

**MASSACHUSETTS MATHEMATICS LEAGUE  
CONTEST 6 – MARCH 2010  
ROUND 1 ALG 2: SIMULTANEOUS EQUATIONS AND DETERMINANTS**

**ANSWERS**

A) \_\_\_\_\_

B) \_\_\_\_\_

C) ( \_\_\_\_\_ , \_\_\_\_\_ )

**\*\*\*\*\* NO CALCULATORS IN THIS ROUND \*\*\*\*\***

A) Given:  $\begin{cases} y = 17 - 2x \\ y = 3x + 2 \\ y = k \end{cases}$  Determine the value of  $k$  for which this system of lines determines a minimum number of points of intersection.

B) Find all ordered pairs  $(x, y)$  satisfying  $\begin{cases} x - 2y = 1 \\ |x| + |y| = 1 \end{cases}$ .

C) Connecting the points  $A(3, 7)$ ,  $B(-1, 2)$ ,  $C(3x, -x)$  and  $D(10, 1)$  you have an outline of the deck in my backyard. It's a nondescript convex quadrilateral and finding its area is baffling my builders. If  $x > 0$  and  $A$  and  $C$  are opposite vertices, compute the coordinates of  $C$  so the area of my deck is 52 square units.

Note: The area of a triangle with vertices at  $(x_1, y_1)$ ,  $(x_2, y_2)$  and  $(x_3, y_3)$  can be computed

$$\text{as } \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

**MASSACHUSETTS MATHEMATICS LEAGUE  
CONTEST 6 - MARCH 2010  
ROUND 2 ALG1: EXPONENTS AND RADICALS**

**ANSWERS**

A) \_\_\_\_\_

B) \_\_\_\_\_

C) \_\_\_\_\_

**\*\*\*\*\* NO CALCULATORS IN THIS ROUND \*\*\*\*\***

A) Given  $\sqrt{2^4(5^{-2}-13^{-2})} = \frac{A}{B}$ , a reduced fraction. Compute  $A + B$ .

Note:  $a^{-n}$  is equivalent to  $\frac{1}{a^n}$

B) Compute  $\left(\frac{2\sqrt{5}-\sqrt{10}}{\sqrt{10}-\sqrt{5}}\right)^2$

C) Given:  $\sqrt{\frac{9}{16}-\frac{9}{25}} = \frac{3}{4}-\frac{3}{x}$  for some integer value of  $x$ .

Compute  $\sqrt{\frac{2^x \cdot 4^{3x+4}}{8^{2x+2}}}$ .

**MASSACHUSETTS MATHEMATICS LEAGUE  
CONTEST 6 – MARCH 2010  
ROUND 3 TRIGONOMETRY: ANYTHING**

**ANSWERS**

A) \_\_\_\_\_ °

B) \_\_\_\_\_

C) \_\_\_\_\_

**\*\*\*\*\* NO CALCULATORS IN THIS ROUND \*\*\*\*\***

A) Solve for  $\theta$ ,  $0^\circ \leq \theta < 360^\circ$ .

$$\sin \theta \cos \theta \tan(90^\circ - \theta) \cot(90^\circ + \theta) \sec(270^\circ + \theta) \csc(270^\circ - \theta) = \cot(180^\circ - \theta)$$

B) Robert Wadlow, the world's tallest man, stands  $x$  feet tall. Your task is to compute  $x$  given the following information:

At 4:00 PM, Robert's shadow is 10 feet longer than his shadow at 2:00 PM.

At 4:00 PM the sun makes a  $30^\circ$  with the ground, but it made a  $60^\circ$  at 2:00 PM.

Recall: An exact answer is required.  
The given information is summarized in the diagram.



C) Given:  $\begin{cases} \sin A = \frac{k}{\sqrt{41}} \\ \cos A = \frac{k+1}{\sqrt{41}} \end{cases}$  and  $k > 0$ . Compute  $\sec\left(A - \frac{5\pi}{2}\right) \cdot \cos(A - 5\pi)$ .

**MASSACHUSETTS MATHEMATICS LEAGUE  
CONTEST 6 - MARCH 2010  
ROUND 4 ALG 1: ANYTHING**

**ANSWERS**

A) \_\_\_\_\_

B) \_\_\_\_\_ years

C) \_\_\_\_\_

**\*\*\*\*\* NO CALCULATORS IN THIS ROUND \*\*\*\*\***

A) Determine all possible positive integral values of  $x$  for which  $236_8 = 134_x$ .  
The subscripts denote the base in which the numeral is written.

B) Alice and her mother have the same birthday (MM/DD).  
Alice is 18 years younger than her mother.  
When her mother was as old as Alice is now, her mother was twice as old as Alice was then.  
In how many years from now will the ratio of Alice's age to her mother's age be 7 : 10?

C) Compute the two rational roots of the following quadratic:

$$4A(48A + 1) = 5(16 - 201A).$$

**MASSACHUSETTS MATHEMATICS LEAGUE  
CONTEST 6 - MARCH 2010  
ROUND 5 PLANE GEOMETRY: ANYTHING**

**ANSWERS**

A) \_\_\_\_\_

B) \_\_\_\_\_ : \_\_\_\_\_

C) \_\_\_\_\_

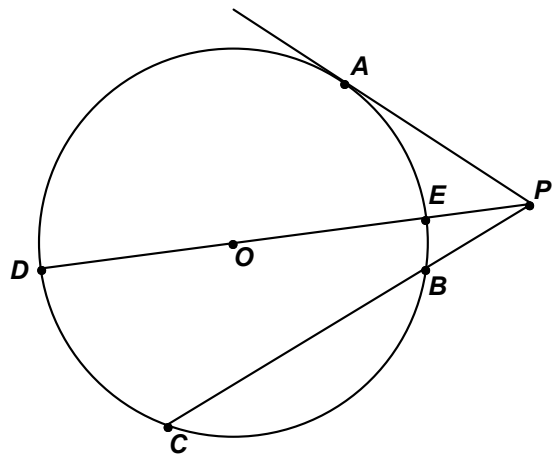
**\*\*\*\*\* NO CALCULATORS IN THIS ROUND \*\*\*\*\***

A) Ten coplanar lines determine  $N$  points of intersection.  
Compute the maximum value of  $N$ .

