$ and the lower right be $A_{2,2}$. Then, because λ has geometric multiplicity 1 in each, Theorem 5.1 requires only one linear condition upon the entries of $A_{1,2} \in \mathcal{P}_{1,2}$, namely $y_1^T A_{1,2} x_2 = 0$ for y_1 a left eigenvector of $A_{1,1}$ and x_2 a right eigenvector of $A_{2,2}$. If $\mathcal{P}_{1,2}$ has 0 or 2 or more nonzeros it is clear that this linear system may be satisfied, to determine an $A_{1,2}$ and thus $A \in \mathcal{P}$. □

COROLLARY 5.3. *Suppose that* $\mathcal{P}$ *is an n-by-n pattern with precisely two components. If these two components share the same type, then* $\mathcal{P} \in C(\mathcal{F})$ *if and only if its irreducible form satisfies the swath condition.*

Proof. If the components are both type 1, then by taking all their stars to be positive, and scaling them so as to achieve a common Perron root, the Perron-Frobenius theory assures that the hypothesis of Corollary 5.2 is satisfied. The necessity of the swath condition has been shown, in general, in Theorem 2.4 and its sufficiency in this case follows from Corollary 5.2. □

We close by noting that, in general, a certain strengthening of the swath conditions on $\mathcal{P}$ is sufficient for $\mathcal{P} \in C(\mathcal{F})$, by virtue of sufficiency for P$_5$. Thus, P$_5$−P$_{11}$ are equivalent in this event.

THEOREM 5.4. *Let* $\mathcal{P}$ *be an n-by-n pattern with k components in its Frobenius normal form*

$$\mathcal{P}' = \begin{bmatrix} \mathcal{P}_{1,1} & \mathcal{P}_{1,2} & \cdots & \mathcal{P}_{1,k} \\ 0 & \mathcal{P}_{2,2} & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \mathcal{P}_{k,k} \end{bmatrix}.$$

Then $\mathcal{P}'$ *allows positive left and right eigenvectors associated with a common eigenvalue if (1)* $\mathcal{P}_{1,1}, \mathcal{P}_{2,2}, \ldots, \mathcal{P}_{k,k}$ *share a common type and (2) each* $\mathcal{P}_{i,j}, i < j$, *contains 0 or two or more* $*$'s.

REFERENCES

[1] C.R. Johnson and D.P. Stanford. Patterns that allow given row and columns sums. *Linear Algebra and its Applications*, 311:97-105, 2000.