15.083: Integer Programming and Combinatorial Optimization
Midterm Exam

10/26/2009

Problem (1) (50 pts) Indicate whether the following italicized statements are true or false. Provide a supporting argument and/or short proof.

(a) [5 pts] Consider the problem $\max \sum_{j=1}^{n} c_j|x_j|$ subject to $\sum_{j=1}^{n} a_j|x_j| \leq b$.
[T/F]: *The problem can be modeled as a linear integer optimization problem.*

(b) [5 pts] Let $(N, \mathcal{F})$ be a matroid with associated rank function $r(\cdot)$.
[T/F]: *For all pairs of sets $T_1, T_2 \subset N$ with $|T_1| = |T_2|$, we have $r(T_1) = r(T_2)$.*

(c) [5 pts] Let $x^* \in \mathbb{R}^n$ be an optimal solution to $Z_{LP} = \min_{Ay \leq b} c^T y$ and let $x \in \mathbb{Z}^n : Ax \leq b$ be a solution obtained from x^* by a randomized rounding procedure. Suppose $E[c'x] = c'x^* = Z_{LP}$.
[T/F]: *It is possible that $E\left[(c'x - Z_{LP})^2\right] > 0$.*

(d) [5 pts] Let $P = \{x \in \mathbb{R}^7 \mid Ax = b, f'x \geq d\}$ be a polyhderon with $rank(A) = 3$, in which the inequality $f'x \geq d$ defines a face of dimension 3.
[T/F]: *The inequality $f'x \geq d$ can be deleted from P.*

(e) [8 pts] Let $N = \{1, ..., n\}$. Consider the knapsack polytope $P_{KN} = conv\{x \in \{0,1\}^n : \sum_{i=1}^{n} w_i x_i \leq b\}$.

Suppose we identify a minimal cover $C \subseteq N$ with the following properties:

- $\sum_{i \in C} w_i > b$
- $\forall j \in C : \sum_{i \in C : i \neq j} w_i \leq b$
- $\sum_{i \in C} w_i + \max\{w_j : j \in N \backslash C\} - \max\{w_i : i \in C\} \leq b$

[T/F]: *The inequality $\sum_{i \in C} x_i \leq |C| - 1$ defines a facet of P_{KN}.*

(f) [6 pts] Let $\{f_0, f_1, ..., f_m\}$ be nonlinear functions. Consider the binary nonlinear optimization problem:

$$\begin{aligned} Z_{BP} = \min & \sum_{i=1}^{n} f_0(x_i) \\ \text{subject to} & \\ \sum_{i=1}^{n} f_j(x_i) & \leq b_j, \ j = 1, \ldots, m \\ x & \in \{0,1\}^n \end{aligned}$$

[T/F]: *The problem can be reformulated as a linear integer optimization problem.*

(g) [5 pts] Suppose we carry out the lift and project method in the variables $\{x_1, ..., x_k\}$.
[T/F]: *The order of the variables in which we perform the lift and project method may lead to different polyhedra.*

(h) [6 pts] Consider the robust optimization problem:

$$
\boxed{\begin{array}{l}
Z_R = \max c'x \\
\text{subject to} \\
\begin{array}{rcll}
a'x & \le & b & \forall a \in \{0,1\}^n : w'a \le B \\
x & \ge & 0 &
\end{array}
\end{array}}
$$

[T/F]: *Calculating Z_R is NP-hard.*

(i) [5 pts] Referring to (h), let $w = e$ (vector of 1's).
[T/F]: *Calculating Z_R is polynomially solvable.*

Problem (2: A directed cut formulation of MST-25 pts) Given a undirected graph $G = (V, E)$, with $|V| = n$ and $|E| = m$, form a directed graph $D = (V, A)$ by replacing each edge $\{i, j\}$ in E by arcs (i, j) and (j, i) in A. We select a node $r \in V$ as the root node. Let $y_{ij} = 1$ if the tree contains arc (i, j) when we root the tree at node r (in other words the solution will be a tree with directed edges away from the root). Let $\delta^+(S)$ be the set of arcs going out of S. Define:

$$
\begin{aligned}
P_{dcut} &= \Big\{x \in \mathbb{R}^m : 0 \le x_e \le 1, x_e = y_{ij} + y_{ji}, \forall e \in E, \\
&\qquad \sum_{e \in A} y_e = n - 1, \sum_{e \in \delta^+(S)} y_e \ge 1, r \in S, \forall S \subset V, y_e \ge 0\Big\} \\
P_{sub} &= \Big\{x \in \mathbb{R}^m : 0 \le x_e \le 1, \forall e \in E, \sum_{e \in E} x_e = n - 1 \\
&\qquad \sum_{e \in E(S)} x_e \le |S| - 1, \forall S \subset V, S \ne \emptyset, V\Big\}
\end{aligned}
$$

Prove $P_{dcut} = P_{sub}$.

Problem (3: Comparison of relaxations for the TSP-25 pts) Given an undirected graph $G = (V, E)$, consider the following two formulations of the TSP:

$$
1)\ \boxed{\begin{array}{l}
\min \sum_{e \in E} c_e x_e \\
\text{subject to} \\
\begin{array}{rcll}
\sum_{e \in \delta(\{i\})} x_e & = & 2 & \forall i \in V \\
\sum_{e \in \delta(S)} x_e & \ge & 2 & \forall S \subset V, S \ne \emptyset, V \\
x_e & \in & \{0,1\} & \forall e \in E
\end{array}
\end{array}}
$$

$$
2)\ \boxed{\begin{array}{l}
\min \sum_{e \in E} c_e x_e \\
\text{subject to} \\
\begin{array}{rcll}
\sum_{e \in \delta(\{i\})} x_e & = & 2 & \forall i \in V \\
\sum_{e \in E(S)} x_e & \le & |S| - 1 & \forall S \subset V, S \ne \emptyset, V \\
x_e & \in & \{0,1\} & \forall e \in E
\end{array}
\end{array}}
$$

Let Z_{IP} be the common optimal cost of the two formulations. Let Z_1, Z_2 be the optimal cost of the linear relaxation of the two formulations respectively. Let Z_{D1}, Z_{D2} be the values of the Lagrangian duals if we relax the constraints $\sum_{e \in \delta(\{i\})} x_e = 2$ for all $i \ne 1$ in the two formulations. Let Z_{MST} be the cost of the minimum spanning tree with respect to the edge costs c_e. Order the values $Z_1, Z_2, Z_{IP}, Z_{D1}, Z_{D2}, Z_{MST}$.

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