B) An equilateral triangle and a regular hexagon have equal perimeters.  
What is the ratio of the area of the triangle to the hexagon?

C) Given:  $\overline{PA}$  is tangent to circle  $O$ ,  
 $PA = 14$ ,  
 $CB = 3PB$  and  $EP = 2\sqrt{PB}$

Compute the area of circle  $O$ .



**MASSACHUSETTS MATHEMATICS LEAGUE  
CONTEST 6 - MARCH 2010  
ROUND 6 ALG 2: PROBABILITY AND THE BINOMIAL THEOREM**

**ANSWERS**

A) \_\_\_\_\_

B) \_\_\_\_\_

C) \_\_\_\_\_

**\*\*\*\*\* NO CALCULATORS IN THIS ROUND \*\*\*\*\***

A) One container has 5 blue chips and 4 red chips, while another container has 4 blue chips and 5 red chips. One chip is chosen at random from each container. What is the probability that one chip is red and the other is blue?

B) Three mathletes  $A$ ,  $B$  and  $C$  are simultaneously (but independently) trying to solve a difficult math problem. The probability of  $A$  correctly solving the problem is  $\frac{1}{5}$  and the probability of  $B$  correctly solving the same problem is  $\frac{1}{4}$ .

Let  $P$  denote the probability of the problem being solved by at least one of the mathletes. Let  $Q$  denote the probability that the problem is solved by exactly one of the mathletes. If  $P : Q = 8 : 5$ , compute the probability that  $C$  solves the problem correctly.

C) The 5<sup>th</sup> term in the expansion of  $\left(\frac{1}{2}x^2 + Ax^{-1}\right)^7$  is  $\frac{70}{81}x^T$ .

Compute all possible ordered pairs  $(A, T)$ .

Note: The 1<sup>st</sup> term in the expansion is  $\frac{x^{14}}{128}$ .

**MASSACHUSETTS MATHEMATICS LEAGUE  
CONTEST 6 - MARCH 2010  
ROUND 7 TEAM QUESTIONS  
ANSWERS**

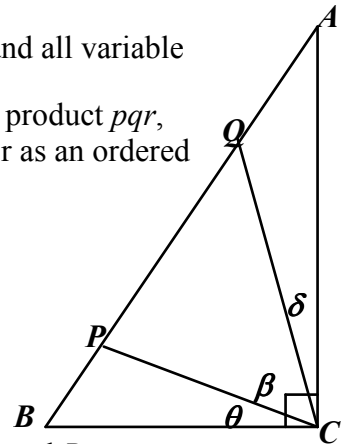
- A) \_\_\_\_\_ D) ( \_\_\_\_\_ , \_\_\_\_\_ )  
 B) ( \_\_\_\_\_ , \_\_\_\_\_ , \_\_\_\_\_ ) E) \_\_\_\_\_  
 C) \_\_\_\_\_ F) \_\_\_\_\_

**\*\*\*\*\* NO CALCULATORS ARE PERMITTED IN THIS ROUND \*\*\*\*\***

A) Let functions  $f$  and  $g$  be defined by  $f(x) = -k|x| + A$  and  $g(x) = 4k|x| - 5B$ .  
 If  $k > 1$ ,  $A = \begin{vmatrix} 1 & k \\ -k & 1 \end{vmatrix}$ ,  $B = \begin{vmatrix} k & 1 \\ 1 & k \end{vmatrix}$  and the functions  $f$  and  $g$  have the same zeros, compute the area of the region bounded by  $f$  and  $g$ .

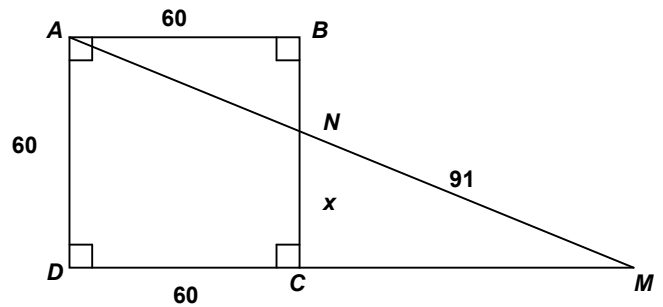
B) Suppose  $\frac{\sqrt[4]{2 \cdot 3^a} \cdot \sqrt[6]{3 \cdot 2^b}}{\sqrt[3]{12}} = \sqrt{2^p \cdot 3^q}$ , each radical is in simplest form and all variable denote positive integers. Compute the number of distinct values of the product  $pqr$ , the maximum product and the minimum product. Express your answer as an ordered triple (count, max, min).

C) Points  $P$  and  $Q$  lie on the hypotenuse of right triangle  $ABC$ .  
 If  $\sin \theta = \frac{3}{5}$  and  $\sin \delta = \frac{7}{25}$ , compute the value of  $\sin \beta$ .



D) Two bottles of equal volume contain rubbing alcohol and water. The ratios of alcohol to water in the two bottles are  $3 : 1$  and  $A : B$ , where  $A$  and  $B$  are relatively prime integers. The contents of the two bottles are mixed and the new ratio of alcohol to water is  $27 : 13$ . Determine the ordered pair  $(A, B)$ .

E) Given square  $ABCD$  below with sides as indicated. Compute  $x$ , the length of  $\overline{NC}$ . Diagram is not necessarily drawn to scale.



F) Let  $N$  be the positive difference between the two largest coefficients in the expansion of  $(3a + 2b)^{11}$ . Determine the prime factorization of  $N$ . When writing the product, list the primes from smallest to largest.

