Problem in standard form: outline

- Prep work for the Simplex method for solving LPs: problems in standard form
  - Basic solutions
  - Bases; primal and dual basic solutions
  - Adjacent basic solutions

Some linear algebra facts

Let $\mathbf{A} \in \Re^{m \times n}$: $\mathbf{A} = [\mathbf{A}_1 \ \mathbf{A}_2 \ \cdots \ \mathbf{A}_n] = \begin{bmatrix} \mathbf{a}_1^T \\ \mathbf{a}_2^T \\ \vdots \\ \mathbf{a}_m^T \end{bmatrix}$

- Row rank of $\mathbf{A}$: dimension of subspace spanned by $\mathbf{a}_1^T, \ldots, \mathbf{a}_m^T$
- Column rank of $\mathbf{A}$: dimension of subspace spanned by $\mathbf{A}_1, \ldots, \mathbf{A}_n$
- The two ranks above are the same for any matrix; called $\mathrm{rank}(\mathbf{A}) \leq \min(m, n)$

Theorem 2.5 (modification)

Let $\tilde{P} = \{\mathbf{x} \in \Re^n : \mathbf{A}\mathbf{x} = \mathbf{b}\}$ be a nonempty set, where $\mathbf{A} \in \Re^{m \times n}$, with rows $\mathbf{a}_1^T, \ldots, \mathbf{a}_m^T$. Suppose that $\mathrm{rank}(\mathbf{A}) = k < m$ and rows $\mathbf{a}_1^T, \ldots, \mathbf{a}_k^T$ are linearly independent. Then

$$\tilde{Q} = \{\mathbf{x} : \mathbf{a}_i^T \mathbf{x} = b_i, \ i = 1, \ldots, k\} = \tilde{P}.$$

Duals of equivalent LPs are equivalent

Theorem 4.2

Suppose that we have transformed a linear programming problem Π_1 into another linear programming problem Π_2, by a sequence of transformations of the following types:

- Replace a free variable with the difference of two nonnegative variables.
- Replace an inequality constraint by an equality constraint involving a nonnegative slack variable.
- If some row of the matrix $\mathbf{A}$ in a feasible standard form problem is a linear combination of the other rows, eliminate the corresponding equality constraint.

Then the duals of Π_1 and Π_2 are equivalent, i.e., they are either both infeasible, or they have the same optimal cost.

Basic solutions of polyhedra in standard form

$$P = \{\mathbf{x} \in \Re^n : \mathbf{Ax} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0}\}$$

$\mathbf{A} \in \Re^{m \times n}$, and rows of $\mathbf{A}$ are linearly independent (i.e., $m \leq n$)

Theorem 2.4

Consider P in standard form above. A vector $\mathbf{x} \in \Re^n$ is a basic solution if and only if we have $\mathbf{Ax} = \mathbf{b}$, and there exist indices $B(1), \ldots, B(m)$ such that:

(a) The columns $\mathbf{A}_{B(1)}, \ldots, \mathbf{A}_{B(m)}$ are linearly independent;

(b) If $i \neq B(1), \ldots, B(m)$, then $x_i = 0$.

To construct a basic solution:

1. Choose m linearly independent columns $\mathbf{A}_{B(1)}, \ldots, \mathbf{A}_{B(m)}$
2. Let $x_i = 0,\ i \neq B(1), \ldots, B(m)$
3. Solve the system of m equations $\sum_{k=1}^{m} \mathbf{A}_{B(k)} x_{B(k)} = \mathbf{b}$ for $x_{B(1)}, \ldots, x_{B(m)}$

A basic solution is feasible if $x_{B(1)}, \ldots, x_{B(m)} \geq 0$

Bases: notation and definitions

Let $B(1), \ldots, B(m)$ be a set of (distinct) indices such that $\mathbf{A}_{B(1)}, \ldots, \mathbf{A}_{B(m)}$ are linearly independent

- $B = \{B(1), \ldots, B(m)\}$ — set of *basic indices*, or simply *basis*; two bases are distinct if the sets of basic indices are different
- $\mathbf{A}_{B(1)}, \ldots, \mathbf{A}_{B(m)}$ – *basic columns* (they span $\Re^m$)
- $$\mathbf{B} = \begin{bmatrix} \mathbf{A}_{B(1)} & \ldots & \mathbf{A}_{B(m)} \end{bmatrix} \in \Re^{m \times m}$$
  is a *basis matrix*; invertible
- $x_i,\ i \in B$ — *basic variables*; $x_i,\ i \notin B$ — *nonbasic variables*
- $N = \{j \neq B(1), \ldots, B(m)\}$ — set of *nonbasic indices* $(|N| = n - m)$
- Re-order and group components of $\mathbf{x}$ and $\mathbf{c}$ and columns of $\mathbf{A}$:
  $$\mathbf{x} = \begin{bmatrix} \mathbf{x}_B \\ \mathbf{x}_N \end{bmatrix}, \ \mathbf{c} = \begin{bmatrix} \mathbf{c}_B \\ \mathbf{c}_N \end{bmatrix}, \ \mathbf{A} = [\mathbf{A}_B;\ \mathbf{A}_N] = [\mathbf{B};\ \mathbf{A}_N]$$
- $\mathbf{x} = (\mathbf{x}_B; \mathbf{x}_N) = (\mathbf{B}^{-1}\mathbf{b}; \mathbf{0})$ — basic solution of P corresponding to basis B

