Chapter 22

Transformation Geometry

Our earlier work with symmetry in Chapter 18 used movements such as reflections and rotations, with a focus on a given figure. These movements do not change the size or shape of the figure and are often called **rigid motions**. In this chapter we first introduce a third type of rigid motion, the *translation*, and then we look more carefully at the different rigid motions. Finally, we present the idea of combining rigid motions, and in doing so, we introduce a fourth type of rigid motion, the *glide-reflection*. In the last section of this chapter, we extend some of the earlier work with tessellations, symmetry, and similarity.

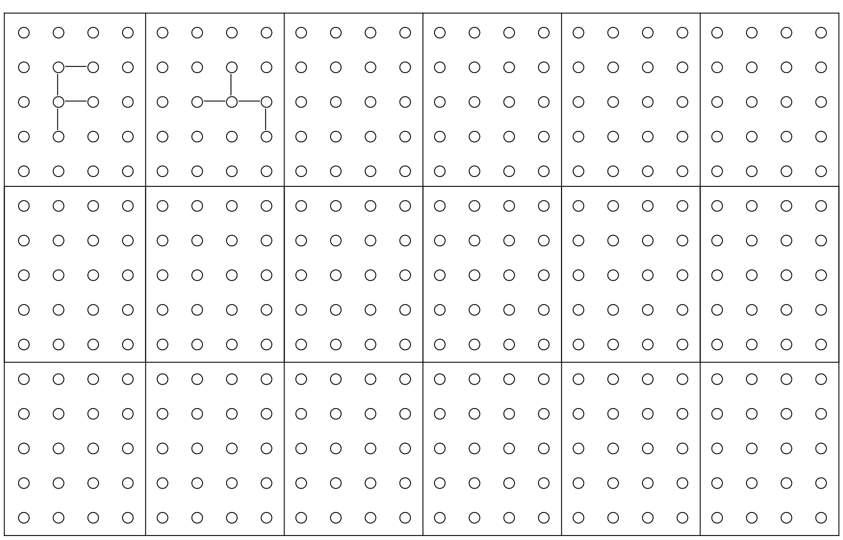
SEE INSTRUCTOR NOTE 22 for the topics covered and for more information about transformation geometry. The Geometer's Sketchpad lessons on Transformation Geometry fit well here. See http://sdmp-server.sdsu.edu/nickerson/.

22.1 Some Types of Rigid Motions

Activity 1[i] (Preliminary Homework) Let's Be Pick-y

Materials: An asymmetric cardboard region (e.g., a side view of a face, an F shape) is handy; perhaps a transparency with the answer to the first Activity.

Find all the different shapes that can be made from four toothpicks, subject to these two rules: (1) Toothpicks can touch only at endpoints, and (2) after the first toothpick, each toothpick either is perpendicular to the toothpick it touches or continues in the same straight line as the toothpick it touches. Record the different shapes you find, one shape to a box. Shapes are not different if they can be made to match. To get you started, two different shapes are shown.



Let's Be Pick-y…There are 16 different 4-toothpick shapes.

SEE INSTRUCTOR NOTE 22.1A for answers, for (fast) ways of checking in class, and for related activities.

It is likely that you mentally or physically "moved" a four-toothpick shape to test whether it was different from one you already had. These mental or physical movements, leading to a type of mathematics called **transformation geometry,** are called **rigid motions** (or **isometries**--*iso-* means same and *metron* means measure).

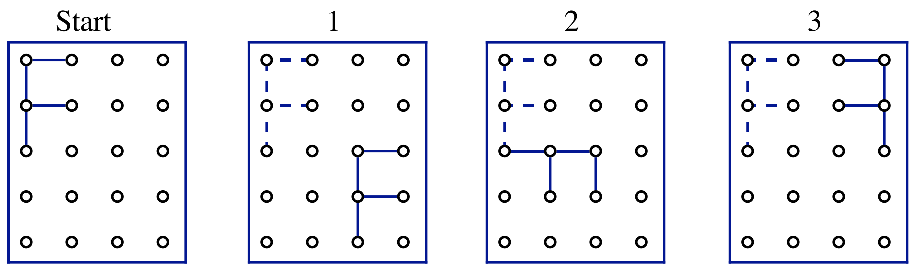
SEE INSTRUCTOR NOTE 22.1B on the terminology and the relevance of the material.

As you may know, a more elegant definition requires only invariant lengths, with invariant angle sizes being provable.

A **rigid motion**, or **isometry**, is a movement that does not change lengths and angle sizes.

Think About…

What are some different types of rigid motions? What language would you use to describe how the shape shown in the starting box below could end up at each of the other positions?

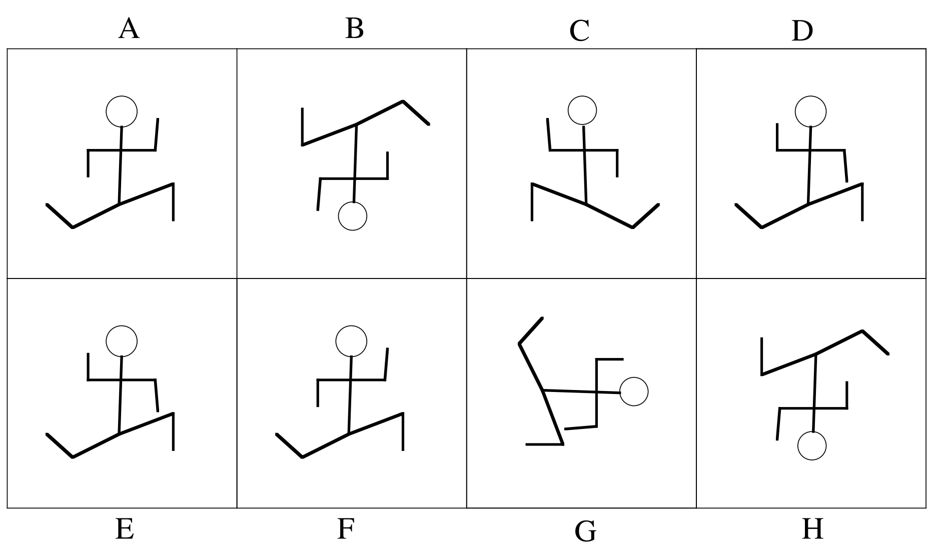


Although you may have used the informal terms *slide*, *turn*, and *flip* or *mirror image* (terms that are sometimes used initially with elementary school children), the technical terms for the rigid motions involved in the Think About are **translation** (see Box 1), **rotation about a point** (see Box 2), and **reflection in a line** (see Box 3).

Activity 2: A Moving Experience…After encouraging your students to try to "move" the shapes mentally, suggest that tracing a shape and moving the copy may help with difficult decisions. There are two shapes here: A (front arm lifted) and D. A-->B by a rotation; A-->C by a reflection; A-->F by a translation; A-->H by a rotation. Shape D-->E by a translation; D-->G by a rotation. There are other possibilities for questions— e.g., B-->H by a translation. Rotation images seem to be the most difficult to recognize. Notice that the glide-reflection, the only other type of 2D isometry, is not introduced here (it comes up in 22.4); don't leave the impression that there are only three types of rigid motions.

Activity 2 A Moving Experience

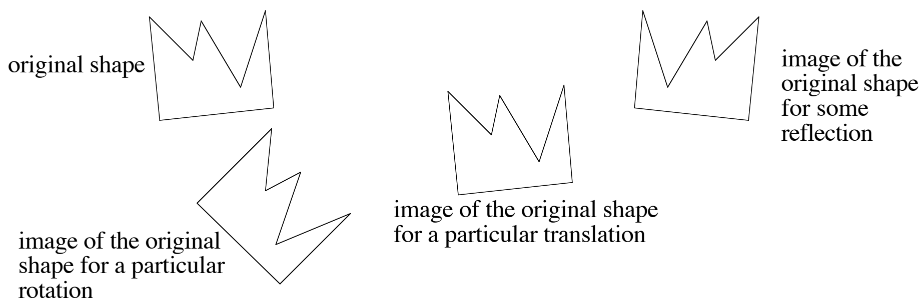
Pretend that the two-dimensional shapes below are on transparencies for an overhead projector. What shapes are the same? Which rigid motions, or isometries, help you decide?



So far, we have seen these types of isometries of the plane: *reflections* (shown in Activity 2, going from A to C), *rotations* (going from A to B), and *translations* (going from A to F). If you started with shape A and imagined shape A being moved to shape F, you could call shape F the **image** of shape A under a translation: original ---> image. Because the shapes are identical, the words *original* and *image* help communicate which shape is the starting shape and which is the ending shape. Although we often focus on a particular shape, each rigid motion affects *every* point in the plane, not just the points in that shape.

Although we use "image," we use "original" instead of "pre-image."

So much focus is on individual shapes that it is easy to forget that these rigid motions do affect each point, not just the ones on the shapes pictured.



What does the phrase *congruent figures* mean to you? Transformations allow a very general view of congruence for all figures, not just for triangles.

This definition formalizes the informal chatter for congruence of 3D shapes in 16.4, if you covered that section. The phrasing here also applies to 3D congruence.

Two figures are **congruent figures** if one is the image of the other for some rigid motion.

This description allows shapes of any sort—segments, angles, polygons, curved regions, even 3D figures—to be congruent. It also corresponds to a physical or a mental manipulation of one shape to see whether it matches the other shape exactly. A consequence of two figures being congruent is that any original part and its image part (referred to as *corresponding parts*, or *matching parts*, in high school) will always have the same measurements, if there is an appropriate measurement.

Take-Away Message…Rigid motions and transformation geometry may be new to you. Their dynamic, motion-based nature clearly can involve much visualization. Because of its importance, visualization is a topic that is receiving increased, explicit attention in the elementary school mathematics curriculum, rather than just being overlooked or left to chance. Furthermore, from the mathematical point of view, transformation geometry is closer to the mainstream of modern mathematics than geometry is *a la* Euclid. Transformations of the plane are special cases of mathematical *functions*, an idea that appears in some form in every area of mathematics.

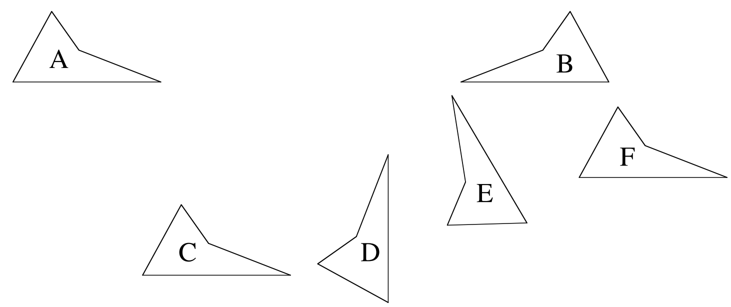
LE 22.1Students do not have answers to 1, 6, and 7.
#7 might serve as a group problem, or a nonroutine problem to be written up over a period of time.

Learning Exercises for Section 22.1

1. Curricula in the middle grades can use these informal terms—*mirror image*, *slide*, *flip*, and *turn*—for the isometries. Which technical term goes with each informal term?

2. Encourage mental work rather than the tracing and physically moving of shapes.

2. a. Try mental manipulation to tell which *single* type of rigid motion transforms shape A to each of the other shapes. It may help to draw a face or some mark on shape A. You may wish to check by tracing a copy of shape A and moving it to another shape.

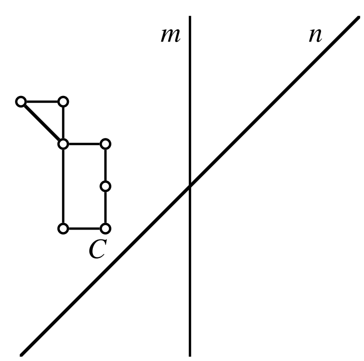


b. Which of the shapes in part (a) are congruent? Explain.

3. Students may be able to generate other real-world examples. Many natural motions involve one or more rigid motions.

3. What rigid motion is involved in each of the following situations? To retain the two-dimensional nature of the work in this section, assume that you are looking at a movie.
   a. A train moving along a straight track
   b. The motion of a fan blade
   c. A child sliding down a playground slide
   d. A clock-hand moving
   e. A skateboarder skating in a circular bowl
   f. A doorknob moving

4. In parts (a)-(d) make a freehand sketch to show the image of the given triangle-rectangle shape for the isometry.
   a. a reflection in line *m*
   b. a reflection in line *n*
   c. a translation 6 cm to the right
   d. a rotation of 90˚ clockwise with center C



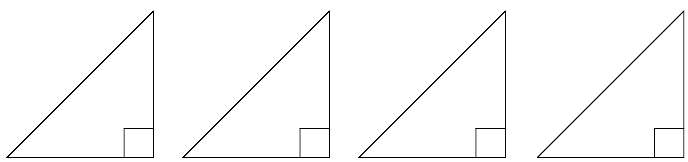
   e. Which of your shapes is congruent to the original shape? Explain.

5. The term **orientation** has a technical meaning that differs from everyday usage of the term. One can assign an orientation to a figure in this way: Pick any three noncollinear points on the figure—call them *P, Q, R*. Then the order *P-Q-R* assigns a clockwise or counterclockwise **orientation** to the figure (it may be helpful to think "*clock* orientation"). Different people, or different choices for the points, can assign a different orientation to the same figure. What is of value is knowing how the orientation of a figure is affected by each type of rigid motion. The orientation of the original and the orientation of its image might be the same, or they might be reversed. Make rough sketches or examine earlier ones to tell how the orientations of an original figure and its image are related for each type of rigid motion.

   a. translation   b. rotation   c. reflection

6. Which of the isometries make sense for "moving" a three-dimensional shape?

7. Given four congruent isosceles right triangles, how many different shapes can you make, using all four triangles each time? Work mentally as much as possible, before you record and name the shapes you find. Which rigid motions did you use?



Exercise 5 introduces "orientation," an idea with a technical meaning that differs from ordinary usage. (You may prefer to use the phrase "clock orientation.") Orientation is useful in deciding what transformation might be involved when it is not visually obvious.

Again, #7 is non-trivial. You may wish to assign the more elaborated version, Shapes from Four Triangles, in the Auxiliary Materials. That version was prepared as the basis for a possible graduation portfolio entry.

22.2 Finding Images for Rigid Motions

Earlier, your main task was to decide what rigid motion or isometry was involved when you were given a shape and its image. In this section, your task is to find the image when you are given a shape and the type of rigid motion. With practice, drawing the image freehand can work out satisfactorily, or the structure provided by using grid paper can help in drawing images. In classrooms you may come across either computer software for finding the images of given figures for rigid motions or plastic devices for finding the images for reflections.

However, the paper-tracing methods of this section offer a low-tech (and therefore slower) means of finding images quite accurately and in a way that shows the motion involved. For making neater and more accurate drawings, use a ruler or the edge of a 3" by 5" card to draw line segments. Each paper-tracing method that follows uses two pieces of translucent paper. The thinnest paper you have can also work all right if you use a felt-tip pen.

First, we illustrate a paper-tracing method for finding the image of a given figure for a particular translation. One way to describe a particular translation is to tell how far to move and in what direction. These two

Materials needed for 22.2: Translucent paper (see note below); square grid paper, isometric grid paper; MIRAs if desired.

If you have access to software that can show isometries, you may wish to give demonstrations or ideally to substitute work with the software for most of this chapter.

The plastic devices mentioned are called MIRAs.

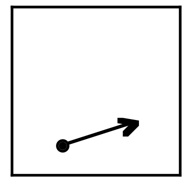
pieces of information can be communicated by a translation arrow, commonly called a **vector**. The length of the vector tells how far to translate, and the direction the vector is pointing tells the direction in which to translate.

Some instructors like to say more about vectors, such as their use in physics.

Paper that is prepared to separate meat or hamburger patties (“patty paper”) works very well for tracing; it may be available at a commercial supply store. If *Patty Paper Geometry*, by Serra, is available, you may wish to refer to it. Warn students to look ahead to predict where the image will be so that all the steps for a particular motion are on one sheet.

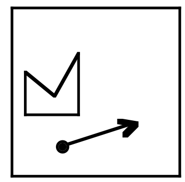
TRANSLATION

Step 1. Draw the translation arrow, or vector, on Paper 1. Put a heavy dot where the vector starts.



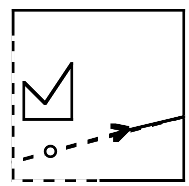
Paper 1

Step 2. Draw an original shape on Paper 1. (Plan the shape so its image will fit on the paper. It does not have to be a polygon; a face, a heart , an animal, or anything else works fine.)



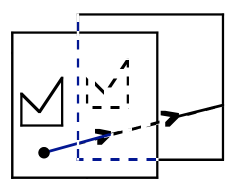
Paper 1

Step 3. Put Paper 2 on top, trace the starting dot of the vector, draw the line through the vector, and trace the original figure.



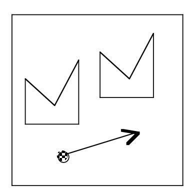
Paper 2 on top of Paper 1

Step 4. Put Paper 2 under the other paper. Keeping the line and the vector aligned, translate Paper 2 until the dot is at the vector head.



Paper 2 underneath Paper 1

Step 5. Trace the image onto Paper 1. (It is a good idea to label the original shape and the image as such.)



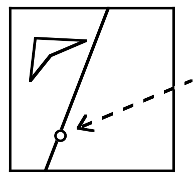
Paper 1

To find the image of a given shape for a reflection, you must know the line of reflection. Next, we illustrate a paper-tracing method for finding the reflection image of a shape.