**MASSACHUSETTS MATHEMATICS LEAGUE  
CONTEST 6 – MARCH 2010 ANSWERS**

**Round 1 Alg 2: Simultaneous Equations and Determinants**

- A) 11                                      B)  $(1, 0), \left(-\frac{1}{3}, -\frac{2}{3}\right)$                                       C)  $(9, -3)$

**Round 2 Alg 1: Exponents and Radicals**

- A) 113                                      B) 2                                      C) 64

**Round 3 Trigonometry: Anything**

- A)  $135^\circ, 315^\circ$                                       B)  $5\sqrt{3}$  feet                                      C)  $-\frac{5}{4}$  (or  $-1.25$ )

**Round 4 Alg 1: Anything**

- A) 11                                      B) 6                                      C)  $-\frac{16}{3}, \frac{5}{64}$

**Round 5 Plane Geometry: Anything**

- A) 45                                      B) 2 : 3                                      C)  $252\pi$

**Round 6 Alg 2: Probability and the Binomial Theorem**

- A)  $\frac{41}{81}$                                       B)  $\frac{4}{5}$  (.8 or 0.8)                                      C)  $\left(\pm\frac{2}{3}, 2\right)$

**Team Round**

- A)  $\frac{500}{3}$  (or equivalent)                                      D)  $(3, 2)$
- B)  $(9, 420, 12)$                                       E) 35
- C)  $\frac{3}{5}$  (.6 or 0.6)                                      F)  $2^5 \cdot 3^7 \cdot 11^1$  (the last exponent is not required)



**MASSACHUSETTS MATHEMATICS LEAGUE  
CONTEST 6 – MARCH 2010 SOLUTION KEY**

**Round 1**

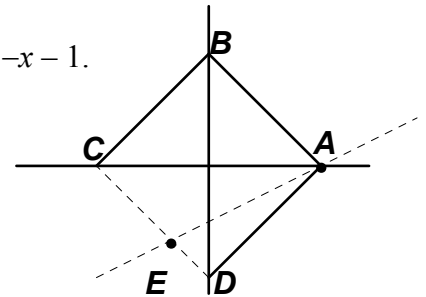
A) If the three lines were parallel, then no points of intersection would be determined. This is not possible, since the first two lines have different slopes and the third line is horizontal. The (unique) minimum will occur when the horizontal line passes through the point of intersection of the first two lines.  $17 - 2x = 3x + 2 \rightarrow x = 3 \rightarrow y = 11 \rightarrow k = \underline{11}$ .

B) The 1<sup>st</sup> condition describes a line  $y = \frac{-x+1}{2}$  (slope of  $-1/2$  and an  $x$ -intercept of  $(1, 0)$ ).

The 2<sup>nd</sup> condition describes square connecting (in order):  $(1, 0)$ ,  $(0, 1)$ ,  $(-1, 0)$  and  $(0, -1)$ . Clearly, **(1, 0)** is a solution.

The equation of the side of the square in quadrant 3 lies on the line  $y = -x - 1$ .

$$\frac{x-1}{2} = -x-1 \rightarrow x-1 = -2x-2 \rightarrow 3x = -1 \rightarrow \left(-\frac{1}{3}, -\frac{2}{3}\right).$$



C)  $Area(ABCD) = Area(\triangle ABD) + Area(\triangle BCD)$

Using the determinant method,

$$Area(ABCD) = \frac{1}{2} \begin{vmatrix} 3 & 7 & 1 \\ -1 & 2 & 1 \\ 10 & 1 & 1 \end{vmatrix} + \frac{1}{2} \begin{vmatrix} -1 & 2 & 1 \\ 3x & -x & 1 \\ 10 & 1 & 1 \end{vmatrix} =$$

$$\frac{1}{2}((6-1+70)-(-7+20+3)) + \frac{1}{2}((x+3x+20)-(6x-10x-1)) = 52$$

$$\rightarrow \frac{1}{2}(59+(8x+21)) = 52 \rightarrow 8x+80 = 104 \rightarrow x = 3 \rightarrow \underline{(9, -3)}.$$

Alternative #1:

The area is given by  $\frac{1}{2} \begin{vmatrix} 3 & 7 \\ -1 & 2 \\ 3x & -x \\ 10 & 1 \\ 3 & 7 \end{vmatrix}$ . This “determinant” is evaluated by the

*lattice* method. Take the absolute value of the difference between

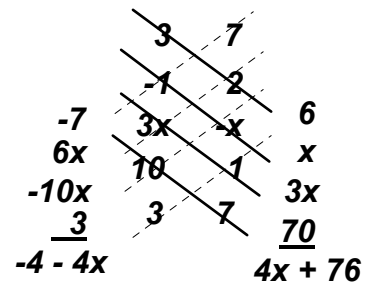
the sum of the diagonal down products and  
the sum of the diagonal up products.

$$\text{Thus, the area is } \frac{1}{2}(4x+76-(-4-4x)) = 52 \rightarrow x = 3 \rightarrow \underline{(9, -3)}.$$

This technique works for finding the area of any convex polygonal region, given given a clockwise or counterclockwise listing of the coordinates of its vertices, repeating the starting coordinates. See if it works for some concave regions!

Does it work for all polygonal regions – convex or concave?

The simplicity and generality of this method is astounding.



**MASSACHUSETTS MATHEMATICS LEAGUE  
CONTEST 6 – MARCH 2010 SOLUTION KEY**

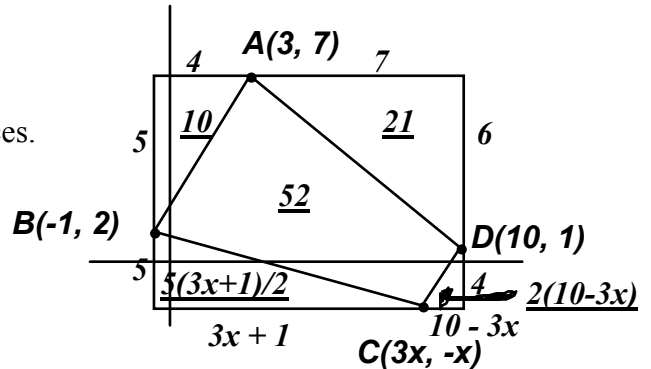
**Round 1 - continued**

Alternative #2 (Utilizing only basic area formulas)  
Study the diagram at the right. Draw a pair of verticals  
and a pair of horizontals through opposite pairs of vertices.

$$10 + 21 + \frac{5(3x+1)}{2} + 2(10-3x) + 52 = 10(11)$$

$$\rightarrow \frac{5(3x+1)}{2} + 2(10-3x) = 110 - 83 = 27$$

$$\rightarrow 15x + 5 + 40 - 12x = 54 \rightarrow 3x = 9 \rightarrow x = 3 \rightarrow \underline{(9, -3)}.$$



Alternate #3

Here is a (more involved) solution which utilizes  
the triangle area formula for any quadrilateral, convex or concave,  
namely half the product of the diagonals times the sine of the included

angle. We start with an area formula for the triangle  $\frac{1}{2}absin\theta$ ,

where  $\theta$  denotes the included angle.