Correspondence of bases and basic solutions

- Not one to one!
  - A basis uniquely determines a basic solution
  - Converse not true (consider, e.g., $\mathbf{b} = \mathbf{0}$)
- Recall: a BS is *degenerate* if the number of active constraints is greater than n.

Definition 2.11

Let $\mathbf{x}$ be a basic solution of $P = \{\mathbf{x} \in \Re^n : \mathbf{A}\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0}\}$. $\mathbf{x}$ is a *degenerate* basic solution if more than $n - m$ of the x_i's are 0.

Example:

$$A = \begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 6 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \ b = \begin{bmatrix} 8 \\ 12 \\ 4 \\ 6 \end{bmatrix}$$

$\mathbf{B}^1 = [\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_7]$ vs. $\mathbf{B}^2 = [\mathbf{A}_1, \mathbf{A}_4, \mathbf{A}_3, \mathbf{A}_7]$

Primal standard form problem, its dual, and their basic solutions

$$\text{(P)}\ \min\ \mathbf{c}^T\mathbf{x}\quad \text{s.t.}\ \mathbf{A}\mathbf{x}=\mathbf{b},\ \mathbf{x}\geq\mathbf{0} \qquad\qquad \text{(D)}\ \max\ \mathbf{b}^T\mathbf{p}\quad \text{s.t.}\ \mathbf{p}^T\mathbf{A}\leq\mathbf{c}^T$$

- Let $B=\{B(1),\ldots,B(m)\}$ be a basis of the primal LP (P)
- Corresponding to this basis, we have:
  - $\mathbf{B}$: a basis matrix
  - $\mathbf{x}=(\mathbf{x}_B;\mathbf{x}_N)=(\mathbf{B}^{-1}\mathbf{b};\mathbf{0})$ — *primal* basic solution corresponding to basis B
  - $\mathbf{p}^T=\mathbf{c}_B^T\mathbf{B}^{-1}$ — ???
- **Claim:** $\mathbf{p}$ defined as above
  - is a basic solution to the dual LP (D), and
  - is complementary to the primal basic solution $\mathbf{x}$ corresponding to the same basis B
    - ...and thus they have the same objective function values

Optimality conditions

$$\mathbf{x}=(\mathbf{x}_B;\mathbf{x}_N)=(\mathbf{B}^{-1}\mathbf{b};\mathbf{0}),\ \mathbf{p}^T=\mathbf{c}_B^T\mathbf{B}^{-1}$$

Theorem 3.1 (paraphrased)

Consider a basis B and corresponding basic solutions $\mathbf{x}$ and $\mathbf{p}$; define

$$\bar{\mathbf{c}}^T\triangleq\mathbf{c}^T-\mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{A}=\mathbf{c}^T-\mathbf{p}^T\mathbf{A}.$$

(a) If $\mathbf{x}_B\geq 0$ and $\bar{\mathbf{c}}\geq\mathbf{0}$, then $\mathbf{x}$ and $\mathbf{p}$ are optimal basic solutions

(b) If $\mathbf{x}$ is optimal and nondegenerate, then $\bar{\mathbf{c}}\geq\mathbf{0}$ and $\mathbf{p}^T=\mathbf{c}_B^T\mathbf{B}^{-1}$ is optimal.

Definition 3.3: An optimal basis

A basis matrix $\mathbf{B}$ is said to be *optimal* if

(a) $\mathbf{B}^{-1}\mathbf{b}\geq\mathbf{0}$, and

(b) $\bar{\mathbf{c}}^T=\mathbf{c}^T-\mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{A}\geq\mathbf{0}$.

Adjacent basic solutions and adjacent bases

- For a general problem: two basic solutions to a set of linear constraints in $\Re^n$ are *adjacent* if there are $n-1$ linearly independent constraints that are active in both of them
  - If both adjacent basic solutions are feasible, the line segment joining them is called an *edge* of the feasible set
- For standard form problem: two bases are *adjacent* if they share all but one basic column