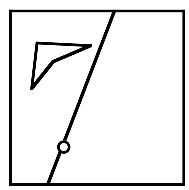
You may prefer to have the students fold on the line of reflection and then trace. If the paper is thin enough to see through two thicknesses, folding so that the original shape is on the outside requires tracing only once to get the image on the same side of the paper as the original.

REFLECTION

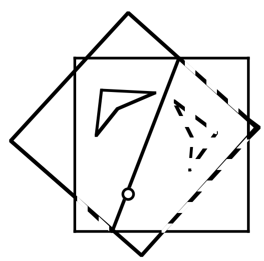
Step 1. Draw an original shape and the line of reflection on Paper 1. Put a heavy dot on the line of reflection for a reference point. (Plan so that the image will fit on the paper.)



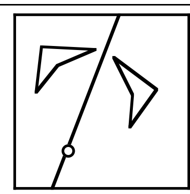
Step 2. Put Paper 2 on top, and trace the figure, the line of reflection, and the reference point.



Step 3. Flipping Paper 2 over and putting it underneath, align the lines of reflection and the reference points.



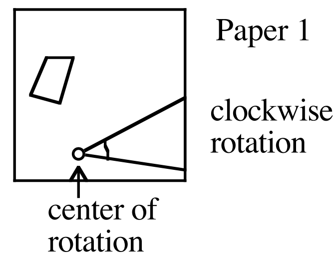
Step 4. Trace the image onto Paper 1. Again, it is a good idea to label the original and its image.



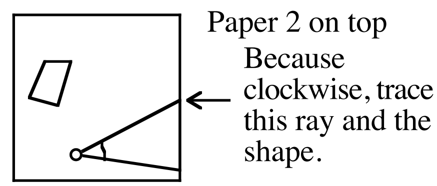
Finally, to find the image of a given shape for a rotation, you must know (1) where the rotation is centered (the **center** of the rotation) and (2) an **angle** that shows the size and direction, clockwise or counterclockwise, of the rotation. We illustrate a paper-tracing method for accurately finding the rotation image of a given shape.

ROTATION

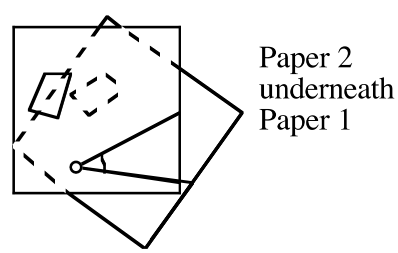
Step 1. Draw an original shape on Paper 1. Pick a point for the center of the rotation, and draw the angle of the rotation so that its vertex is at this center. Plan ahead so the image will fit on Paper 1.



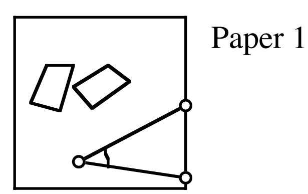
Step 2. With Paper 2 on top of Paper 1, trace the shape, the center of rotation, and the starting ray—the ray of the angle that would be rotated to give the second ray of the angle.



Step 3. Put Paper 2 underneath and align everything. With your pencil tip at the center of the rotation, turn the underneath paper in the proper clock direction until the starting ray is aligned with the second ray of the angle of rotation.

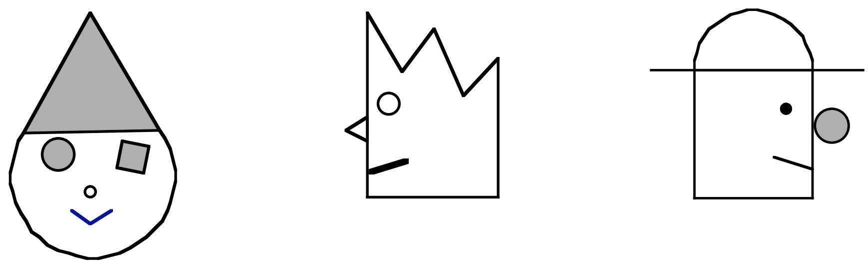


Step 4. Trace the image onto Paper 1. (Again, it is a good idea to label the original and the image.)



Activity 3 Let's Move It

The aim of this activity is to make certain you can find images accurately using the illustrated methods (or perhaps some other methods your instructor may prefer). Each person in your group should choose one of the figures below, and then find its images for a translation, a reflection, and a rotation. (Do them separately because drawings can get cluttered quickly.) Choose your own vector (translation arrow), line of reflection, and center and angle of rotation.



You will practice the paper-tracing methods more in the Learning Exercises or on your own, as well as practice drawing images freehand and with dot paper.

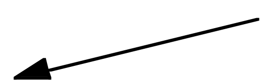
Take-Away Message…Paper-tracing methods can help to find accurate images for the different types of rigid motions—translations, reflections, and rotations.

Learning Exercises for Section 22.2

1. Find the image for each indicated rigid motion using paper-tracing.

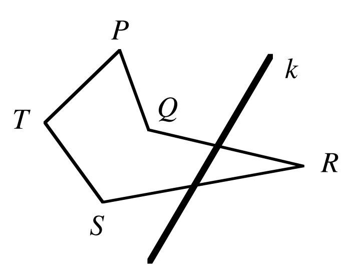
22.2 Students do not have answers or *Hints* for 3, 12, and 13.

a. The image of a shape of your choice, for a translation with the vector shown at the right.



b. The image of a shape of your choice for a rotation with a center of your choice and with a 90˚ clockwise angle.

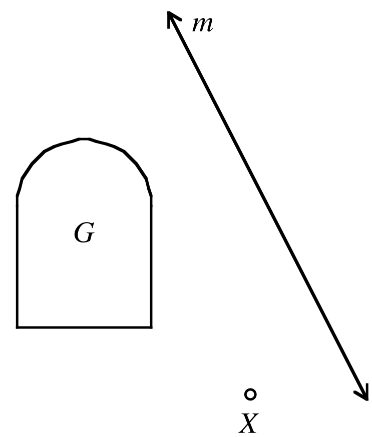
c. The image of the following shape for a reflection in the heavy line segment *k*.



d. The image of a shape of your choice for a rotation with center on the shape and with a 90˚ counterclockwise angle.

2. Copy and sketch *freehand*, as accurately as you can, the image of shape G for the rigid motion given in parts (a)-(c).

   a. A translation 5 cm east (label the image *H*)
   b. A 90° rotation clockwise, center at *X* (label the image *I*)
   c. A reflection in line *m* (label the image *J*)
   d. Check your freehand drawings.
   e. Which type of rigid motion is most difficult for you to visualize images?

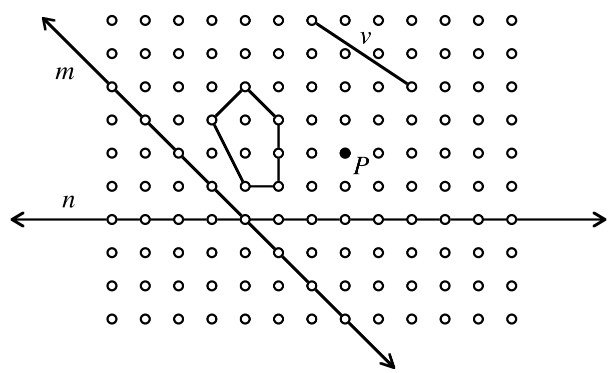


3. Choose another figure from Activity 3 and use tracing paper to find its images for a translation, a reflection, and a rotation. You can choose your own vector, line of reflection, and center and degrees of rotation.

4. Grid paper is commonly used in elementary school to help locate images for rigid motions.

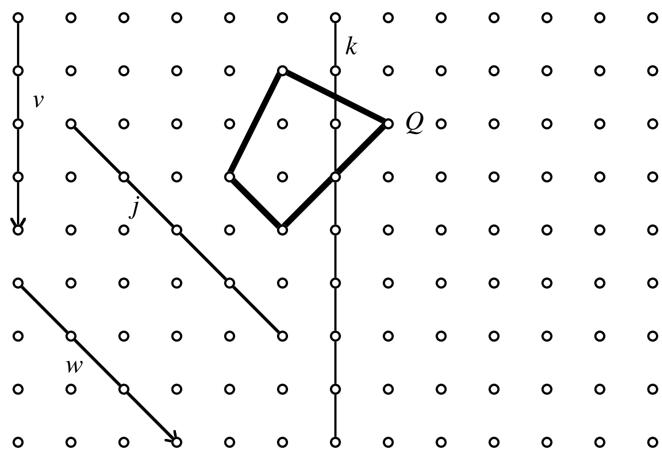
For Exercise 4, you might wish to make transparencies of the drawing so that you can use them for demonstration purposes.

   a. Find the image S' of the given pentagon for a translation 4 units west.
   b. Find the image S" of the pentagon for a rotation of 180 degrees, center *P*.
   c. Find the image S''' of the pentagon for a reflection in line *m*.
   d. Find the image S"" of the pentagon for a reflection in line *n*.
   e. Find the image S""' of the pentagon for a translation with vector *v*.

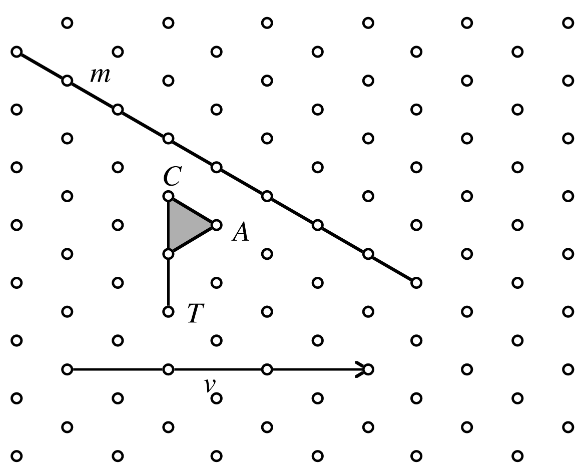


5. Find and label the image of the given quadrilateral for each rigid motion described.

   a. image A' for a translation with vector *v*
   b. image A" for a translation with vector *w*
   c. image A''' for a reflection in line *j*
   d. image A"" for a reflection in line *k* (shade it lightly)
   e. image A""' for a rotation of 90˚ counterclockwise, center *Q*

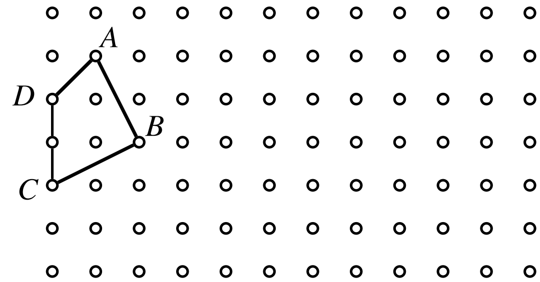


6. The regularity of isometric dot paper is also helpful in finding images. Find the image of the flag shape *CAT* for each rigid motion described.
   a. Translation with vector *v*
   b. Reflection in line *m*
   c. Rotation, 120˚ clockwise, center *C* (label it part (c))
   d. Rotation, 90˚ counterclockwise, center *T* (label it part (d))
   e. Rotation, 60˚ counterclockwise, center *A* (label it part (e))

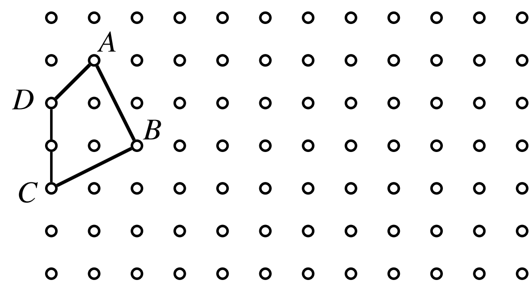


7. Copy the given quadrilateral *ABCD* on dot paper and sketch its image for a translation 6 spaces east. Label the image *A'B'C'D'*, where *A'* is the image of *A*, *B'* of *B*, and so on. Draw the line segments joining *A* and *A'*, *B* and *B'*, *C* and *C'*, and *D* and *D'*. What do you notice? Does this make sense?

Exercises 7, 8, and 9 are aiming toward analyzing the rigid motions, to be followed up in the next section. These ideas may be useful to non-visualizers, as a means of dealing with particular transformations.



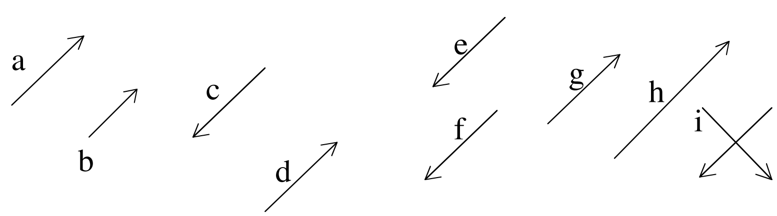
8. Using the same ABCD from Learning Exercise 7, sketch its image for a 90° clockwise rotation, using vertex *C* of the quadrilateral as the center of rotation. Label the image *A*"*B*"*C*"*D*", where *A*" is the image of *A*, *B*" of *B*, and so on.



   a. Join each of *A*, *B*, and *D* to its image with a line segment. Are all these line segments the same length? Does this make sense for a rotation?
   b. Join each point *A, A", B, B", D* and *D*" to the center *C* with a line segment. Which of these line segments have the same length? Does this make sense for a rotation?
   c. Examine the angles made by joining a point to the center C and then to the image of the point. Are all these angles the same size? How large is each of these angles? Does this make sense for a rotation?

9. Examine one of the reflections you made in previous exercises. Pick out three or four points on the original shape, and join each point to its image with a line segment. Are all these line segments the same length? What does appear to be true? Does this make sense for a reflection?

10. Which of the following vectors are describing the same translation? Explain.



11. A rotation actually affects every point of the plane (imagine a sheet of stiff plastic). Which of the following rotations will have the same effect on every point of the plane? The center in each case is the same. Explain your decision.
    a. rotation of 90° clockwise
    b. rotation of 90° counterclockwise
    c. rotation of 180° clockwise
    d. rotation of 180° counterclockwise
12. Dot paper has some limitations when you wish to involve only points at the dots as vertices.
    a. Experiment with square-dot paper to see which reflection lines allow the use of only points at the dots (being able to use the dots makes finding the images easier).
    b. Experiment with square-dot paper to see which rotations allow the use of only points at the dots (which, again, makes finding the images easier).
    c. Experiment with isometric paper in ways similar to those in parts (a) and (b).
13. You and a friend are looking at a two-dimensional shape in a plane. Your friend is visualizing where the image of the shape would be for different rigid motions. For the questions that follow, what information would you need so that you could draw the image in the very same place your friend is visualizing it?
    a. Suppose that your friend is visualizing the image of the given shape for some translation. If you want to draw the image in exactly the spot your friend imagined it, what information about the translation would be essential?
    b. Suppose that your friend is visualizing the image of a given shape for some reflection. If you want to draw that image in exactly the spot your friend imagined it, what information about the reflection would be essential?
    c. Suppose that your friend is visualizing the image of a given shape for some rotation. If you want to find that image in exactly the spot your friend imagined it, what information about the rotation would be essential?

#13 can be viewed as preparation for Section 22.3. With luck someone may offer something like the "Key Relationships" in Section 22.3.

22.3 A Closer Look at Rigid Motions

Section 22.3 is fairly dense in content. The first part lists the defining characteristics for some of the rigid motions. These are often useful in making fairly careful drawings without tracing, in constructing images or missing parts, in checking images, and in coping with visual tasks, for weak visualizers. Given a figure and its image, they also enable the construction of the elements of the rigid motion.

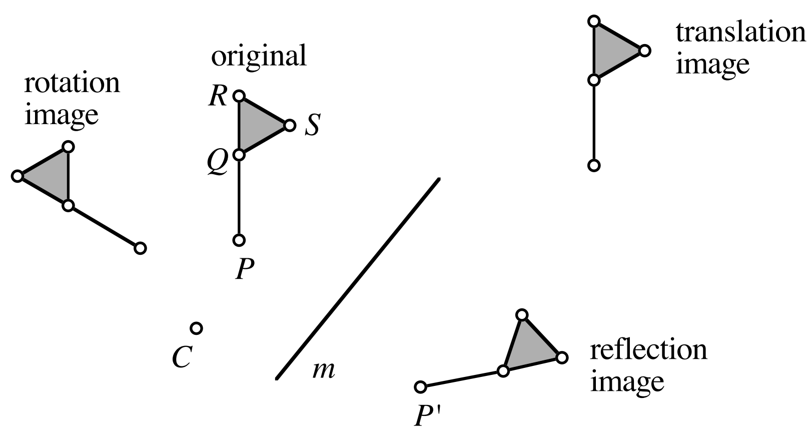
Some previous exercises should have suggested the following key relationships for the rigid motions we have seen. They are key relationships because, mathematically, knowing how a rigid motion affects each point is the most basic knowledge. In addition, these key relationships give a means of checking the results of rigid motions and of guiding the sketching of images. For someone who has difficulty with visualization, the key relationships give an alternative to strictly visual approaches. These key relationships also help in describing a rigid motion when only a shape and its image are given.

A common notation is to let P' represent the image of point P.

Type of rigid motion	**Need to know**	**Key relationships**
Reflection in line m	Line m	Line m is the perpendicular bisector of the line segment joining P and P'.
Rotation of $x°$ with center C	Center C, x, and the clock direction	1. For each point P, the distance from the center C to P is the same as the distance from C to P'. 2. Every $\angle PCP'$ has $x°$.
Translation	Distance and direction, or vector	All segments $\overline{PP'}$ have the same length and are parallel (the length and direction are summarized by the vector).

Activity 4 Practicing the Key Relationships

Using the original shape and its images shown below, illustrate each of the key relationships given in the table with one or more points. A sample point P and its image P' are shown.