Since  $\sin(180 - \theta) = \sin \theta$ , we have  $\text{Area}(ABCD) =$

$$\frac{1}{2} \sin \theta (ac + ad + bd + bc) = \frac{1}{2} \sin \theta (a(c+d) + b(c+d)) =$$

$$\frac{1}{2} \sin \theta ((a+b)(c+d)) = \frac{1}{2} \sin \theta \cdot AC \cdot BD. \text{ Let's show that } (9, -3) \text{ produces an area of } 52.$$

Using the distance formula,  $AC = \sqrt{(-6)^2 + 10^2} = \sqrt{136} = 2\sqrt{34}$  and  $BD = \sqrt{11^2 + (-1)^2} = \sqrt{122}$

The slope of  $\overline{AC} = -\frac{5}{3}$  and the slope of  $\overline{BD} = -\frac{1}{11}$ .

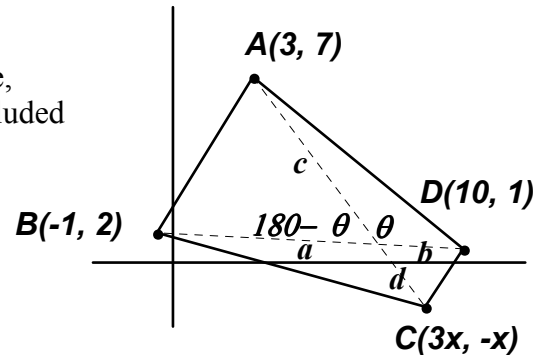
We need the angle between the two lines. If  $\theta$  denotes the acute angle, then  $\tan \theta = \left| \frac{m_2 - m_1}{1 + m_1 m_2} \right|$ .

$$\text{In this case, } \tan \theta = \frac{-\frac{1}{11} + \frac{5}{3}}{1 + \frac{1}{11} \cdot \frac{5}{3}} = \frac{-3 + 55}{33 + 5} = \frac{52}{38} = \frac{26}{19} \rightarrow \sin \theta = \frac{26}{\sqrt{1037}} \text{ and the area is}$$

$$\frac{1}{2} \cdot \frac{26}{\sqrt{1037}} \cdot 2\sqrt{34} \cdot \sqrt{122} \text{ But does this equal } 52 \text{ ?????}$$

$$\frac{1}{2} \cdot \frac{26}{\sqrt{1037}} \cdot 2\sqrt{34} \cdot \sqrt{122} = \frac{1}{2} \cdot \frac{26}{\sqrt{61} \cdot \sqrt{17}} \cdot 2(\sqrt{2} \cdot \sqrt{17}) \cdot (\sqrt{2} \cdot \sqrt{61}) = 13(4) = \underline{52}.$$

Amazing (albeit tedious) solution. The basic area formulas above are worth remembering.



**MASSACHUSETTS MATHEMATICS LEAGUE  
CONTEST 6 – MARCH 2010 SOLUTION KEY**

**Round 2**

$$A) \sqrt{2^4(5^{-2}-13^{-2})} = \sqrt{2^4\left(\frac{1}{5^2}-\frac{1}{13^2}\right)} = \sqrt{2^4\left(\frac{13^2-5^2}{5^2 \cdot 13^2}\right)} = \sqrt{2^4\left(\frac{12^2}{5^2 \cdot 13^2}\right)} = \frac{4 \cdot 12}{5 \cdot 13} = \frac{48}{65}$$

$$\rightarrow A + B = \underline{113}.$$

$$B) \left(\frac{2\sqrt{5}-\sqrt{10}}{\sqrt{10}-\sqrt{5}}\right)^2 = \frac{20-4\sqrt{50}+10}{10-2\sqrt{50}+5} = \frac{30-4\sqrt{50}}{15-2\sqrt{50}} = \frac{30-20\sqrt{2}}{15-10\sqrt{2}} = \frac{10(3-2\sqrt{2})}{5(3-2\sqrt{2})} = \underline{2}.$$

C) First and foremost -  $\sqrt{\frac{9}{16}-\frac{9}{25}} \neq \frac{3}{4}-\frac{3}{5}$  !!! Squaring both sides,

$$\sqrt{\frac{9}{16}-\frac{9}{25}} = \frac{3}{4}-\frac{3}{x} \rightarrow \frac{9}{16}-\frac{9}{25} = \left(\frac{3}{4}-\frac{3}{x}\right)^2 = \frac{9}{16} - 2\left(\frac{3}{4}\right)\left(\frac{3}{x}\right) + \frac{9}{x^2} = \frac{9}{16} - \frac{9}{2x} + \frac{9}{x^2}$$

$$\rightarrow \frac{9(25-16)}{16(25)} = \frac{9}{16} - \frac{9}{2x} + \frac{9}{x^2} \rightarrow \frac{9}{16(25)} = \frac{1}{16} - \frac{1}{2x} + \frac{1}{x^2} \rightarrow$$

$$\frac{9}{16(25)} - \frac{25}{16(25)} = -\frac{1}{25} = -\frac{1}{2x} + \frac{1}{x^2}.$$

Multiplying through by  $-50x^2$ ,  $2x^2 = 25x - 50 \rightarrow 2x^2 - 25x + 50 = (2x - 5)(x - 10) = 0$

$$\rightarrow x = 10.$$

$$\sqrt{\frac{2^x \cdot 4^{3x+4}}{8^{2x+2}}} = \sqrt{\frac{2^x \cdot 2^{6x+8}}{2^{6+6x}}} = \sqrt{2^{x+2}} \quad \text{Since } x = 10, \text{ we have } \sqrt{2^{12}} = 2^6 = \underline{64}.$$

**MASSACHUSETTS MATHEMATICS LEAGUE  
CONTEST 6 – MARCH 2010 SOLUTION KEY**

**Round 3**

- A) Rather than evaluating by direct substitution, let's simplify the expression and show that, if the expression is defined, the product  $P$  is invariant, i.e. has the same value, namely 1. Using reduction formulas,  $\theta$  is a reference value (angle) in quadrant 1 and when we see  $(90 - \theta)$  – think Q1,  $(90 + \theta)$  – think Q2,  $(270 - \theta)$  – think Q3,  $(270 + \theta)$  – think Q4. The mnemonic ASTC identifies the sign of the trig functions in quadrants 1 – 4.

Simplifying the lefthand side, we have  $\sin \theta \cos \theta (\cot \theta) (-\tan \theta) (\csc \theta) (-\sec \theta)$

Regrouping,  $(\sin \theta \cdot \csc \theta) (\cos \theta \cdot -\sec \theta) (\cot \theta \cdot -\tan \theta) = 1 \cdot -1 \cdot -1 = \underline{1}$ .

Thus,  $\cot(180 - \theta) = -\cot(\theta) = 1 \rightarrow \theta$  belongs to  $45^\circ$  family in quadrants 2 and 4  $\rightarrow \underline{135^\circ, 315^\circ}$ .

*As a 13 year old Boy Scout – 7' 4"*

- B) Let  $s$  denote the length of Shorty's shadow at 2:00. Then:

$$\frac{x}{s} = \tan 60^\circ \rightarrow x = s\sqrt{3}.$$

$$\text{At 4:00 } \frac{x}{s+10} = \tan 30^\circ = \frac{1}{\sqrt{3}} \rightarrow x\sqrt{3} = s+10.$$

$$\text{Multiplying by } \sqrt{3}, 3x = s\sqrt{3} + 10\sqrt{3} \rightarrow 3x = x + 10\sqrt{3} \\ \rightarrow x = \underline{5\sqrt{3}}.$$

Note: Wadlow was a real person and his actual maximum adult height was 8 feet 11.1 inch.

He died at the age of 22 and is acknowledged to be the modern world's tallest man.

Goto [http://www.maniacworld.com/worlds\\_tallest\\_man.htm](http://www.maniacworld.com/worlds_tallest_man.htm) to learn more or Google Robert Wadlow.

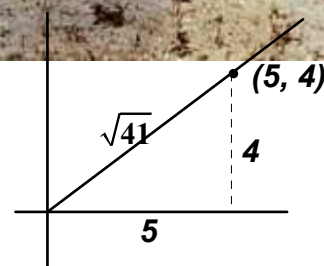


$$\text{C) Since } \begin{cases} \sin A = \frac{k}{\sqrt{41}} \\ \cos A = \frac{k+1}{\sqrt{41}} \end{cases}, \frac{k^2}{41} + \frac{(k+1)^2}{41} = 1.$$

$$2k^2 + 2k = 40 \rightarrow k^2 - k - 20 = (k+5)(k-4) = 0. \quad k > 0 \rightarrow k = 4.$$

$$\text{Simplifying, } \sec\left(A - \frac{5\pi}{2}\right) \cdot \cos(A - 5\pi) =$$

$$\sec\left(A - \frac{\pi}{2}\right) \cdot \cos(A - \pi) = \sec\left(\frac{\pi}{2} - A\right) \cdot \cos(\pi - A) = \csc A \cdot -\cos A = -\frac{\cos A}{\sin A} = -\cot A \rightarrow \underline{-\frac{5}{4}}.$$



**MASSACHUSETTS MATHEMATICS LEAGUE  
CONTEST 6 - MARCH 2010 SOLUTION KEY**

**Round 4**

A)  $2(8^2) + 3(8) + 6 = 1x^2 + 3x + 4 \rightarrow 158 = x^2 + 3x + 4 \rightarrow x^2 + 3x - 154 = (x - 11)(x + 14) = 0$   
 $\rightarrow x = \underline{11}$ ,  ~~$14$~~ .

B) Mom and Alice are now  $x + 18$  and  $x$  years old respectively. Clearly, 18 years ago, Mom was as old as Alice is now and Alice was only  $x - 18$  years old.

$x = 2(x - 18) \rightarrow x = 36$ . Thus, in  $n$  years the required ratio is satisfied, namely  $\frac{36+n}{54+n} = \frac{7}{10}$   
 $\rightarrow 360 + 10n = 378 + 7n \rightarrow n = \underline{6}$ .

C)  $4A(48A+1) = 5(16-201A) \rightarrow 192A^2 + 4A = 80 - 1005A \rightarrow 192A^2 + 1009A - 80 = 0$

No doubt the coefficients are intimidating, BUT...

Since the roots are rational, this quadratic must factor.

How do we avoid tedious trial and error?

The key observations are:

$192 = 3 \cdot 2^6$ ,  $80 = 5 \cdot 2^4$  (each with exactly one odd factor) and  
the coefficient of the middle term of the trinomial is **odd**.

Since the QI in FOIL produces the middle term, we must have an odd outer product and an even inner product (or vice versa). Thus, the factorization must be  $(3A + 16)(64A - 5) = 0$

$\rightarrow A = \underline{-\frac{16}{3}, \frac{5}{64}}$ .

**Round 5**

A) Two coplanar lines determine at most one point of intersection.

A third line will determine at most two more, if it crosses both of the existing lines.

Likewise, each subsequent line adds a maximum number of new intersection points if it crosses each of the existing lines.

Thus, for 10 lines  $N_{\max} = 1 + 2 + 3 + 4 + \dots + 9 = 9(10)/2 = \underline{45}$ .

B) Let  $p$  = perimeter.