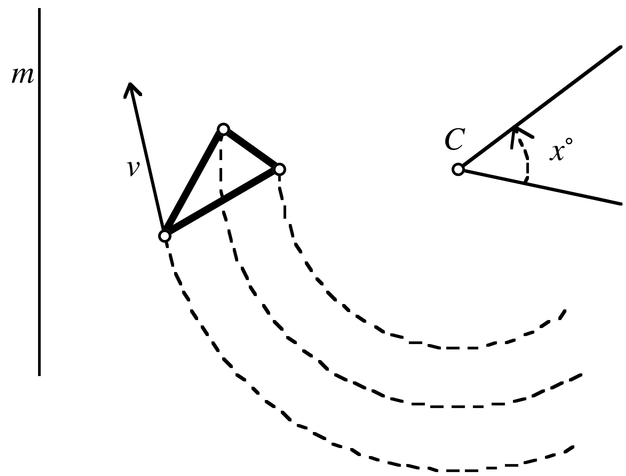


Activity 5 Constructing Images

With a compass and straightedge or tracing paper, construct the images of the given triangle for the rigid motion described. (Hint: It may be a good idea to copy the pertinent parts for each on separate paper, because the drawing will get messy.)

a. a reflection in line m
b. a clockwise rotation of x° with center C
c. a translation with vector v.

Activity 5. Constructing…: You may ask for just rough sketches. You may need to review the constructions.

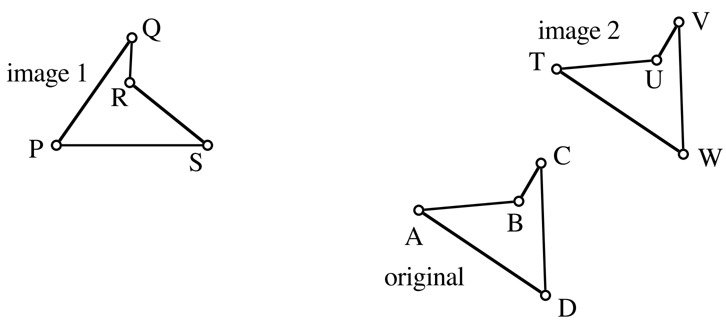


So far, the bulk of the work with rigid motions has involved finding the image, given a figure and information about a rigid motion. An important variation is this situation: Given a figure and its image, determine the information needed about the rigid motion involved. The key relationships help in describing a rigid motion *fully*.: If the motion is a reflection, the key relationships help to find the line of reflection; if a rotation, the key relationships help to find the center and the angle; and if a translation, they help to find the vector.

Think About...

Which rigid motions give image 1 and image 2 below? (Describe the rigid motion fully. For example, if it is a reflection, indicate the line of reflection.)

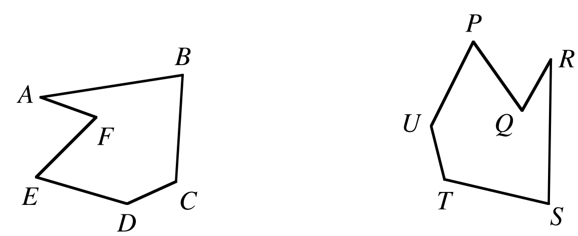
Think About… You may want your students to construct the line of reflection.



Activity 6   Another Lost Rigid Motion

SEE INSTRUCTOR NOTE 22.3A for a discussion of Activity 6. (Needed: Ex. 12d in 21.1)

Which rigid motion gives the result shown below? Describe the rigid motion fully. For example, if it is a reflection, indicate the line of reflection.



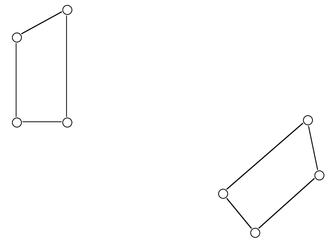
Take-Away Message…Key relationships for the rigid motions give the underlying mathematical relationships for points and their images. The key relationships also can help in making free-hand sketches or constructions of images of shapes. Furthermore, the key relationships can help to locate the full information about an unknown rigid motion.

Learning Exercises for Section 22.3

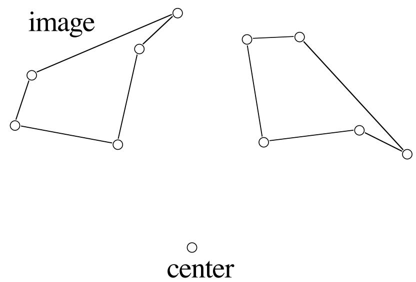
LE 22.3 The student version has answers or *Hint*s for all the exercises. You may wish to ask for constructed solutions for #1 and 2b, as well as 3.

1. Draw a triangle with a ruler (and not too close to the edge of your paper). Then do the following.
   a. Use the key relationships to draw accurately the image of the triangle, for a translation 3.5 cm east. Write down how you used the key relationships. (*Hint*: Where are the images of the vertices of the triangle?)
   b. Use the key relationships to draw accurately the image of the triangle for a rotation of 90° counterclockwise, with center at your choice of point. Write down how you used the key relationships.
   c. With a new triangle and your choice for line *m*, use the key relationships to draw accurately the image of the triangle for the reflection in line *m*. Write down how you used the key relationships.
2. Trace the given shapes in each part. Then draw accurately the information needed for the motion described. (You may find it helpful to label points and their images.) Tell how you used the key relationships.
   a. The vector for the translation giving the following shape and its image. (Does it matter which shape is the original and which one is the image?)

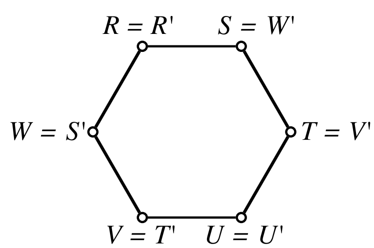
b. The line of reflection for the reflection giving the following shape and its image. (Does it matter which shape is the original and which one is the image?)



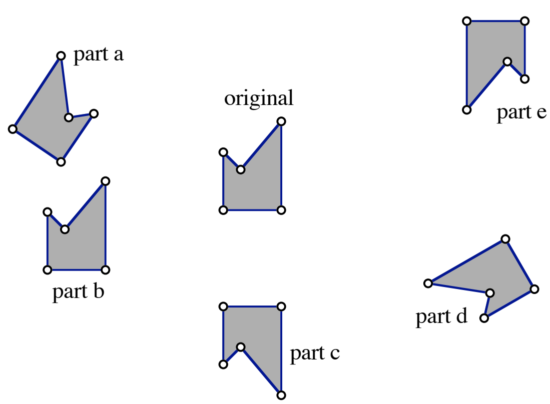
c. The angle and its clock direction for the rotation giving the following shape and its image; measure the angle also. (Does it matter which shape is the original and which one is the image?)



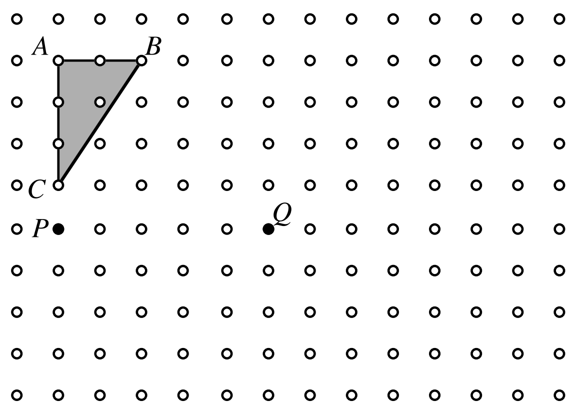
d. The line of reflection for the reflection of hexagon RSTUVW giving the image indicated.



3. The shapes in parts (a)-(e) are different images of the given original shape. Trace the original and the image separately for each part to allow room for work. Identify the rigid motion involved, and then describe the rigid motion fully.

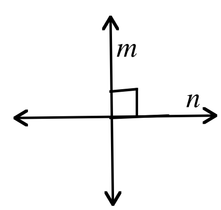


4. Notice that in Learning Exercise 2(d), point R is its own image. If the image of a point is the point itself, that point is called a **fixed point**. Tell where *all* of the fixed points in the plane are, if there are any, for the following rigid motions.
   a. the reflection in some line k
   b. the translation 4 cm north
   c. the rotation with center C and angle 55° clockwise (*Hint*: There is more than 0 fixed points.)
   d. the rotation with center Q and angle 180°
   e. the rotation with center M and angle 360°
5. Copy the given figure and grid.
   a. Show the image of triangular region ABC for a 90° clockwise rotation with center P. Label it A'B'C'. (The key relationships may be helpful.)
   b. Now find the image of A'B'C' (notice the primes) for a 90° *counter*clockwise rotation with center Q. Label it A"B"C".

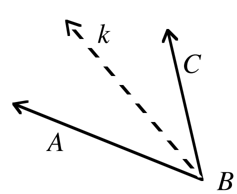


Exercise 5c gives a bit of experience with composition of rigid motions, the focus of the next section. This composition (in part (c)) is a translation.

   c. What single rigid motion will give A"B"C" as the image of the original ABC? Describe it as completely as possible. (For example, if it is a reflection, what would be the line of reflection?)
6. a. If lines m and n are perpendicular, what is the image of line n for a reflection in line m?



   b. If k is the bisector of $\angle ABC$, what is the image of $\overrightarrow{BA}$ for a reflection in line k? The image of $\overrightarrow{BC}$?



22.4 Composition of Rigid Motions

Just as 8 and 2 can be *combined* to give another number—possibly10, 6, 16, or 4, depending on whether the *combining* is done by adding, subtracting, multiplying, or dividing—rigid motions can be *combined* to give another rigid motion. The resulting isometry is called the **composition** of the two rigid motions (just as 16 is called the *product* of 8 and 2, for example).

SEE INSTRUCTOR NOTE 22.4A about the glide-reflection and the composition of rigid motions.

The diagram in Figure 1 is an example of the composition of these two rigid motions: the translation, move 3 cm east and the reflection in line *k*.

It is handy to have a cardboard cut-out of some shapes, to help give chalkboard options quickly and to make board work look more exact. Having a shape with curves gives a reminder that rigid motions can deal with more than polygons.

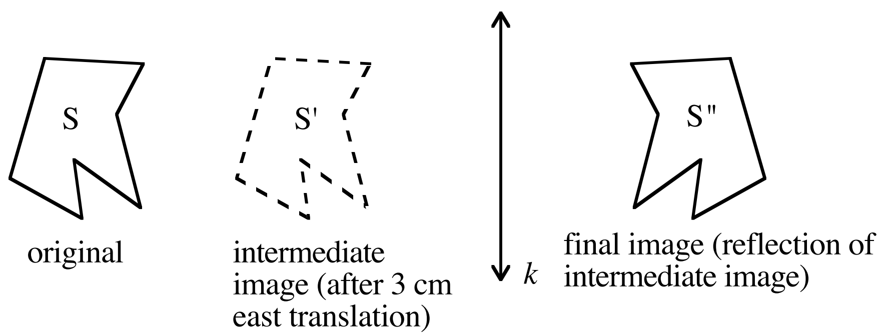


Figure 1

Although each of these two motions affects every point in the plane, let us focus on shape S. To arrive at the composition of these two rigid motions, first do the translation to get S'. The second motion, the reflection, then *starts* with S' as its original and gives S" as the image of S'.

As you know, composition of functions is more general than this definition. Compositions involving a size change come up in the next section.

The **composition** of two rigid motions is the *single* rigid motion that takes the original S to the final S".

In this case, it looks as though a reflection in some new line *m*—a different line from line *k*—would give S" as the image of S—that is, the composition of the two given motions would be this single reflection. See Figure 2. (Notice that we use the original line *k* for the reflection and not its translation image, which is not even pictured here.)

You may need to review how line *m* can be determined from the original S and the final S".

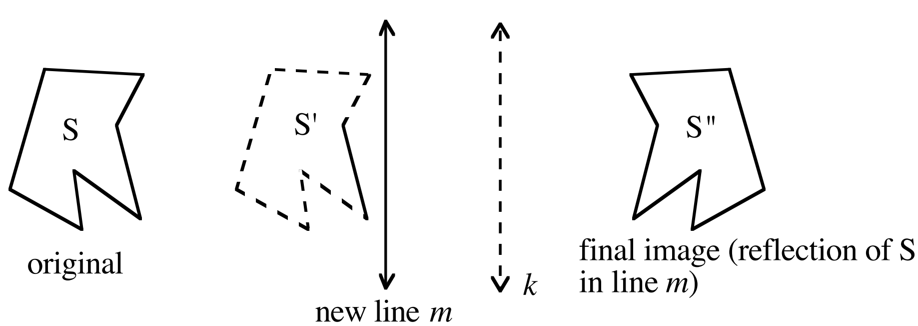


Figure 2

Now let us look at how to express the composition of two rigid motions. Just as we write 8 + 2 = 10 (or 8 – 2 = 6, and so on), we could write:

(reflection in line *k*) ∘ (translation 3 cm east) = (reflection in line *m*).

Notice the order in which the motions are written; this convention is important to observe when working with compositions.

In summary, the composition of two rigid motions is achieved by doing one rigid motion first, and then using the image from that motion as the *original* for the second rigid motion. Whatever motion would be the net effect of that combination on the original is a single motion, the composition of the two rigid motions. Using the symbol ∘ to represent this way of combining two rigid motions, we can write

(second motion) ∘ (first motion) = (the composition of the motions).

One helpful way of reading the symbol ∘ is to pronounce it as *after*. The order of recording the first and second motions may seem backwards, but it is the conventional order in advanced mathematics.

One of the exercises deals with why the line and vector are so restricted: The order in which one does the reflection and the translation does not matter if the line and the vector are parallel. If they are not parallel, the composition is still (usually) a glide-reflection, but different orders for the translation and reflection will give different compositions. (If the line of reflection is perpendicular to the line of the vector, the composition is a reflection.)

Discuss this (rather complicated) figure, or use a similar one.

Just as 287 + 95 can be determined to be 382, a major question is how—or even whether—the composition of two rigid motions can be predicted from only information about the rigid motions. One composition that is of particular value occurs with the composition of a translation and a reflection. For theoretical reasons, the line of reflection must be *parallel* to the line of the vector of the translation, but the distance of the translation, or the length of the vector, can be any length. Suppose the translation is given by vector *v* in Figure 3, and the line of reflection is line *n* (notice that line *n* is parallel to the line the vector is on).

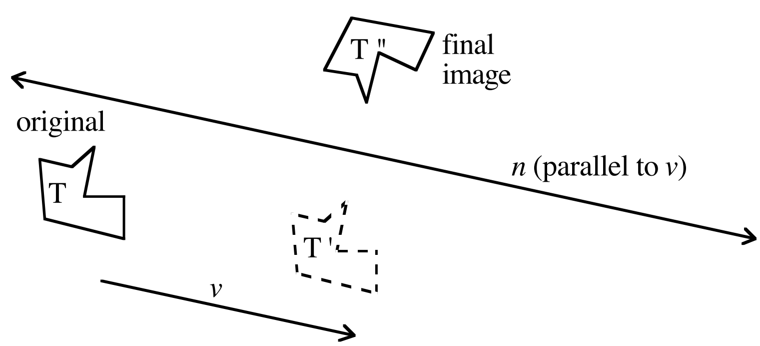


Figure 3

This type of composition has a name: a **glide-reflection.** Notice that the final image T" is not the image of T for any single reflection, rotation, or translation.

Whenever a translation and a reflection in a line parallel to the vector of the translation are combined, the resulting composition is a **glide-reflection**.

Glide-reflections are regarded as *single* rigid motions, even though they are defined in terms of two motions. This usage may seem strange, and it opens up the question of which other compositions will have special names. But there is a good reason for identifying glide-reflections. Adding just glide-reflections to the list of types of rigid motions makes the list *complete*. It will not be necessary to add any more types of rigid motions to the list.

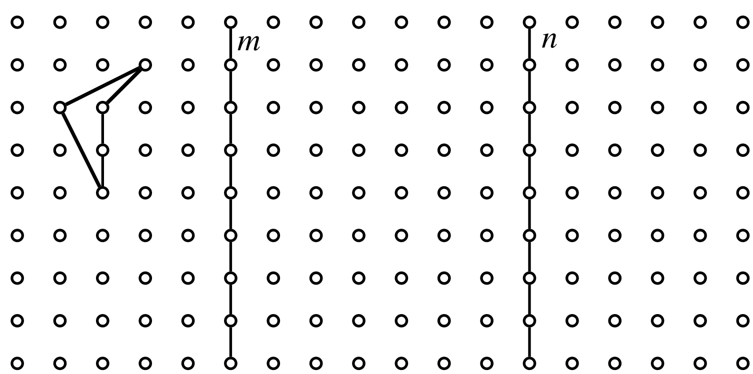
Any composition of rigid motions can be described by a *single* reflection, rotation, translation, or glide-reflection.

This last statement takes some time to sink in. It asserts, for example, that the composition of 100 reflections in 100 different lines is the same as a *single* one of the rigid motions. It also assures us that, given any two copies of the same planar figure, one copy can be transformed into the other by a *single* one of the motions.

Activity 7: Composition of… The composition will be a translation (14 spaces east). Ask if having the starting figure between the lines, or to the right of them, would change their minds.

Activity 7 Composition of Reflections in Two Parallel Lines

Carry out the composition (reflection in n) o (reflection in m) and form a conjecture as to what type of rigid motion the composition of two reflections in parallel lines might always be.



Take-Away Message…Composition of rigid motions, which is performed by following one motion by another, gives a sort of arithmetic for motions. One type of composition in particular, the glide-reflection, completes the list of rigid motions necessary to describe every composition of rigid motions of the plane. Certain compositions have predictable descriptions as single rigid motions.