Area of triangle =  $\frac{1}{2}(\text{base})(\text{height})$

$$= \frac{1}{2} \left(\frac{p}{3}\right) \left(\frac{p}{3}\right) \frac{\sqrt{3}}{2}$$

$$= \frac{p^2 \sqrt{3}}{36}$$

Area of hexagon =  $6(1/2)(\text{base})(\text{height})$

$$= 6 \left(\frac{1}{2}\right) \left(\frac{p}{6}\right) \left(\frac{p}{6}\right) \frac{\sqrt{3}}{2}$$

$$= \frac{p^2 \sqrt{3}}{24}$$

Ratio of triangle to hexagon is  $\underline{2:3}$

**MASSACHUSETTS MATHEMATICS LEAGUE  
CONTEST 6 - MARCH 2010 SOLUTION KEY**

**Round 5 - continued**

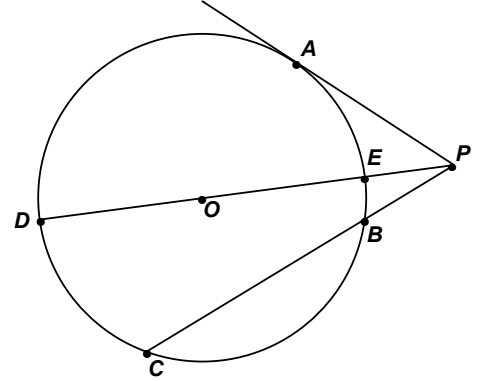
C) Let  $PB = x$  and  $BC = 3x$ . According to the tangent-secant relation,  $PA^2 = PB(PC)$ , i.e. outer(outside + inner)

$$x(4x) = 14^2 \rightarrow x = 7 \rightarrow PE = 2\sqrt{7}.$$

$$\text{Let } OE = OD = r. \text{ Then } 14^2 = 2\sqrt{7}(2\sqrt{7} + 2r).$$

$$\rightarrow 196 = 28 + 4\sqrt{7}r \rightarrow r = \frac{168}{4\sqrt{7}} = \frac{42}{\sqrt{7}} = 6\sqrt{7}.$$

Thus, the area of circle  $O$  is  $\pi(6\sqrt{7})^2 = \underline{\underline{252\pi}}$ .



**Round 6**

A) The desired outcome could be either  $R_1B_2$  or  $B_1R_2$ .

OR  $\rightarrow$  ADD

$$\text{Thus, the probability is } \frac{4}{9} \cdot \frac{4}{9} + \frac{5}{9} \cdot \frac{5}{9} = \frac{16+25}{81} = \underline{\underline{\frac{41}{81}}}.$$

B)  $P = 1 - P(\text{none of them solve the problem}) = 1 - P(\text{not } A) \cdot P(\text{not } B) \cdot P(\text{not } C) = 1 - \frac{4}{5} \cdot \frac{3}{4} \cdot (1-x)$

$$= \frac{2+3x}{5} = \frac{8+12x}{20}.$$

$Q = P(A \text{ not } B \text{ not } C \text{ or not } A \text{ } B \text{ not } C \text{ or not } A \text{ not } B \text{ } C) =$

$$\frac{1}{5} \cdot \frac{3}{4} \cdot (1-x) + \frac{4}{5} \cdot \frac{1}{4} \cdot (1-x) + \frac{4}{5} \cdot \frac{3}{4} \cdot x = \frac{7}{20}(1-x) + \frac{12x}{20} = \frac{7+5x}{20}.$$

$$\text{Thus, } P : Q = 8 : 5 \rightarrow \frac{8+12x}{7+5x} = \frac{8}{5} \rightarrow 40 + 60x = 56 + 40x \rightarrow 20x = 16 \rightarrow x = \underline{\underline{\frac{4}{5}}}.$$

C) The 5th term is  $\binom{7}{4} \left(\frac{1}{2}x^2\right)^3 (Ax^{-1})^4 = \frac{7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3} \cdot \frac{1}{8} \cdot A^4 \cdot x^6 \cdot x^{-4} = \frac{35A^4}{8}x^2 = \frac{70}{81}x^T$

$$\rightarrow T = 2 \text{ and } A^4 = \frac{16}{81} \rightarrow (A, T) = \left(\pm \frac{2}{3}, 2\right).$$

**MASSACHUSETTS MATHEMATICS LEAGUE  
CONTEST 6 - MARCH 2010 SOLUTION KEY**

**Team Round**

A) The graph of the bounded region is represented in the diagram at the right:

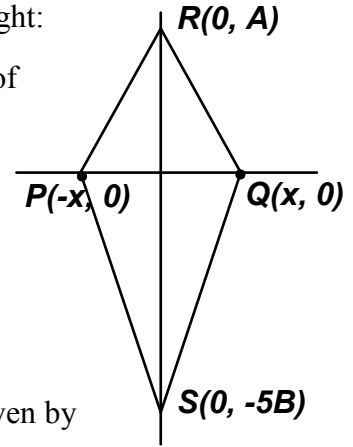
$Q(x, 0)$  is  $\left(\frac{A}{k}, 0\right)$  from  $f$ 's point of view and  $\left(\frac{5D}{4k}, 0\right)$  from  $g$ 's point of view. Evaluating the determinants, equating and canceling out the common factors of  $k$  in the denominator,

$$\frac{1+k^2}{k} = \frac{5(k^2-1)}{4k} \rightarrow 4+4k^2 = 5k^2-5 \rightarrow k^2 = 9 \rightarrow k = 3.$$

$P\left(-\frac{10}{3}, 0\right)$ ,  $Q\left(\frac{10}{3}, 0\right)$ ,  $R(0, 10)$  and  $S(0, -40)$

$PRQS$  is a quadrilateral w/perpendicular diagonals; hence its area is given by

$$\frac{1}{2}d_1d_2 \rightarrow \frac{1}{2}PQ \cdot RS = \frac{1}{2}\left(\frac{20}{3}\right)(10 - (-40)) = \frac{10}{3} \cdot 50 = \frac{500}{3}.$$



B) 
$$\frac{\sqrt[4]{2 \cdot 3^a} \cdot \sqrt[6]{3 \cdot 2^b}}{\sqrt[3]{12}} = \frac{(2 \cdot 3^a)^{\frac{1}{4}} (3 \cdot 2^b)^{\frac{1}{6}}}{(2^2 \cdot 3)^{\frac{1}{3}}} = \frac{(2 \cdot 3^a)^{\frac{3}{12}} (3 \cdot 2^b)^{\frac{2}{12}}}{(2^2 \cdot 3)^{\frac{4}{12}}} = \left(\frac{2^3 3^3 a^3 2^2 b^2}{2^8 3^4}\right)^{\frac{1}{12}} = (2^{2b-5} 3^{3a-2})^{\frac{1}{12}}$$

$$= \sqrt[12]{2^{2b-5} \cdot 3^{3a-2}}.$$

Since this expression must equal  $\sqrt[12]{2^p 3^q}$ , we know that  $r = 12$ ,  $p = 2b - 5 < 12$ ,  $q = 3a - 2 < 12$ . Since each of the original radicals was also in simplest form,  $a = 1, 2, 3$  and  $b = 1, 2, 3, 4, 5$ . Thus,  $p = 1, 3$  or  $5$  and  $q = 1, 4, 7$ . Since  $pqr = 12pq$ , the minimum value occurs for  $(p, q) = (1, 1)$  and the maximum occurs for  $(p, q) = (5, 7)$ . (count, Max, min) = **(9, 420, 12)**.

C) Think  $\theta: 3-4-5$  and  $\delta: 7-24-25$

Let  $\alpha = \theta + \delta$ . Then

$$\cos \alpha = \cos(\theta + \delta) = \cos \theta \cos \delta - \sin \theta \sin \delta = \frac{4}{5} \cdot \frac{24}{25} - \frac{3}{5} \cdot \frac{7}{25} = \frac{75}{125} = \frac{3}{5} \rightarrow \sin \alpha = \frac{4}{5}$$

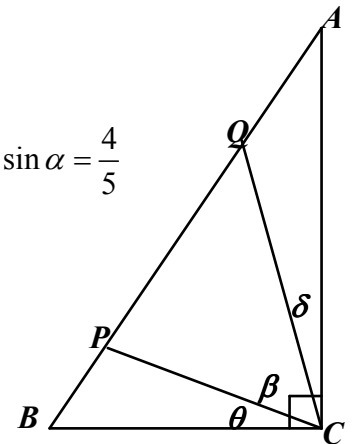
$$\sin(\beta + (\theta + \delta)) = \sin(\beta + \alpha) = \sin \beta \cos \alpha + \cos \beta \sin \alpha = \sin(90^\circ) = 1$$

$$\text{If } \sin \beta = k, \text{ then } \cos \beta = \sqrt{1-k^2} \rightarrow k\left(\frac{3}{5}\right) + \sqrt{1-k^2}\left(\frac{4}{5}\right) = 1$$

$$\rightarrow 4\sqrt{1-k^2} = 5-3k \rightarrow 16(1-k^2) = 25-30k+9k^2 \rightarrow$$

$$25k^2 - 30k + 9 = (5k-3)^2 = 0 \rightarrow k = \sin \beta = \frac{3}{5}.$$

Does this imply that  $\overline{CP}$  actually bisects  $\angle QCB$ ?



**MASSACHUSETTS MATHEMATICS LEAGUE  
CONTEST 6 - MARCH 2010 SOLUTION KEY**

**Team Round - continued**

- D) Bottle #1 contains 4 parts and bottle #2 contains  $(A + B)$  parts.  
To insure equal volumes – multiply the bottle #1 ratio by  $(A + B) \rightarrow 3(A + B) : (A + B)$   
and the bottle #2 ratio by 4  $\rightarrow 4A : 4B$ .

The alcohol : water ratio, after mixing, is  $(7A + 3B) : (A + 5B) = 27 : 13$   
 $\rightarrow 91A + 39B = 27A + 135B \rightarrow 64A = 96B \rightarrow 2A = 3B$ .

The only relatively prime pair of values satisfying this equality is  $(A, B) = \underline{(3, 2)}$ .

- E) Since  $\triangle NBA \sim \triangle NCM$ , we require

$$\frac{NC}{CM} = \frac{NB}{BA}$$

Solution #1: (Banking on a hunch)

A triangle with sides 5 – 12 – 13 is a right triangle and since  $91 = 7(13)$ , we might guess that  $NC = x = 7(5) = 35$ ,

$$CM = 7(12) = 84, \quad \frac{NC}{CM} = \frac{5}{12}$$

$$NB = 60 - 35 = 25 \text{ and } \frac{NB}{BA} = \frac{25}{60} = \frac{5}{12}, \text{ so our hunch was correct, } x = \underline{35}.$$

In the absence of a hunch, here are two analytical solutions.

Solution #2: (Thanks to Andrew Geng – Westford Academy/M.I.T.)

Let  $y = 60 - x$ . By applying the Pythagorean Theorem and similarity relations:

$$\frac{60^2 + y^2}{91^2} = \frac{y^2}{x^2} \rightarrow (60^2 + y^2)x^2 = 91^2 y^2 \quad (***)$$

Rewriting  $60^2 + y^2$  as  $(60 - y)^2 + 120y$  and noting that  $60 - y = x$ , we have  $60^2 + y^2 = x^2 + 120y$ .