LE 22.4. The student version has answers or hints for all the exercises. Several further the "theory" part of transformation geometry (rare in K-6). You may choose to use these as in-class activities. #13 is not theoretical.

Learning Exercises for Section 22.4

1. Make measurements on the example of composition given in Figure 2 to see if there is some relationship between line *k*, line *m*, and the 3 cm east translation.

You may wish to start or imitate Exercise 2 in class to be sure that students have the idea of composition.

2. Using your choice of original shape, find the composition of the two rigid motions described in each part. What single rigid motion does the composition seem to be?

   a. (translation, 4 cm east) ∘ (translation, 2.8 cm west)
   b. (translation, 3 cm northeast) ∘ (translation, 6 cm southwest)
   c. (reflection in line *n*) ∘ (reflection in line *m*), when lines *n* and *m* are parallel
   d. (reflection in line *n*) ∘ (reflection in line *m*), when lines *n* and *m* are not parallel
   e. (rotation, center P, $x°$ clockwise) ∘ (rotation, center P, $y°$ clockwise)
   f. (rotation, center Q, $a°$ clockwise) ∘ (rotation, center Q, $b°$ counterclockwise)
   g. (rotation, center R, 50° clockwise) ∘ (rotation, center S, 60° clockwise), where R and S are different points
   h. (rotation, center T, 40° clockwise) ∘ (reflection in line *p*), where line *p* goes through point T
   i. (rotation, center T, 40° clockwise) ∘ (reflection in line *q*), where line *q* does not go through point T

You may need to review orientation of a figure. Exercise 3b can be useful in narrowing the search for the rigid motion relating two congruent figures.

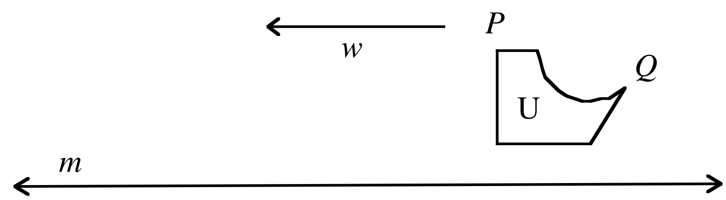
3. a. How does a glide-reflection affect the (clockwise or counterclockwise) orientation of a figure?
   b. Summarize how the different types of rigid motions affect the orientation of a figure.
   c. Does a glide-reflection give an image that is congruent to the original shape? Explain your thinking.

Exercise 4 uses 3b. Students might need some help in getting started on Exercise 4.

4. Using only orientation as a guide, what two types of rigid motions might each of the following possibly be? (*Hint*: See Learning Exercise 3(b).)

   a. The composition of a translation followed by a rotation
   b. The composition of a translation followed by two different reflections
   c. The composition of 4 different reflections in different lines
   d. The composition of 17 different reflections
   e. The composition of an even number of reflections; an odd number
   f. The composition of 3 different rotations, followed by 7 different translations, followed by 9 different glide-reflections

5. a. Copy and find the image of shape U for the glide-reflection given by the translation with vector *w* and the reflection in line *m*. Use paper-tracing if you wish.



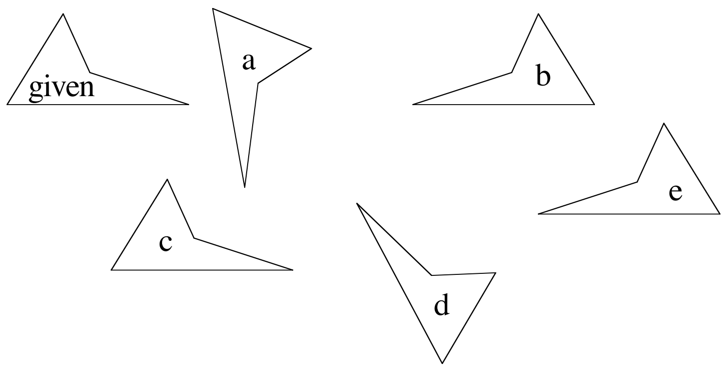
b. Label the (final) image of P as P' and of Q as Q'. With a ruler draw the line segments joining P and P' and joining Q and Q'. How does the line of reflection seem to be related to these line segments? Check with other points and their images. This result is a key relationship for a point and its image for a glide-reflection.

Exercise 5b can be used in finding the line of reflection for a glide-reflection, given a figure and its image.

6. a. In finding the composition of two rigid motions, does it matter in which order the rigid motions are done, in general? Explain.
   b. With the translation and reflection that define a glide-reflection, does it matter in which order the motions are done? Your finding is the reason why glide-reflections are defined in such a particular way.

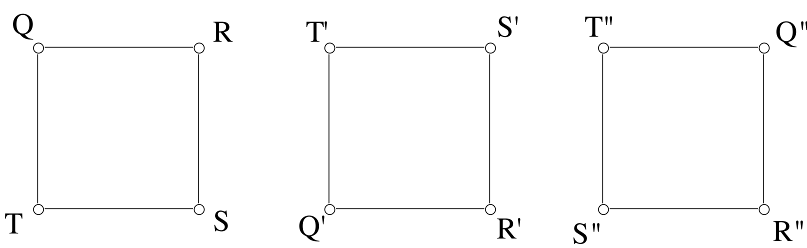
Non-commutativity of composition comes up in Exercise 6.

7. What single type of rigid motion gives each lettered shape as the image of the given one? Describe each of the rigid motions as fully as you can (for example, if it is a reflection, what line is the line of reflection?). Explain how you decided.

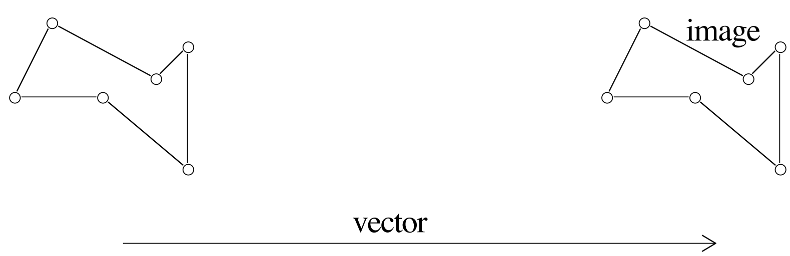


7. If you ask for constructions, parts (d) and (e) involve glide-reflections, and it may be a stretch (using #5b) to construct the line and vector.

8. For each image of QRST, describe the rigid motion that gives the given image as indicated.

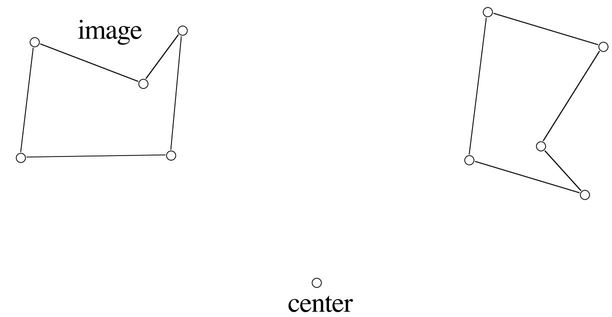


9. Trace the drawing and find two reflections so that their composition will give the same image as the original rigid motion.
   a. This translation



Exercise 9 deals with the converses of Exercises 2c and 2d, suggesting that any translation or any rotation could be replaced by a composition of two reflections, thus giving the non-intuitive relationship of Exercise 11.

b. This rotation



c. Find *other* pairs of reflecting lines for parts (a) and (b).

10. Make conjectures based on Learning Exercises 2(c), 2(d), and 9, and then gather more evidence.

11. Give an argument for this statement: Any rigid motion can be accomplished with at most three reflections. (*Hint*: Learning Exercise 10)

12. How many reflections, at minimum, might be needed to achieve the same effect as each composition described? Explain your reasoning.
   a. A composition of 17 reflections in different lines
   b. A composition of 24 reflections in different lines

13. Which rigid motion(s) could be used in describing each situation?
   a. footprints in the sand
   b. one's right hand in climbing a ladder
   c. one's two hands in climbing a ladder
   d. turning a microwave dish
   e. tuning in a different radio station
   f. adjusting a thermostat

15. You may wish to introduce the idea of the identity transformation.

14. Does a glide-reflection have any fixed points? That is, is there any point whose image under a glide-reflection is the original point?

15. a. What rigid motion is (reflection in line k) $\circ$ (reflection in line k)?
   b. What rigid motion is (translate 10 cm north) $\circ$ (translate 10 cm south)?

16. Experiment to find the single rigid motion equal to the composition of three reflections in parallel lines. Then describe that rigid motion in terms of the original lines (make some measurements on the original lines).

Section 22.5 revisits 2D congruence, symmetry, tessellations, and similarity, with an emphasis on how transformation geometry (including size changes) relates to these topics. You may wish to ask what your students know about functions; some may remember the $f(x)$ notation, which you can compare to a $T(P) = P'$ notation.

22.5 Transformations and Earlier Topics

Now that we have all the categories of rigid motions of the plane, we can review congruence, symmetry, tessellations, and similarity, discussing their relation to the types of transformations. Although many of the relationships carry over to three-dimensional geometry, we will work only with two dimensions here because most of our transformation geometry has been in two dimensions, with rigid motions (translations, rotations, reflections, glide-reflections) and with size transformations (size changes).

Congruence of two shapes, defined earlier as some rigid motion that gives one of the shapes as the image of the other, continues to hold, even when glide-reflections are involved. Do you see why?

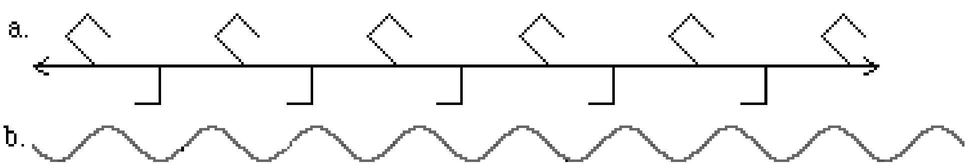
Discussion 1 Each Part…

Without looking at an example, explain why a shape and its glide-reflection image must be congruent. (Hint: What is a glide-reflection?)

The symmetries we studied earlier were either reflection symmetries or rotational symmetries, so a natural question is, Are there figures which have translation or glide-reflection symmetries?

Discussion 2 A Long, Long Trail

Describe the reflections and rotations that are symmetries of each of the following figures. (Both patterns continue indefinitely.) Look for translations and/or glide-reflections that are also symmetries of the figures.



Discussion 2. During *A Long, Long Trail*, be sure that the students realize that there are infinitely many symmetries for each type of rigid motion. When the students are focusing on the set of points as a whole, they can overlook the fact that two rigid motions may give different images for at least one point (and usually for most) and so are different symmetries.

From Discussion 2 we see that each type of rigid motion is a candidate for being a symmetry of a figure. Symmetries of figures are special cases of transformations.

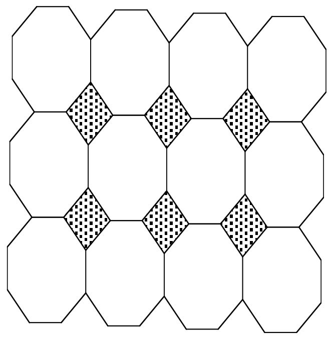
Tessellations too can be viewed with transformations (and congruence and symmetry) in mind.

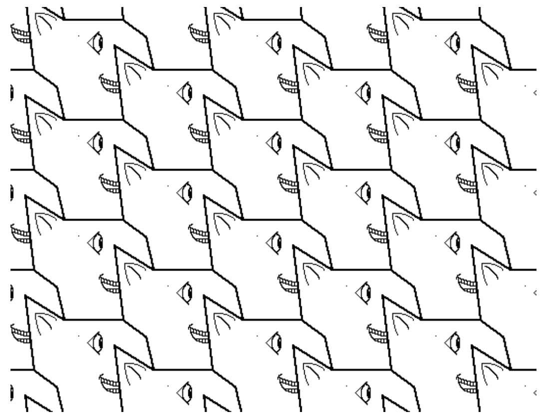
Skip this part on tessellations if your students have not encountered tessellations.

Discussion 3 What Isometries?

Keeping in mind that tessellations continue indefinitely, describe some rigid motions that give the same pattern for each of the tessellations suggested in the following two figures. Look for translations and glide-reflections as well as reflections and rotations.

Discussion 3. What Isometries?
The first has both types, but the second has only translation symmetries. Point out translations along "slants," if they are not noticed.





Rigid motions cannot themselves account for enlargements or reductions, of course. Size transformations must be involved if there is a change in size. The terms *similarity* and *similar figures* allow the composition of a size transformation and any rigid motion.

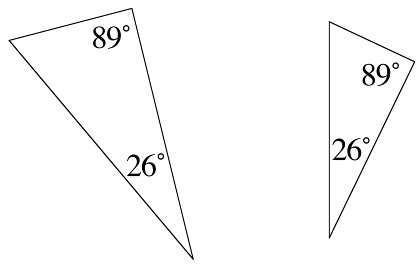
> Two figures are **similar figures** if one is the image of the other under (1) a size transformation or (2) the composition of a size transformation and a rigid motion.

You will note that the justification for AA is very weak, mathematically speaking. Issues are, by how much does one rotate, and will the lines joining corresponding vertices indeed pass through a single point? And if the triangles have different orientations (in the clockwise, counter-clockwise sense), a reflection must be involved. You can judge whether to slur over these points.

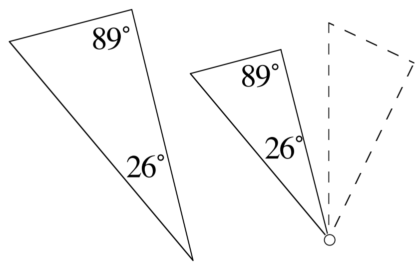
Rigidly moving an image after a size change does not negate the similarity; it just makes it more difficult to find the center of the size change. Fortunately, in most work with similar figures, the facts that the all ratios of pairs of corresponding lengths equal the scale factor and corresponding angles are the same size, are of greater importance than the location of the center of the size change.

Recall that for triangles there is a simple test for similarity: Two triangles are similar if *two* angles in one have the same sizes as two angles in the other. How can that be? A moment's thought makes clear that the third pair of angles must also have the same size because the sizes of the three angles in every triangle add to 180°. But how does having two pairs of angles the same size make certain that there is a size transformation? The following sketches give one way to convince yourself that there will be a size transformation relating two triangles that are not congruent. (Do you remember how to find the scale factor before you have the center?)

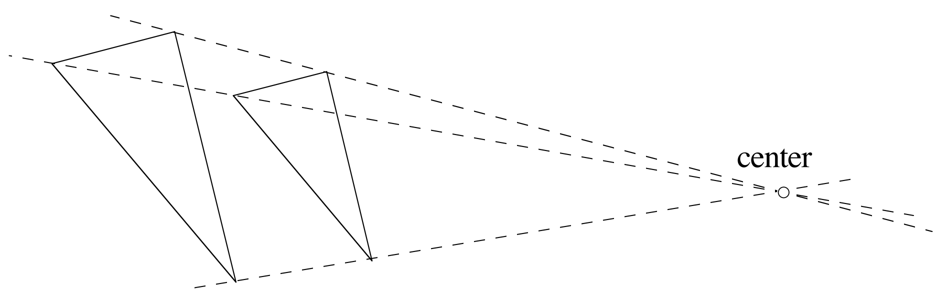
The two triangles have two pairs of angles with the same size.



Rotate one triangle so that the two sides of one angle are parallel to the sides of the corresponding angle in the other triangle.



Join corresponding vertices to locate the center of the size transformation. (Can you find the scale factor another way now?)



Other situations might involve a reflection rather than a rotation.

Discussion 4 And in Space?

Can three-dimensional congruence, similarity, symmetry, and tessellations also be treated by a study of three-dimensional transformations?

SEE INSTRUCTOR NOTE 22.5A regarding Discussion 4.

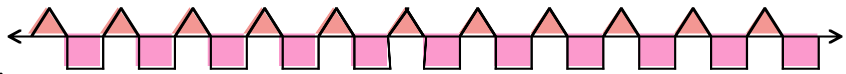
Take-Away Message…The usual secondary school topics of two-dimensional similarity and congruence, as well as symmetry and tessellations, can be treated in a very general fashion with transformations.

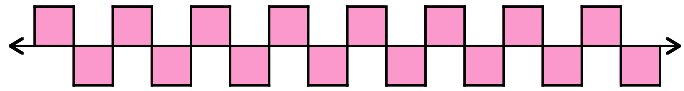
LE 22.5. The student version does not answers or *Hint*s for 10, 12, and 13. Samples of Islamic art give venues for reviewing transformations, symmetry, and tessellations.

Exercise 5 repeats most of an earlier exercise, but now possible symmetries include translations and glide-reflections.

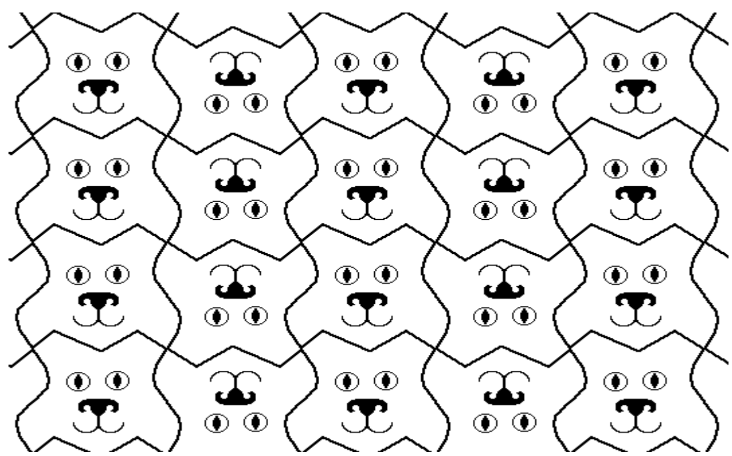
Learning Exercises for Section 22.5

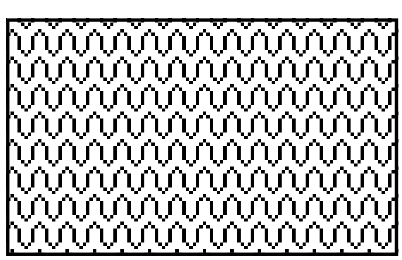
1. Describe all the symmetries possible in each diagram. (An arrow means the pattern continues.)

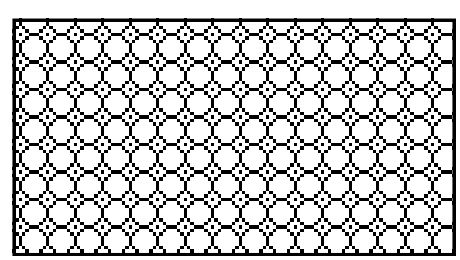
a. 

b. 

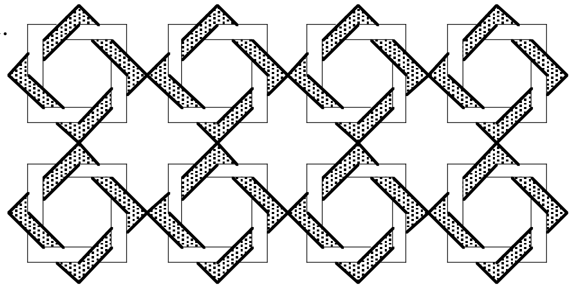
2. Examine both statements given in each part. Is each statement true? Explain your decisions.
   a. If a shape has translation symmetry, then the shape is infinite.
      If a shape is infinite, then the shape has translation symmetry.
   b. If a shape has glide-reflection symmetry, then the shape is infinite.
      If a shape is infinite, then the shape has glide-reflection symmetry.
3. Which types of symmetry does each 2D shape have?
   a. a line segment b. a ray c. a line d. an angle
4. Design a figure that has translation symmetry. Does your figure also have glide-reflection symmetry?
5. Which transformations are symmetries for the tessellations given by the following (infinite) patterns?

a. 

b. 

c. 

d.



6. Information is given about angle sizes in the following four triangles. Which of the triangles, if any, are similar? Explain how you know.

Triangle a: 65° and 32°	Triangle b: 65° and 35°
Triangle c: 35° and 80°	Triangle d: 65° and 83°

7. Describe the image of a shape under the composition of a size transformation with scale factor 1 and a rotation of x°, both with the same center.

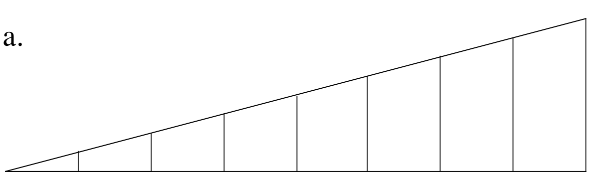
8. Use the meaning of congruence from the transformation view to answer these.
   a. If Shape 1 is congruent to Shape 2, then is Shape 2 congruent to Shape 1? Explain.
   b. Is every shape congruent to itself? Explain.
   c. If Shape 1 is congruent to Shape 2 and Shape 2 is congruent to Shape 3, is Shape 1 congruent to Shape 3? Explain.

9. Use the meaning of similarity from the transformation view to answer these.
   a. If Shape 1 is Similar to Shape 2, is Shape 2 similar to Shape 1? Explain.
   b. Is every shape similar to itself? Explain.
   c. If Shape 1 is similar to Shape 2 and Shape 2 is similar to Shape 3, is Shape 1 similar to Shape 3? Explain.

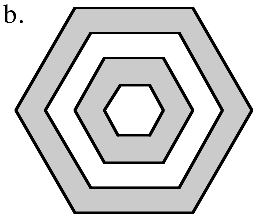
10. If it is possible, give an example of two triangles such that each pair of matching angles is the same size, but the triangles are not congruent. Must the two triangles be similar? Explain.

11. Discuss each diagram or description with transformations, symmetries, and tessellations in mind.

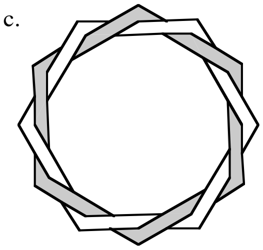
    a.



b.



c.



d. any wallpaper design that is handy

12. Draw two quadrilaterals to show that having all four pairs of corresponding angles the same size is not enough to ensure that the quadrilaterals are similar.
13. *Pattern Blocks* (See Appendix G if you do not have access to Pattern Blocks themselves.) Show patterns that would have different kinds of symmetry. (Don't forget translational and glide-reflection symmetries.)

22.6 Check Yourself

You should be able to work problems like those assigned and to meet the following objectives.

1. Name the four types of isometries in the plane and, given a figure and its image, identify which isometry is involved.
2. Locate the image of a point or figure for given isometry(ies), by using rough but accurate sketches (using key relationships as well as visualization). You may be directed by your instructor to use tracing paper, dot paper, MIRA, or compass-straightedge constructions.
3. State and apply key facts about each type of isometry.
4. Given a figure and its image, identify the type of isometry involved, perhaps using orientation as an aid. In some cases, you may then be asked to give a full description for the transformation. (For example, if a translation is involved, what is its vector?)
5. Tell what a glide-reflection is.
6. Illustrate what *the composition of two motions* means, and find the (final) image of a given figure for a composition of given transformations.
7. Recognize symmetries, including translation symmetries and glide-reflection symmetries, of given figures, and draw a figure that has translation and/or glide-reflection symmetry.
8. Illustrate and give essential features of *similar figures*.

Reference

[i]From Burns, M., & Tank, B. (1988). *A collection of math lessons: From Grade 1 through 3*. Math Solutions Publications.

Chapter 23

Measurement Basics

Of all the occurrences of numbers in daily life, most of them will involve measurements, particularly if you include monetary values. If counts are also included as measurements (as they are in advanced mathematics), virtually all uses of numbers involve measurements. Many types of measurements share certain characteristics or key ideas. We identify these key ideas of measurement in Section 23.1. Although we have regarded length and angle size as familiar concepts in earlier chapters, we will also look at them from the key-ideas-of-measurement viewpoint in Section 23.2.

The length of the treatment of measurement reflects what we believe to be its importance in a teacher's understanding. SEE INSTRUCTOR NOTE 23.1A for suggestions on how to begin this chapter.

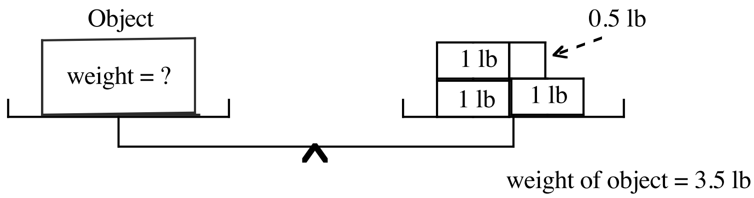
23.1 Key Ideas of Measurement

A given object has several characteristics, many of which we can quantify. For example, a particular woman has an age, a height, a shoe size, a blood pressure, a hair color, a certain amount of self-discipline, some degree of athleticism or degree of friendliness, and many other characteristics. Some characteristics are relatively easy to quantify, whereas others are not.

Equipment: Rulers (metric and English); protractor (chalkboard version if available, although a transparent one works well with an overhead projector if you can keep your hand/body out of the way).

Finding the values of quantities associated with characteristics of objects or events often involves measurement. The term *measurement* may refer to the *process* of finding the value of a quantity, or it may refer to the *result* of that process. For instance, the measurement of the weight of an object might refer to how we measured the weight (for example, with a pan balance) or to the resulting value (for example, 18 pounds). This section covers four key features of the usual kinds of measurement, which are summarized in the Take-Away Message at the end of the section.

First our focus is on the process of measurement**,** particularly *direct measurement*. The **process of direct measurement** of a characteristic involves matching that characteristic of the object by using (often repeatedly) a **unit**, which is a different object having the same characteristic. The unknown weight of an object, for example, might be determined on a pan balance by directly matching it with copies or partial copies of a known unit weight. The measurement (the result) is the number of units needed to match the measured object with respect to the characteristic being measured. The measurement is the value of the quantity.



Measuring devices like the pan balance are often used for measuring weight, although one could roughly measure the unknown weight even more directly by hefting the object in one hand and then adding copies of the unit weight in the other hand until the weight in each hand felt the same. **Indirect measurements** of quantities involve mathematical or scientific principles, such as in determining a weight with the usual bathroom scale or a scale with a spring that compresses or stretches predictably according to Hooke's law.

SEE INSTRUCTOR NOTE 23.1B for some points to bring out for Discussion 1 and the metric system.

Each of "English," "British," "regular," "customary," or "traditional" is used to describe the system (foot, pound, etc.) we inherited from England.

Some of this information on measuring quantities is repeated from Part I: *Reasoning about Numbers and Quantities*.

Discussion 2. Influences...A global economy necessitates much greater standardization. It is neither economical nor practical (production, storage, inventory, display, etc.) to maintain two systems of measurement when one will do.

Discussion 1 Why Standard Units?

1. We usually use **standard units** when giving values of quantities. For example, the English system includes the ounce, pound, ton, inch, yard, mile, and so forth, and the metric system uses the gram, kilogram, metric ton, centimeter, meter, kilometer, and so forth. Why are standard units used?
2. Why are there so many different units for the same characteristic, even within one system of measurement? For example, the English system has the units ounce, pound, and ton for measuring weight.
3. Elementary school curricula often start measuring work with nonstandard units. For example, lengths might be measured with crayons or markers. Why? (Keep in mind the process of measurement.)

Most of the world uses the **metric system**, or **SI** (from *Le Système International d'Unités*). Thus it is surprising that the United States, as a large industrial nation, has clung to the English system. Although the general public has not always responded favorably to governmental efforts to mandate the metric system, international trade is forcing us to be knowledgeable about, and to use, the metric system. Some of the largest U.S. industries have been the first to convert to the metric system from the English system.

Discussion 2 Influences on the Choice of Standard Units

Why has the U.S. public resisted adopting wholeheartedly the metric system? Why do international trade and alliances influence our measurements?

Scientists have long worked almost exclusively in metric units. Part of the reason is that virtually the entire world uses the metric system, but the main reason is how sensible and systematic the metric system is. A basic metric unit is carefully defined (for the sake of permanence and later reproducibility), and larger units and smaller subunits are related to each other in a consistent fashion (making the system easy to work in). In contrast, the English system's units are neither well defined (historically) nor consistently related. For example, the length unit *foot* possibly evolved from the lengths of human feet and is related to other length units in an inconsistent manner: 1 foot = 12 inches; 1 yard = 3 feet; 1 rod = 16.5 feet; 1 mile = 5280 feet; 1 furlong = $\frac{1}{8}$ mile; 1 fathom = 6 feet. Quick!—how many rods are in a mile? A comparable question in the metric system is just a matter of adjusting a decimal point.

The basic SI unit for length (or its synonyms) is the **meter**. (The official SI spelling is **metre**. You occasionally see *metre* in U.S. books.) The meter is too long to show with a line segment here, but two subunits fit easily and illustrate a key feature of the metric system: units smaller and larger than the basic unit are multiples or submultiples of powers of 10.

1 decimeter (0.1 meter)

1 centimeter
(0.01 meter)

Printing processes sometimes distort these lengths, so you should check them.

Furthermore, these subunits have names, such as *decimeter* and *centimeter*, and are consistently formed by putting a prefix on the word for the basic unit. The prefix *deci-* means 0.1, so *decimeter* means 0.1 meter. Similarly, *centi-* means 0.01, so *centimeter* means 0.01 meter. You have probably heard the word *kilometer*. The prefix *kilo-* means 1000, so *kilometer* means 1000 meters. Reversing your thinking, you can see that 10 decimeters are in 1 meter, 100 centimeters are in 1 meter, and 1 meter is 0.001 kilometer.

If you have access to base ten materials, you might check them out to see whether they could be used as visuals for centimeters and decimeters.

Two asides are in order. First, the small cubes in base ten materials often have edges of length 1 centimeter, and the longs are often 1 decimeter long. A meter stick is, of course, 1 meter long, which is slightly more than 1 yard. Second, for pronunciation of units in SI, the United States Metric Association advocates, for consistency, that the accent be on the first syllable: KILL-a-meter, not ki-LOM-ater. (Does anyone pronounce *kilogram* as ki-LOG-ram?) Common usage and some dictionaries, however, do use the non-preferred pronunciation.

An important advantage of SI is the consistent combination of the symbols for the basic units and the prefixes. In the case of the characteric length, the use of the symbol m (for meter) along with a symbol for a prefix (such as d for deci-, c for centi-, and so on) allows us to report a length measurement quite concisely: for example, 18 cm or 1.8 dm. Many of the SI prefixes are given in Table 1; more are given in the Glossary. You should note that the symbols for the metric units do not end with periods (a convention which is also true for most of the unit abbreviations in the English system), and the same symbols are used for singular and plural. For example, m stands for meter and meters, and yd stands for yard and

Different style manuals may differ on the use of periods and the formation of plurals.

SI has prefixes for even larger and smaller units:
yotta-, Y, 10^{24}
zetta-, Z, 10^{21}
exa-, E, 10^{18}
peta-, P, 10^{15}
and
pico-, p, 10^{-12}
femto-, f, 10^{-15}
atto-, a, 10^{-18}
zepto-, z, 10^{-21}
yocto-, y, 10^{-24}.
See the Glossary.

www.physics.nist.gov is a government website with lots of SI information.

yards. In the English system the exception for not using a period is the abbreviation for inch (in.).

Metric Prefix	Metric Symbol	Meaning of Prefix	Applied to Length
tera-	T	1 000 000 000 000 or 10^{12}	Tm
giga-	G	1 000 000 000 or 10^{9}	Gm
mega-	M	1 000 000 or 10^{6}	Mm
kilo-	k	1 000 or 10^{3}	km
hecto-	h	100 or 10^{2}	hm
deka-	da	10 or 10^{1}	dam
(no prefix)		(basic unit) 1 or 10^{0}	m
deci-	d	0.1 or 10^{-1}	dm
centi-	c	0.01 or 10^{-2}	cm
milli-	m	0.001 or 10^{-3}	mm
micro-	µ	0.000001 or 10^{-6}	µm
nano-	n	0.000000001 or 10^{-9}	nm

Table 1. Metric prefixes

If you are new to the metric system, your first job will be to master the prefixes so you can apply them to the basic units for other characteristics. You may know a mnemonic for several of the metric prefixes. For example, the phrase "*k*ing *h*enry *da*nced, *d*rinking *c*hocolate *m*ilk" (although not using good capitalization) does give the most common metric prefixes. The table above shows some of the other metric prefixes. For example, *kilo-* means 1000, so 1 kilometer means 1000 meters; the symbol k for *kilo-* combined with the symbol m for *meter* give the symbol km for *kilometer*.

Here is one final note on SI: Commas are not used in writing numbers. Where we write commas in multidigit numbers, SI would have us put spaces. So 13,438 m would be written as 13 438 m, and 0.84297 km would be written as 0.842 97 km. Some countries use the comma where we use the decimal point, so some agreement is necessary to avoid potential confusions in international trade and other communication. This SI recommendation is not observed in all U.S. elementary school textbooks, however.

Activity 1 It's All in the Unit

1. Measure the width of your desk or table in decimeters. Express that length in centimeters, millimeters, meters, and kilometers.
2. Measure the width of your desk or table in feet. Express that length in inches, yards, and miles.

3. In which system are conversions easier? Explain why.

The next discussion introduces another key idea of measurement.

This activity requires measuring equipment for both the metric and English systems.

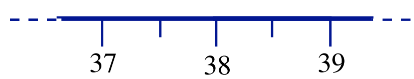
Discussion 3 Are Measurements Exact?

Only one of these statements can be true. Determine which one is true, and explain why is it the only one.

a. "She weighed exactly 148 pounds."
b. "She is exactly 165 centimeters tall."
c. "It is exactly 93,000,000 miles to the sun."
d. "There were exactly 5 people at the meeting."

SEE INSTRUCTOR NOTE 23.1C about the approximate nature of most measurements.

Because actual measurements (except for counts) are only approximations, a given value should not be read to imply that it is more nearly exact than is reasonable. For example, the value 38 pounds implies that the measuring was done only *to the nearest pound*. Thus a reported "38 pounds" might describe a weight that is actually greater than or equal to $37\frac{1}{2}$ pounds and less than $38\frac{1}{2}$ pounds. Deducing that the value 38 pounds is exactly 38×16 = 608 ounces would be risky. In actuality, the object could be as light as 600 ounces ($37\frac{1}{2} \times 16$) or as heavy as just under 616 ounces ($38\frac{1}{2} \times 16$).



Discussion 4 Interpreting a Measurement Given To the Nearest

Discussion 4. Interpreting a Measurement…: Students often need help in interpreting such reports. Encourage the making a drawing of a ruler as an aid.

Suppose a length is reported as $2\frac{3}{4}$ inches, to the nearest $\frac{1}{4}$ inch.

Make a drawing of part of a ruler and argue that the length could be as short as $2\frac{5}{8}$ inches or as long as just under $2\frac{7}{8}$ inches.

Unfortunately, much school work with measurement treats the values as exact, so students can easily form the erroneous impression that the values *are* exact. But just as the ideal constructs of *line segment* and *angle* can be discussed, theoretically perfect measurements can also be discussed. For example, we can *imagine* a square with sides exactly 2 centimeters long, although drawing one even with a finely sharpened pencil is not possible. Likewise, we can *imagine* a line segment with length exactly 2 cm although we cannot actually produce one.