This substitution will make it possible to reduce equation (\*\*\*) to a quadratic.

$$(x^2 + 120y)x^2 = 91^2 y^2 \rightarrow x^4 + 120x^2 y - 91^2 y^2 = 0$$

The quadratic formula can be used to solve for  $x^2$  (in terms of  $y$ ) or alternately, the left side can be factored by noticing that  $91^2$  is the product of  $7^2$  and  $13^2$ , which differ by 120.

$$x^2 = \frac{1}{2} \left( -120y \pm \sqrt{120^2 y^2 + 4 \cdot 91^2 y^2} \right) = -60y \pm \sqrt{60^2 y^2 + 91^2 y^2}$$

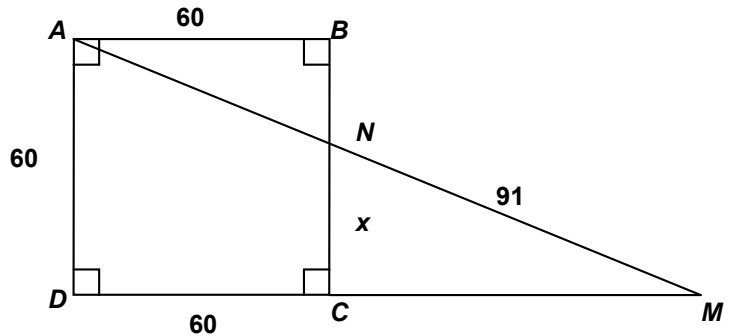
$$= -60y \pm \sqrt{11881y^2} = -60y \pm 109y = -49y, -169y$$

Since both  $x$  and  $y$  must be positive,  $-169y$  can be discarded. Applying the substitution  $y = 60 - x$  and using the quadratic formula (or factoring again) finishes the problem:

$$x^2 = 49(60 - x) \rightarrow x^2 + 49x - 49 \cdot 60 = 0$$

$$x = \frac{1}{2} \left( -49 \pm \sqrt{49^2 + 4 \cdot 49 \cdot 60} \right) = \frac{1}{2} \left( -49 \pm 7\sqrt{49 + 240} \right) = \frac{7}{2} \left( -7 \pm \sqrt{289} \right)$$

$$= \frac{7}{2} (-7 \pm 17) = \underline{35}, \quad \cancel{84}$$





**MASSACHUSETTS MATHEMATICS LEAGUE  
CONTEST 6 - MARCH 2010 SOLUTION KEY**

**Team Round - continued**

E) Solution #3 (Norm Swanson – Hamilton-Wenham/M.I.T.)

Since  $\triangle ABN \sim \triangle MCN$ , let  $BN = kx$ , then

$$x + kx = 60 \rightarrow x = \frac{60}{k+1} \text{ and } CM = \frac{60}{k}$$

By the Pythagorean Theorem, we have

$$\left(\frac{60}{k}\right)^2 + \left(\frac{60}{k+1}\right)^2 = 91^2.$$

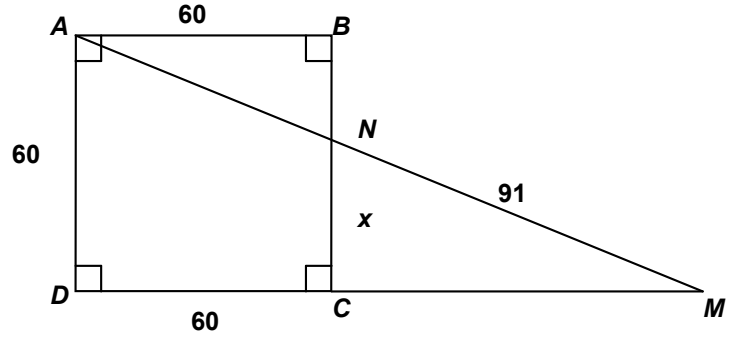
Since  $91 = 7(13)$ , let  $r = \frac{60}{7}$  and rewrite the

equation as  $\left(\frac{r}{k}\right)^2 + \left(\frac{r}{k+1}\right)^2 = 13^2$ . Recalling that 5-12-13 is a pythagorean triple,

we try  $\frac{r}{k} = 12$  and  $\frac{r}{k+1} = 5$  (the larger the denominator, the smaller the fraction).

Since  $x = \frac{60}{k+1}$ , solving for  $k+1$  will give us  $x$ .

$$\frac{r}{k+1} = 5 \rightarrow \frac{60/7}{k+1} = 5 \rightarrow k+1 = \frac{12}{7}. \text{ Thus, } x = \frac{60}{12/7} = \underline{\underline{35}}.$$



**MASSACHUSETTS MATHEMATICS LEAGUE  
CONTEST 6 - MARCH 2010 SOLUTION KEY**

**Team Round - continued**

- F) Which coefficients are the largest coefficients is not so obvious.  
How do we avoid calculating all the coefficients in order to decide?

The combinatorial coefficients are: **1 11 55 165 330 462 462 330 165 55 55 1**

The last six coefficients are identical to the first six since  $\binom{11}{i} = \binom{11}{j}$

whenever  $i, j \geq 0$  and  $i + j = 11$ .

Think of successive combinatorial coefficients in terms of multiplicative factors of

$$11, 5, 3, 2, 7/5, 1, \text{ etc.}$$

Successive coefficients in the expansion of  $(3a + 2b)^{11}$  introduce one more factor of 2 and one less factor of 3, i.e. a multiplicative factor of  $2/3$ .

Successive coefficients will continue to increase as long as the multiplicative factor is greater than 1.

Combining these multipliers we can determine which coefficient is the largest, without a great deal of tedious arithmetic.

The first coefficient is  $\binom{11}{0} 3^0 2^{11} = 2^{11}$ .

The composite multipliers are:

$$11(2/3) = 22/3, \quad 5(2/3) = 10/3, \quad 3(2/3) = 2, \quad 2(2/3) = 4/3, \quad (7/5)(2/3) = \underline{14/15}$$

Specifically, the 5<sup>th</sup> term is  $4/3$  times the 4<sup>th</sup>, but the 6<sup>th</sup> term is only  $14/15$  times the 5<sup>th</sup>. Therefore, the 5<sup>th</sup> and 6<sup>th</sup> terms are the two largest coefficients.

$$c_5 = \binom{11}{4} (3)^7 (2)^4 = 330 \cdot (3)^7 (2)^4 \quad \text{and} \quad c_6 = \binom{11}{5} (3)^6 (2)^5 = 462 \cdot (3)^6 (2)^5$$

$$\rightarrow c_5 - c_6 = 2^4 \cdot 3^6 (330 \cdot 3 - 462 \cdot 2) = 2^4 \cdot 3^6 (990 - 924) = 2^4 \cdot 3^6 \cdot 66 = \underline{2^5 \cdot 3^7 \cdot 11}$$

Notice we did not bother to find the numerical value of either  $c_5$  or  $c_6$  or their difference.