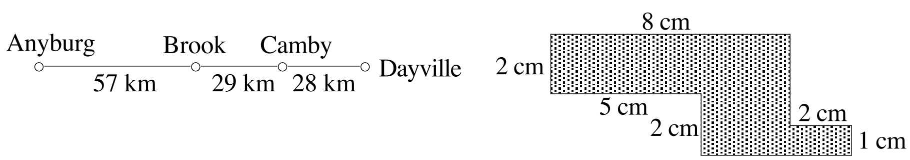
Which of these two ways would give a better measurement for the length of a board: measuring to the nearest inch or measuring to the nearest foot? In a direct measurement, choosing the smaller unit naturally allows the matching to be done more closely. So, usually a measurement with a smaller unit narrows the range of possible values for the measurement and lessens the error.

The numerical part of the value, however, can also imply something about the precision of the measurement. Suppose you have these measurements, all reported for the length of the same object: 2.75 feet, 2 feet 9 inches, and 33.0 inches. Which measurement is most precise, or are all three the same? At first glance, they seem to be exactly the same. But a measurement given as, say, 2.75 feet is implying that the measurement was carried out to the nearest one-hundredth of a foot, so it would be more precise than a measurement reported as 2 feet, 9 inches, which implies that the measurement was carried out only to the nearest inch (or one-twelfth of a foot). On the other hand, 33.0 inches would imply that the measurement was accurate to the nearest one-tenth of an inch (or $\frac{1}{120}$ of a foot). So, 33.0 inches would be the best approximation of the measurements: 2.75 feet, 2 feet 9 inches, and 33.0 inches.

Discussion 5. Measuring Piecemeal... Measuring an object by "cutting" it into separate pieces, measuring each piece, and then adding those measurements is another key idea, often used with length, area, and volume. You may know that (countably) infinitely many pieces can be used, but we emphasize only finite additivity here.

Discussion 5 Measuring Piecemeal

How can you find the distance from Anyburg to Dayville and the area of the z-shaped region?



Your method in Discussion 5 likely involved another key idea of measurement: You can measure an object by thinking of the object as cut into pieces, measuring each piece, and then adding those measurements.

Take-Away Message…Here is a summary of four key ideas of measurement.

1. **(Direct) measurement of a characteristic of an object involves matching the object with a unit, or copies of a unit, having that characteristic. The matching should be based on the characteristic. The number of units, along with the unit, give the measurement, or the value, of the quantity associated with the characteristic.**
2. **Standard units are used because they are relatively permanent and enable communication over time and distance. Nonstandard units may be used in schools to allow a focus on the process of measurement.**
3. **Measurements are approximate. Smaller units give better approximations, although the numerical part of a reported value can have a bearing on the implied accuracy as well. (If we include counts as measurements, however, they can be exact.)**
4. **A quantity can be measured by thinking of the object being measured as "cut" into a (finite) number of pieces, measuring the quantity in each piece, and then adding up those measurements or values.** ♦

Key idea 4 is re-visited in Section 26.2. For example, the overall speed for a trip cannot be determined by thinking of the speed for the first part, and adding that to the speed for the second part.

Learning Exercises for Section 23.1

Overall, the focus is on metric units, but there are some review items for English units, e.g., #13, #22, #24.

Students do not have answers or hints for 1e-i, 5ab, 6b, 7b, 8ad, 9e-h, 10c, 15, 16e-i,18cd, and 21.

1. Some quantities that are commonly measured include length, area, volume, weight, speed, time, temperature, intelligence, student achievement, force.

   Which of the above quantities are you measuring when you measure the following quantities?

   a. how far two people are apart
   b. how much soda is in a glass
   c. how warm is the lake water
   d. how heavy is a football is
   e. how much room is left to draw on, on a piece of paper
   f. how fast you can run
   g. how long it takes to walk to school
   h. how large is a football field
   i. how much carpet is needed for a room
   j. how effective is a teacher

2. What are some possible units for the quantities listed in Learning Exercise 1?

3. What characteristics of a child could be measured with the given device?

   a. a thermometer b. the child's math book c. a stop-watch

4. For which quantities could each term or phrase be a unit? (A dictionary can be useful for some items.)

a. mm of mercury	b. number per square mile	c. acre
d. quire	e. hectare	f. cubit
g. candela	h. stone	i. coulomb
j. rating point (TV)	k. decibel	l. Scoville
m. body mass index (BMI)	n. section	

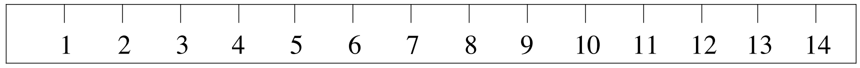
4d. quire: a set of 24 (or sometimes 25) uniform sheets of paper.
l. Scoville units are used to measure the hotness of peppers.

5. Why would each of the following items *not* be satisfactory as a unit for the given quantity?

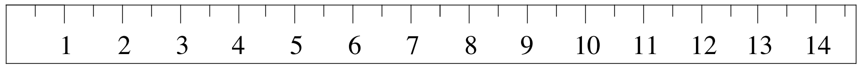
   a. a rubber band, for length
   b. an ice cube, for weight
   c. a garage-sale item, for monetary value
   d. a person's judgment, for temperature
   e. a pinch of salt, for saltiness
   f. a sip, for amount of drink

6. Answer the following items about reports of measurements.
   a. Explain why this claim cannot be accurate: “The fish I caught was exactly 18 inches long!”
   b. “The fish weighed 123 pounds.” What weights for the fish are possible and still have the claim be a true statement?

7. a. Show the shortest and longest lengths that could be reported as 7 units using the ruler.



b. Show the shortest and longest lengths that could be reported as 7.5 units using this ruler.



8. One kind of bathroom scale shows weights to the nearest half-pound.

a. On consecutive days, a person's readings were 151.0 and 151.0 on the scale. Did the person weigh exactly the same amount each day? Explain.

b. What weights would all give a reading of 162.0 on the scale?

c. What weights would all give a reading of 128.5 on the scale?

d. How much "off" could a reading of 116.0 be, on the scale? A reading of 223.5?

9. In parts (a)-(h) tell without calculating whether x is greater than 2000, equal to 2000, or less than 2000. Explain your thinking.

a. x raisins weigh 2000 pounds
b. 2000 raisins weigh x pounds
c. x yards = 2000 miles
d. 2000 yards = x miles
e. x kilometers = 2000 meters
f. 2000 kilometers = x meters
g. x tons = 2000 pounds
h. 2000 tons = x pounds

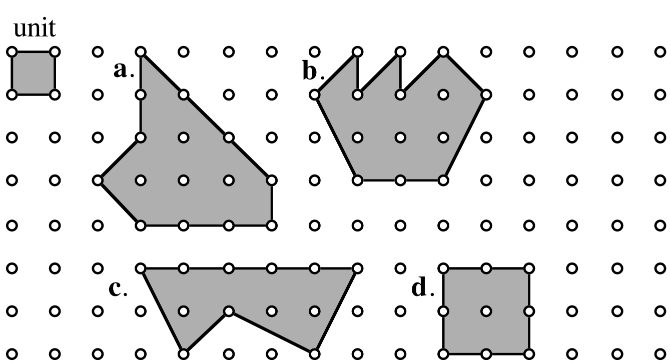
10. In parts (a)-(c), tell which of the made-up units is larger. Tell how you know.

a. 150 ags = 100 aps
b. 260 bas = 1.31×10^3 bos
c. 0.19 cons = 0.095 cins

11. Tell how one of the four key ideas of measurement could be used in determining the weight of a wiggly puppy.

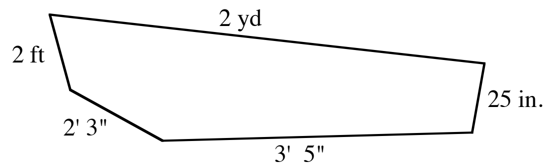
12. Give the number of unit square regions there are in each figure. Explain your reasoning, especially how you used key ideas of measurement.

12bc. Student may get correct answers by "eye-balling," but the better technique of pieces-of-rectangles comes up in Section 24.1.



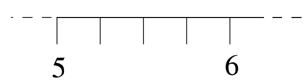
e. Let the unit region be the region shown in Figure (d). What would be the values for the number of units in Figures (a), (b), and (c)? Explain your reasoning.

13. Sections of low picket fence come ready-made and cost $2.89 for a section 27 inches long. How much would it cost to put a fence along the borders of a flower patch shaped like the drawing below? (What key idea or ideas of measurement did you use?)



14. a. Add these lengths: $5\frac{3}{4}$ inches, $7\frac{7}{8}$ inches, and $3\frac{1}{2}$ inches.

b. Make pieces of rulers (like the one below for $5\frac{3}{4}$ inches) for each measurement given in part (a). Then show where the shortest and the longest lengths that could be described as $5\frac{3}{4}$ inches (or $7\frac{7}{8}$ inches or $3\frac{1}{2}$ inches) would end.



c. Add the *least* lengths that each of the measurements in part (a) could be describing.

d. Add the *greatest* lengths that each of the measurements in part (a) could be describing.

e. What is the range of values possible for the sum of the measurements in part (a)?

15. One teacher's gradebook computer program shows scores to the nearest whole point but keeps several decimal places in its memory. Use that information to explain why the gradebook program might give a sum of 241 for scores of 82, 73, and 87.

16. Think of the sizes of the units (think metric) to help you complete the following equalities of measurements.

a. 76.3 cm = _____ m	b. 2.7 m = _____ cm
c. 19 mm = _____ cm	d. 4.62 cm = _____ mm
e. 0.62 km = _____ m	f. 108 m = _____ km
g. 8.7 dm = _____ cm	h. 29 cm = _____ dm

i. A person 155 cm tall is _____ meters tall.

17. For some purposes, 1 second (1 s) of time is too large a unit. Fast computers, for example, need nanoseconds to describe times. Review the metric prefixes and complete the following.

a. ___ ms = 1.3 s	b. 143 ns = ___ s
c. 500 ms = ___ s	d. 2500 μs = ___ s

For Exercise 18, if you have access to 1 kg and 1 g masses, have the students pass them around. Most non-science students do not have much sense of the sizes of those units, and can gain a greater appreciation of, say, 1 mg from having handled 1 g.

18. For historical reasons, the kilogram, rather than the gram, is the metric system's basic unit for mass. A raisin has a mass of about 1 gram (1 g). Complete the following conversions.

    a. 153.2 g = ___ kg   b. 3.4 g = ____ mg

    c. 2.17 kg = ____ g   d. 56 mg = ____ g

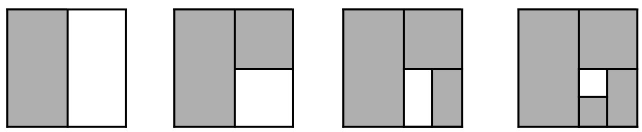
19. A map has the scale 1 000 000 : 1.

    a. In reality, how far apart in kilometers are two cities that are 8 cm apart on the map?

    b. Two locations that are actually 150 kilometers apart should be how far apart on the map?

20. Sometimes one can measure something by cutting it into an *infinite* number of pieces and then adding the measurements of all the pieces. What will the following infinite sum equal?

    $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots = ?$ (*Hint*: Use the diagrams for an idea.)



and so on

21. When the French were designing the metric system in the late 1700s, they invited other countries to participate. England said that "reform of weights and measures was considered 'almost impracticable'".[i] Why would anyone think that?

22. Decide whether each calculation is correct. Correct any calculation that is not correct. (*Recall*: 1 pound (lb) = 16 ounces (oz) and 1 gallon = 4 quarts.)

    a. $\begin{array}{r} \overset{4}{\not 5}\,.\,{}^{1}05 \text{ m} \\ -\ \underline{3\ .\ 52 \text{ m}} \\ 1\ .\ 53 \text{ m} \end{array}$

    b. $\begin{array}{r} \overset{4}{\not 5} : {}^{1}05 \text{ p.m.} \\ -\underline{3 :\ 5\ 2 \text{ p.m.}} \\ 1 :\ 5\ 3 \end{array}$

    c. $\begin{array}{r} \overset{5}{\not 6} \text{ lb } {}^{1}3 \text{ oz} \\ -\ \underline{2 \text{ lb } \ 8 \text{ oz}} \\ 3 \text{ lb } \ 5 \text{ oz} \end{array}$

    d. $\begin{array}{r} \overset{5}{\not 6}\,.\,{}^{1}3 \text{ kg} \\ \underline{-\,2\ .\ 8 \text{ kg}} \\ 3\ .\ 5 \text{ kg} \end{array}$

    e. $\begin{array}{r} \overset{3}{\not 4} \text{ gal } {}^{1}1 \text{ qt} \\ \underline{-\qquad 9 \text{ qt}} \\ 3 \text{ gal } \ 2 \text{ qt} \end{array}$

    f. $\begin{array}{r} \overset{1}{2} \text{ hr } \overset{1}{3}5 \text{ min} \\ \underline{+\qquad 75 \text{ min}} \\ 3 \text{ hr } \ 10 \text{ min} \end{array}$

23. The following lists examples of nonmetric length units used in England at different times or in different settings:

    cubit, rod, ell, fathom, foot, furlong, hand, inch, knot, mile, pace, yard.

    With those units in mind, describe one advantage of using the metric system approach to length units over the English system.

24. In earlier times in England, the following were units for measuring volume:[i]

2 mouthfuls	= 1 jigger
2 jiggers	= 1 jackpot
2 jackpots	= 1 gill
2 gills	= 1 cup
2 cups	= 1 pint
2 pints	= 1 quart
2 quarts	= 1 pottle
2 pottles	= 1 gallon
2 gallons	= 1 peck
2 pecks	= 1 half-bushel
2 half-bushels	= 1 bushel
2 bushels	= 1 cask
2 casks	= 1 barrel
2 barrels	= 1 hogshead
2 hogsheads	= 1 pipe
2 pipes	= 1 tun

a. How many cups are in a gallon?
b. Express each unit in terms of quarts.
c. What feature of this system of volume units is somewhat like those of the metric system? What feature is different?

25. In 1893 the U.S. yard unit was defined to be equal to $\frac{3600}{3937}$ meter. Then in 1959 it was defined as 0.9144 meter. By how much did the new foot differ from the old foot?

26. A search engine should show whether these are still active web sites.

26. (*computer*) Two Websites that focus on measurement are given. Browse the sites to see what they offer.
a. U. S. Metric Association–http://lamar.ColoState.edu/~hillger/
b. National Institute of Standards and Technology (until 1988, the National Bureau of Standards)–http://nist.gov. You might settle on http://physics.nist.gov/GenInt/contents.html.

Section 23.2 Some students may need help in using a protractor (see Appendix F) and occasionally there will be a student who has trouble reading a ruler, especially one marked in English units.

Equipment needed: Rulers (metric and English); protractor, (chalkboard version if available, although a transparent one works well with an overhead projector is you can keep your hand/body out of the way), trundle wheel if available.

23.2 Length and Angle Size

Although we have treated length and angle size as familiar, they can also be looked at from the viewpoint of the key ideas of measurement. The basic classroom tools for measuring length and angle size are the ruler and the protractor. Children often use (simple) rulers as early as the first grade; they learn about protractors later, perhaps in Grade 5.

There are many learning exercises for this chapter. You may choose to do some as in-class activities.

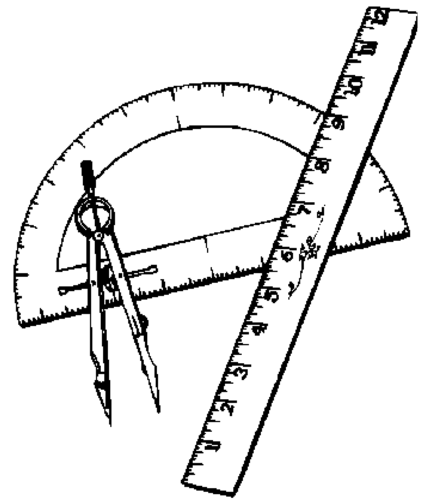
It is probably worth mentioning that although length words are occasionally used to refer to time ("How long will it take?" "How lengthy will the session be?"), length and time are two quite different quantities.

Height (pronounced "hite", not "hite-th"), width, depth, girth, thickness, perimeter, circumference, distance, base, altitude, radius, diameter,... are possibilities. (One might argue that the last four refer to segments.) Students may suggest area or volume units, so think about how you (or the class) might react to such responses.

Throughout this chapter encourage students to establish personal benchmarks for common units of measure.

Length

We speak of the length of a piece of wire or a rectangle in two ways. The term *length* might refer to the quality or attribute we are focusing on, or it might refer to the measurement of that quality. The context usually makes clear which reference is intended.



However, what might be puzzling to elementary school students is that there are several terms, all of which refer to the *length* attribute of different objects.

Discussion 6 Aliases for Length

Height and *width* are words that refer to lengths. What other terms refer to the same attribute as length does?

All the key ideas of measurement from Section 23.1 are useful in working with length. For example, early work in elementary grades often involves nonstandard units—paper clips, chains of plastic links, or pencils.

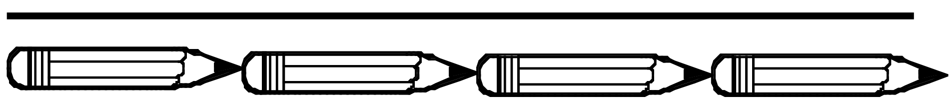


Figure 1

You should keep the approximate nature of measurements in mind. The heavy line segment shown in Figure 1 is about 4 "pencils" long, because four copies of the (same) pencil unit just about match the length of the line segment. If we measured to the nearest *half*-pencil, the measurement here would still be 4 pencils, and so we could say "4 pencils, to the nearest half-pencil" to communicate the greater precision.

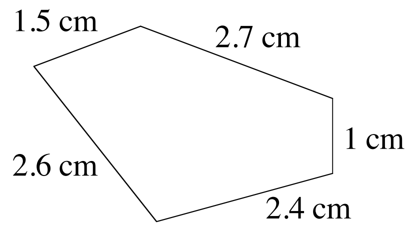
The meter and the metric prefixes from the metric system give a variety of length units. If you have never established personal comparisons, or benchmarks, for some of the metric units, you should begin to do so. What does approximately 1 millimeter, 1 centimeter, 1 decimeter, 1 meter, and 1 kilometer mean to you? For many people, the width of a particular fingernail is a good benchmark for 1 cm.

1 decimeter (0.1 meter)

1 centimeter
(0.01 meter)

Estimating measurements is a very practical skill that is often neglected in school. Having benchmarks for units can help in estimating, say, the width of a room in meters or the perimeter of a piece of paper in decimeters.

The length of a polygon is usually called its **perimeter** (*peri-* means around; *metron* means measure). (Recall that a polygon is made up of just the line segments.)

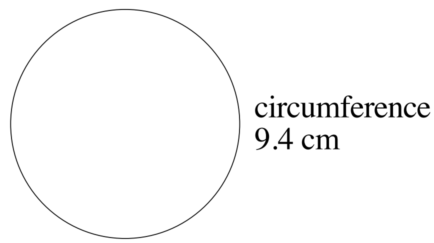


perimeter = 10.2 cm

Think About…What key idea of measurement is often involved in finding the perimeter of a polygon?

Think About… Measure a whole by measuring its parts, and adding those measurements.

The length of a circle is usually called its **circumference** (*circum-* means around; *ferre* means to carry). In everyday language, both *perimeter* and *circumference* may be used to refer to the sets of points, as well as the length measurement.



Measuring the lengths of crooked curves presents difficulties. With short curves we can do the practical thing: Place a flexible measuring tape or a piece of string closely on the curve and then straighten out the tape or string and measure it. There are also small, wheeled devices that can be used to trace over a route on a map to find measurements of lengths. You may have seen an auto accident scene where a trundle wheel (a device made of a wheel) was used to measure distances.

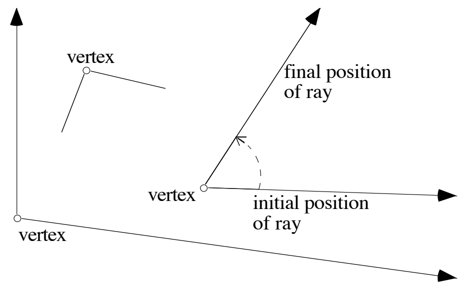
You may have available a trundle wheel for measuring; these are not unusual in elementary schools.

One way to find lengths of curves that pays off is to approximate the curve by line segments and then observe what happens each time as smaller and smaller line segments are used. As all the segments get smaller and smaller, getting as close to zero length as you arbitrarily want, the sum of the lengths of all the line segments will get closer and closer to the (ideal) exact length of the curve. (Not only is this process plausible, it is also applied in the study of calculus.)

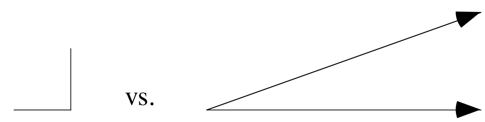
Figure 2

In Figure 2, the shorter segments in the drawing on the right should give a better approximation of the length of the curve than the segments in the drawing on the left.

Angles and Angle Size



Let us turn our attention to angles and their measurement. As you may know, angles can be viewed in two different ways—(1) statically, as two rays or segments (the **sides** of the angle) starting from the same point (the **vertex**) or (2) more dynamically, as the result of a ray rotating from an initial position to a final position. The second way allows you to speak of measuring angles in a clockwise or counter-clockwise direction.



Which of the two angles shown to the right is larger? The size of an angle is determined by the opening, or the amount of rotation, or turn, involved in the angle. So the first angle has greater size. The visible length of the sides of the angle is irrelevant.

The key ideas of measurement also apply to the measurement of angle size, of course. The standard unit used in the K–8 curriculum is the **degree** (°). Across the world, societies that were advanced enough to study astronomy, such as the Mesopotamians (in what is now Iraq) and the Mayans (in what is now Mexico and Central America), knew that there were about 365 days in a solar year, and they settled on 360 as a good number with which to define a unit for angle size. Hence, a full turn of a ray about its vertex goes through 360° ("a circle has 360°"). Choosing 360˚ for a full turn means that the degree is a fairly small unit. However, for angles used in ocean navigation or astronomy, even a degree is too large a unit. To obtain smaller units, each degree can be divided into 60 equal pieces, with each piece called a **minute** ('). Each minute can then be further divided into 60 **seconds** ("). (Notice that *minute* and *second* are the same words and relationships used with time, but here are about angle size.) The metric system recognizes the degree as a unit, but the more official SI unit is called the radian (about 57.3˚). Radians do not usually appear until high school trigonometry, where assigning positive and negative signs to angle measurements to indicate counter-clockwise and clockwise turns, respectively, also appears.

1 radian = ($360/2\pi$) degrees

Because the degree is a relatively small unit, you could use a nonstandard unit larger than a degree, as illustrated in Figure 3, for introductory classroom work with angle-size measurement. In the drawing the size of angle X is approximately $4\frac{1}{2}$ units, and we could write $m(\angle X) = 4\frac{1}{2}\,units$, where the letter *m* stands for the measurement of the angle. Indeed, rather than saying *size of the angle*, we could say *measure of the angle*.

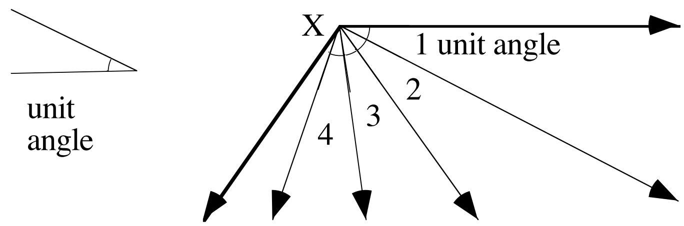
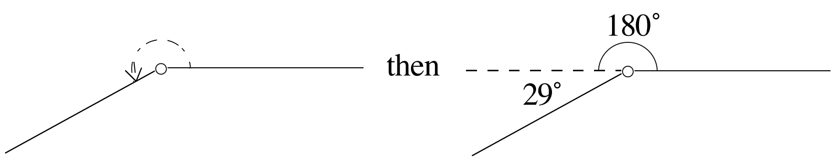


Figure 3

As you know, the usual classroom protractor shows 180°, the size of a half-turn. If you wish to measure an angle whose size is greater than 180°—for example, the counter-clockwise angle shown in the following sketch—you might think of the given angle as divided into two angles: the 180° angle and a smaller angle. Measuring the smaller angle with the protractor then enables you to tell the size of the whole angle, which is 209°.



Think About...

Do you see another way to use the protractor to get the size of the counter-clockwise angle?

Measure the obtuse angle, and subtract from 360˚.

You already know some important results about angle measurements. For example, the sizes of two supplementary angles add to 180˚. And the sum of the sizes of the angles in any triangle is 180˚. This last result leads to the more general result for *n*-gons: The sum of the sizes of the angles of an *n*-gon is $(n - 2)180˚$. So, for example, the sizes of the 8 angles in any octagon add up to $(8 - 2)180˚$, or 1080˚.

Think About...

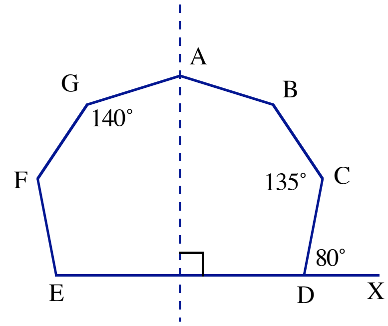
How could you show that each angle in an *equiangular* octagon has 135˚?

Think about... Exercise 27 pursues this $(n - 2)180/n$ line.

Several ideas about lengths or angle sizes can be involved in a single problem. For example, recall how corresponding lengths and angle sizes are related in congruent polygons and polyhedra or in similar polygons and polyhedra. With congruent polygons and polyhedra, corresponding lengths are equal, as are the sizes of corresponding angles. For similar polygons and polyhedra, corresponding angles are the same size, but every two corresponding sides give the same ratio (the scale factor) for their lengths.

EXAMPLE 1

The dashed line is a line of symmetry for heptagon ABCDEFG. Some angle sizes are given in the drawing. Find the sizes of the other angles of the heptagon. (*Recall*: The sum of the sizes of the angles of an *n*-gon is $(n-2)180^\circ$.)



SOLUTION Because the dashed line is a line of symmetry, $\angle B$ is the same size as $\angle G$, or 140˚. Similarly, $\angle F$ has size 135˚. Because the segment EDX is straight and $\angle CDX$ has size 80˚, $\angle D$ and its image $\angle E$ each has 100˚. The only angle unaccounted for is ∠A. All the angles of the heptagon will total (7 – 2)180˚, or 900˚. The angles already determined total 140 + 140 + 135 + 100 + 100 + 135 = 750˚, which leaves 150˚ for ∠A.

As with length, some ability at estimating angle size is often useful. Most people have a good mental picture of 90°, 180°, and 270° (if they have thought about the fact that 90˚ + 180˚ = 270°). Thinking of a half or a third of 90° (a right angle) can help in estimating angle size.

Take-Away Message…The four key ideas of measurement from Section 23.1 all apply to length and angle size. There are many words that refer to the same characteristic as does *length*, including *perimeter* and *circumference*. Angle sizes are usually given in degrees, with 360˚ in a full turn and 180˚ in a half-turn. Several results about angle sizes enable us to find unknown measurements from known ones. ♦

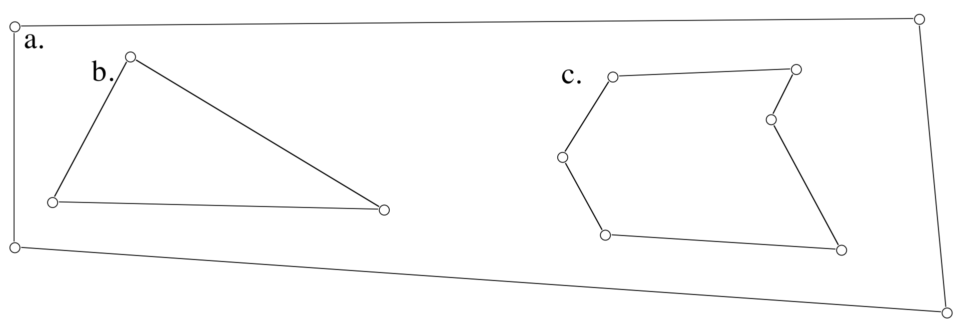
LE 23.2 You will have to pick-and-choose among the many exercises here.

Students do not have answers or hints for 2, 3, 5, 6a-g, 7c, 12cd, 14abc, 15, 16c-f, 18c, 21, 22b, 24, 27d, 39, 40, and 41a.

Some nice-to-know vocabulary words are introduced in the exercises:
22 (vertical angles);
23 (the angles with parallel lines);
36 (dihedral angle).

Learning Exercises for Section 23.2

1. Find the perimeter of each figure. (*Suggestion*: Use metric units.)



   d. Find the perimeter of each face of Shape A and Shape B from your kit of polyhedra.

2. Explain why the statement, "My brand-new pencil is exactly $7\frac{1}{2}$ inches long" is not completely true.

3. Show a segment that would measure 6 cm to the nearest centimeter but 6.5 cm to the nearest half-centimeter. How many possibilities are there?

4. A person has a broken ruler with the first 3 inches missing. However, she has measured several segments by noting where on the ruler the two end points of each segment were. Find the lengths as she did, using the following information about where the two endpoints were. Can you determine them mentally?

a. 4 inches and $7\frac{3}{4}$ inches   b. $5\frac{7}{8}$ inches and $9\frac{1}{4}$ inches

c. $6\frac{7}{16}$ inches and 10 inches

5. What is your personal benchmark for each length?

a. 1 millimeter   b. 1 centimeter   c. 1 decimeter

d. 1 meter   e. 1 kilometer

6. Estimate the length of each segment in metric units, and then measure each to check your estimating powers. (Notice that you can make up your own examples for further practice.)

a. ____

b. __________________________________

c. ___

d. ___________

e. the perimeter of the room you are in

f. the height of the ceiling in the room you are in

g. the length of a car you know

h. the diameter of a quarter

i. the height of this piece of paper

j. the perimeter of this piece of paper

6. Lengths should be: a. 10 cm; b. 7 cm; c. 0.7 cm; d. 2.3 cm

7. An imagined polygon has perimeter 24 cm with each side having a length that is a whole number of centimeters. For each given polygon, what are all the possibilities for the lengths of its sides?

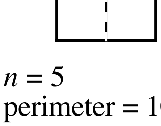
a. a quadrilateral   b. a rectangle   c. a pentagon

#7ac have many possibilities, so you may ask for fewer.

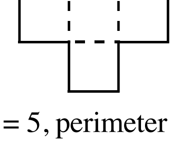
8. If n identical square regions are placed so that touching regions share a whole side, what is an expression for the *maximum* perimeter possible? Use the side of the square as the unit of length.

8. It is not at all obvious to many students that shapes with the same area (the number of squares) can have *different* perimeters.

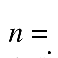
Sample trials



$n = 5$
perimeter = 10



$n = 5$, perimeter = 12



$n = 1$
perimeter = 4

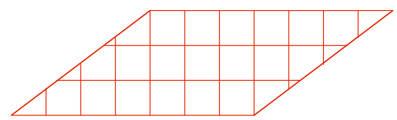
9. a. How does one find the shortest distance from a point to a line, as in the drawing to the left below? How does one find the distance between two parallel lines, as in the right below?

If you did compass and ruler constructions in Chapter 21, Exercise 9 gives a review opportunity. The "No" for 9b is assumed later.

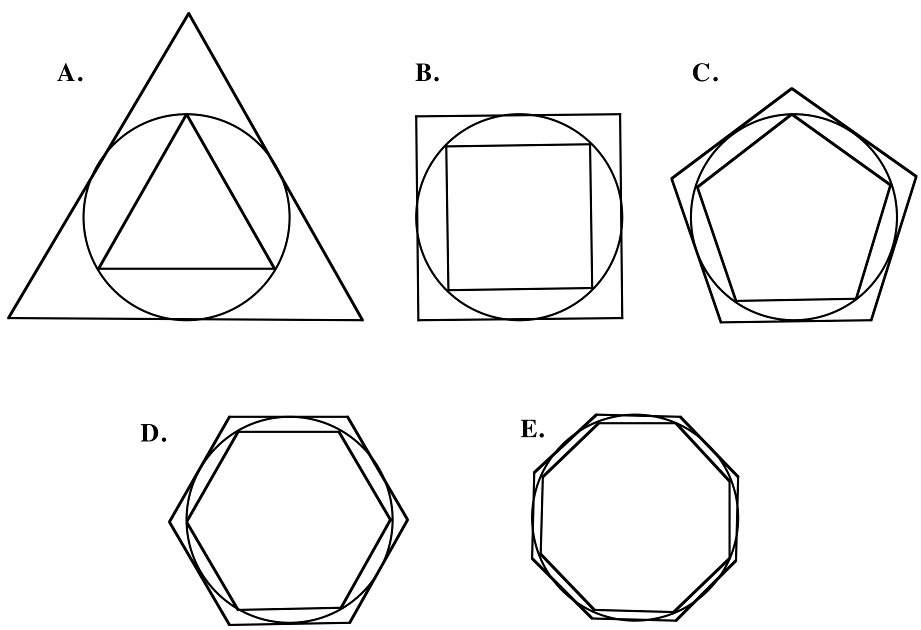
b. For parallel lines, does it matter *where* you measure the distance between them?

10. What *length* can you measure to find out how many (horizontal) rows of squares are in the following shape?

Exercise 10 is preparatory for the development of area formulas.



11. For the five figures below, the same size circle is shown with an inside polygon and an outside polygon. Check that the perimeters of the outside polygons are getting smaller and the perimeters of the inside polygons are getting larger as you go from Figure A to Figure E.



12. Complete each of the following measurement equalities.

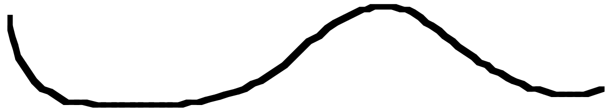
Looking forward, #39 has some conversion within the English system, for contrast to #12.

a. 1.2 km = ___ m　　b. 5.3 m = ___ cm

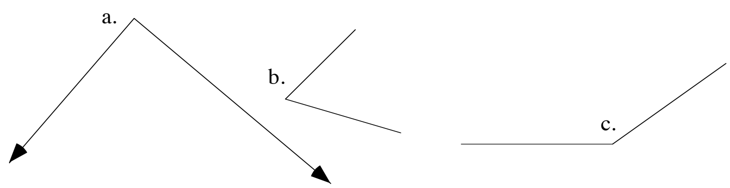
c. 62 mm = ___ cm　　d. 3.25 dm = ___ cm

e. We write "3.4 cm + 4.3 cm = 7.7 cm," but the sum could be as small as ___ cm.

13. Approximate the length of the curve first with segments 2 cm long, then with segments 1 cm long, and finally with segments 0.5 cm long. What would you estimate the length of the curve to be?



14. Estimate the size of each given angle. Then copy and measure to check your estimates. (*Note*: With some protractors, you may need to extend the sides of the angles. Also, as with line segments, you can make up your own angles for more estimation practice.)



d. Classify each of the angles in parts (a), (b), and (c) as being acute, right, obtuse, or straight.

15. a. How many degrees must you turn a doorknob to open a door? Estimate and then check.
    b. A hippopotamus can open its mouth about 130°. Sketch an angle to show your estimate of such an angle, and then check by measuring.

16. How many degrees does the minute hand of a clock go through for each given time span?

a. from 4:00 to 4:45	b. from noon to 12:35
c. from 1:00 to 1:05	d. from 8:48 to 9:17
e. from 2:00 to 3:00	f. from 2:00 to 3:30
g. from 6:36 to 8:19	h. in a whole day

17. How many seconds are in a full 360° turn? One second is what part of a full turn?

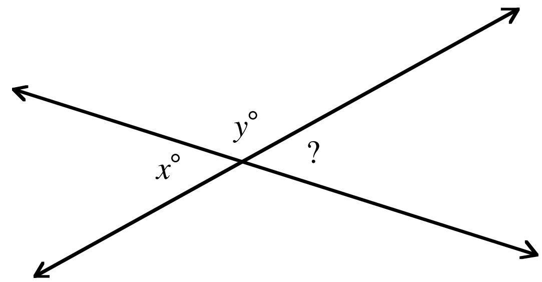
18. How many degrees of longitude separate locations with the following degrees of longitude?
    a. Location 1: 42° 16' 23" west longitude; location 2: 60° west longitude
    b. Location 3: 115° 43" west longitude; location 4: 68° 32' west longitude
    c. Location 5: 34° 52' west longitude; location 6: 19° 48' *east* longitude

19. Why are time zones roughly 15° of longitude wide?

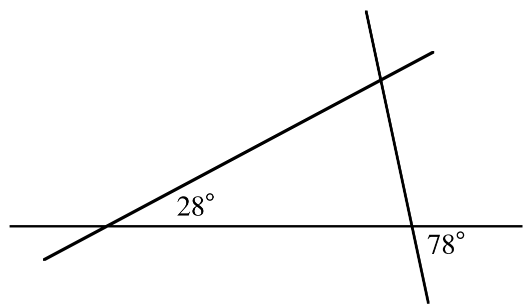
20. The equatorial circle is about 25,000 miles long. What is the approximate length, in miles, of a 1° arc on the equatorial circle? A 1 minute arc? A 1 second arc?

21. Sketch angles with the following sizes by estimating: 30˚, 45˚, and 150˚. Check your estimates by measuring your angles.

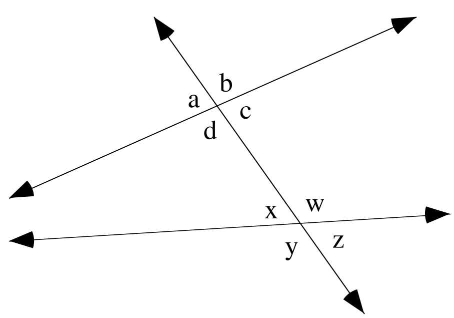
22. a. When two lines cross, they form pairs of angles that always have the same size. These angles are called **vertical angles** or **opposite angles**. Give a justification that a pair of vertical angles will always be the same size (without relying on checking a particular example or examples); that is, why must the ? angle in the drawing below have x°? (*Hint*: How is y related to the ? and x?)



b. Without using a protractor, give the sizes of all the angles with sizes not given in the drawing below.



23. a. If two lines cross, four angles are formed. If a third line crosses two other lines, eight new angles are formed and can be grouped into four pairs of **corresponding angles**, which are angles located in corresponding positions, such as b and w in the following diagram. Give the other three pairs of corresponding angles here.



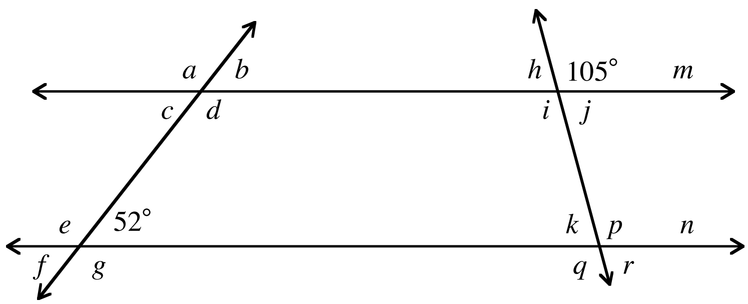
b. Are the two angles in a corresponding pair always the same size?

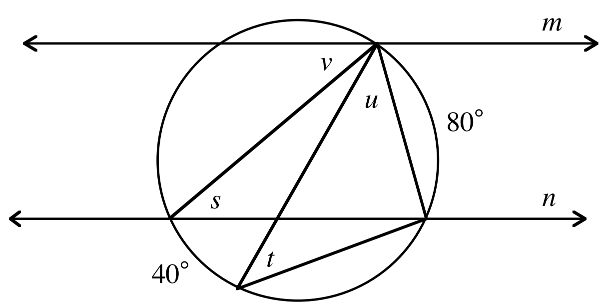
c. Suppose that the third line cuts two *parallel* lines. Experiment to see whether there is any relationship between the sizes of the two angles in a corresponding pair in this case. (It is easy to get parallel lines by using the lines on notebook paper.)

d. Angles such as d and w shown in the previous drawing are called **alternate interior angles.** Give another pair of alternate interior angles. How do they compare in size if the lines are parallel?

24. In the diagrams below, lines *m* and *n* are parallel. Give the sizes of the lettered angles.

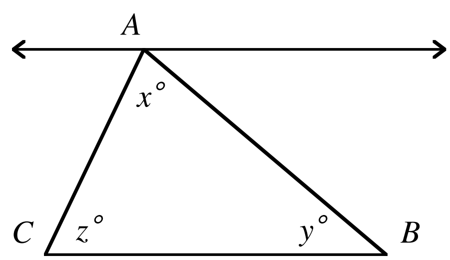
#24a-r,v and #25a are applications of #23. 24s-u use inscribed angles (Section 21.1).





25. You know that the sizes of the angles of a triangle sum to 180°. Yet, direct measurements of the angles of a triangle may be "off," even when carefully measured.

a. Give an argument for the sum being 180° based on using angles in parallel lines rather than direct measurement. Use the following diagram in stating your argument.



b-e. Give the missing angle size(s) in the triangles described.

b. Two angles have sizes 72° and 39°.

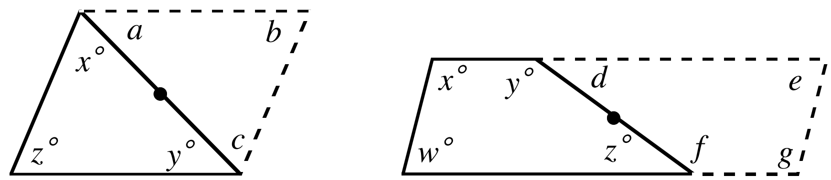
c. A right triangle has an angle of 41°.

d. A triangle has angles of 56° 39' and 62° 43'.

e. An isosceles triangle has an angle of 20°. (There are two possibilities.)

f. In every right triangle, what is the sum of the sizes of the acute angles?

26. In the following drawing, the triangle and the trapezoid are rotated 180° with the marked midpoints of sides as centers of the rotations. Give the measurements of angles (*a*)–(*g*), in terms of w, x, y, z.



27. Recall that the sum of all the (interior) angles of an *n*-gon is

$(n-2)180°$.

a. If the *n*-gon is regular or just equiangular, what is the size of each angle?

b. Make a table showing the sizes of the angles in each equiangular *n*-gon from $n = 3$ to $n = 12$.

27de require a modicum of algebraic skill, in solving, say, $(n - 2)180 = 4500$.

c. As the number of sides of a regular *n*-gon gets larger and larger, what can you say about the size of each angle?

d. The angle sum for some polygon is 4500°. How many sides does the polygon have?

e. The angle sum for another polygon is 2520˚. How many angles does the polygon have?

28. In a tessellation of the plane with regular polygonal regions, each vertex will involve the same arrangement of polygons, with the sizes of the angles at each vertex totaling 360°.

a. Use the results from Learning Exercise 27(b) to argue that a tessellation involving exactly one kind of regular polygon is possible for only equilateral triangles, squares, and regular hexagons.

b. A tessellation of the plane can involve more than one kind of regular polygon. Give some possibilities for such a tessellation, considering only the angles at a vertex. For example, because 90˚ + 90˚ + 120˚ + 120˚ = 360˚, perhaps two squares and two hexagons could give a tessellation.

Exercises 29ab give the gist of the argument as to why there can be at most five types of regular polyhedra.

29. Recall from Section 16.5 that there are only five different kinds of convex regular polyhedra.

a. Consider a vertex of a regular polyhedron. The sum of the angles with their vertices all at one vertex of the polyhedron must be less than 360°. Explain why.

b. Argue that there are only five possible regular polyhedra.

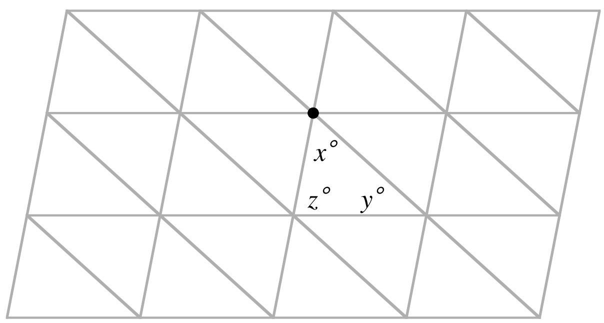
c. A semiregular polyhedron may involve more than one kind of regular polygon at each vertex, although each vertex must have the same arrangement. Give some possibilities for a semiregular polyhedron, considering only the angles at a vertex.

30. Skateboarders, skiers, or snow-boarders may talk about "doing a 1080." What does that mean?

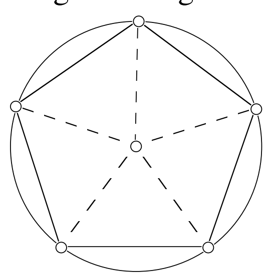
31. Three angles of quadrilateral STAR have sizes 63˚ 45', 110˚ 25', and 120.25˚. Another quadrilateral is similar to STAR, with a scale factor 2.5. What are the sizes of the angles in the second quadrilateral?

32. Tell whether the following statement is true or false, and explain your thinking: For all circles, arcs with the same number of degrees always have the same length.

33. Some teachers like to use tessellations to suggest some important geometric results. For example, fill in the measurements of the other angles that have the highlighted point as the vertex to get a justification that the sum of the angles of a triangle is 180°, that is, $x + y + z = 180$.

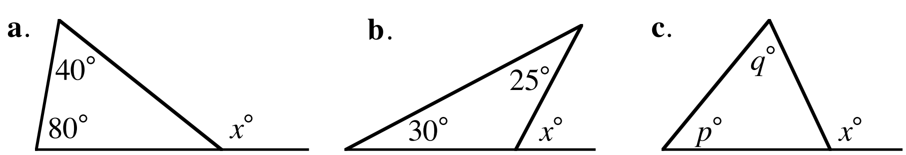


34. a. Any regular polygon can be fit "inside" some circle so that all the vertices of the polygon are on that circle. The center of the circle is the center of the regular polygon. Because the sides of a regular polygon are all equal, the arcs and the central angles they give must be equal. How large is each central angle for an equilateral triangle? A square? A regular pentagon? A regular hexagon? A regular *n*-gon?

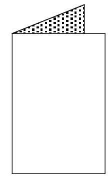


b. How could a teacher draw a regular heptagon with the help of a compass and a protractor?

35. In parts (a)-(c), find the number of degrees *x* in the exterior angle.



36. Just as angles in a plane are made when lines intersect, **dihedral angles** in space are made when two planes intersect:



a dihedral angle

How might one measure a dihedral angle? For example, would it make sense to talk about a 90° dihedral angle? Explain your reasoning. (Giving your ideas, rather than looking up information, is of interest here.)

37. Tell how you could get an angle of 94° if all you had was a wooden angle of 19° (and paper and pencil).

37. This problem is more difficult than it first appears. 95° can be found by using the wooden angle 5 times. To find 1° (in order to remove it from the 95° angle), you may copy 19 wooden angles to obtain 361°, then remove 360°.

38. a. Although the 100-yard dash is common in the United States, the 100-meter dash is used in international track meets. One yard = 0.9144 meter. How do the 100-m dash and the 100-yd dash compare? Give two answers, one for the additive comparison (difference) and one for the multiplicative comparison (ratio).

b. Repeat part (a) for the mile (1760 yards) versus the 1500-meter run.

39. Complete each of the following measurement equalities. (Conversions for some English system length units are given in the Glossary, under **English system**.)

a. 4 miles = _______ yd   b. 1320 yd = _______ mi

c. 1320 ft = _______ mi   d. 1.5 mi = _______ ft

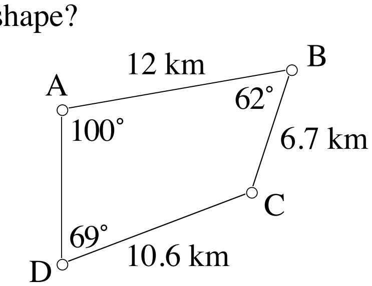
e. 12 yd = _______ in.   f. 84 in. = _______ yd

40. Technical words with everyday meanings or connotations can naturally be confusing to children (and others). Give different meanings or connotations for these.

a. degree   b. yard   c. meter

d. regular   e. right angle (vs. ____ angle)

41. a. A hidden shape is congruent to quadrilateral ABCD shown in the following diagram. What do you know about specific measurements on the hidden shape?



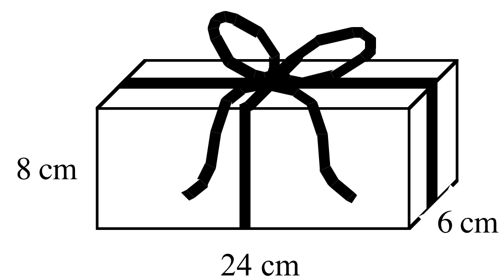
b. Another hidden shape is similar to the given quadrilateral. What can you say about specific measurements on this hidden shape?

42. In the late 1700s, a Frenchman involved in the creation of the metric system thought the *grade*, a unit for angle size equal to one-hundredth of a right angle, would be useful.

a. How many grades would be in a full circle?

b. How many degrees would be in a centigrade?

43. You want to wrap the package below with fancy ribbon as pictured. How many centimeters of ribbon do you need? Allow 25 cm for the bow.

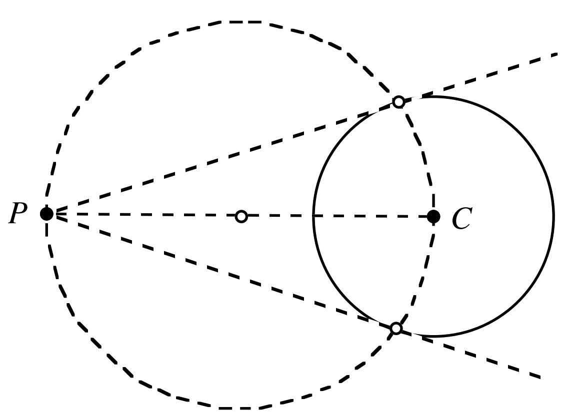


44. a. With a compass and straightedge, construct a circle and a pair of perpendicular diameters for the circle. Join the endpoints of the diameters. What special quadrilateral do the endpoints appear to give? Can you justify that it *is* that shape?

b. With a compass, draw a circle. Then, starting at some marked point, mark off points 1 radius apart around the circle. Join the points in order. What shape do these points appear to give?

c. How might you construct a regular octagon and a regular 12-gon? (*Hint*: Use parts (a) and (b).)

45. Here is how to construct with compass and straightedge the two tangents to a circle from a point *outside* the circle: Use the segment joining the outside point *P* to the center *C* of the circle as the diameter of another circle. Where the two circles meet indicates how to draw the two tangents to the first circle. (The details of the construction are omitted in the drawing below.) Why does this method work?



23.3 Check Yourself

You should be able to work exercises like those assigned and to meet the following objectives.

1. Describe in general terms what the process of measurement involves.
2. State and apply the four key ideas of measurement, and recognize when they have been used.
3. State the range of values that a given measurement covers. (For example, a length reported as 15.6 cm could be as short as 15.55 cm or as long as up to 15.65 cm.)
4. Recognize the "inverse" relationship between the size of the unit and the numerical part of the measurement.
5. Apply the four key ideas of measurement to length (or its special synonyms such as perimeter, height, circumference, and so on) and to angle size.
6. Use your personal benchmarks to estimate given lengths or given segments or lengths of named objects.
7. Estimate angle sizes and lengths, and use a protractor to measure given angles and a ruler marked in English or metric units to measure given segments.
8. Use adjectives with angles correctly (for example, right angle, acute angle, obtuse angle, straight angle, supplementary angle, central angle, vertical angles, alternate interior angles, corresponding angles, exterior angle, inscribed angle, and so on), and deal with the relationships covered (for example, angles with parallel lines, angles of a triangle, and so on).
9. Convert among metric units for length, convert among English units for length, and deal with angle sizes in degrees, minutes, and seconds.
10. Determine the total of the sizes of the angles in a polygon having a given number of sides, and the size of each angle in an equiangular or regular polygon having a given number of sides.

Reference

[i]Klein, H. A. (1974). *The world of measurements*. New York: Simon and Schuster. (The quote in Learning Exercise 21 in Section 23.1 is from page 112.)