MATH 251

Final Examination

April 29, 2013

FORM A

Name:————————

Student Number:————————

Section:————————

This exam has 16 questions for a total of 150 points. **In order to obtain full credit for partial credit problems, all work must be shown. For other problems, points might be deducted, at the sole discretion of the instructor, for an answer not supported by a reasonable amount of work.** The point value for each question is in parentheses to the right of the question number. A table of Laplace transforms is attached as the last page of the exam.

YOU MAY NOT USE A CALCULATOR ON THIS EXAM. PLEASE TURN OFF AND PUT AWAY YOUR CELL PHONE AND ALL OTHER MOBILE DEVICES.

Do not write in this box.

1 - 10:——————

11:——————

12:——————

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16:——————

Total:——————

1. (6 points) Consider the autonomous equation

$$y' = y^2(25 - y^2).$$

Suppose $y(77) = \alpha$. What values of α guarantee that $\lim\limits_{t \to \infty} y(t) = 0$?

(a) $-5 < \alpha < 5$

(b) $-5 < \alpha \leq 0$

(c) $0 \leq \alpha < 5$

(d) $-\infty < \alpha < \infty$

2. (6 points) Which initial or boundary value problem below is guaranteed to have a unique solution according to the Existence and Uniqueness theorems?

(a) $\cos(t)y' + ty = 0$, $y(\frac{\pi}{2}) = -1$.

(b) $y'' + t^3 y' + \dfrac{1}{(t-1)^2}y = 0$, $y(1) = -2$, $y'(1) = 6$.

(c) $t^2 y'' - 2ty' + 3y = 0$, $y(\sqrt{2}) = \pi^2$, $y'(\sqrt{2}) = e^3$.

(d) $y'' + y = 0$, $y'(0) = 0$, $y'(\pi) = 0$.

3. (6 points) Solve explicitly the following initial value problem:

$$y' = e^{2y} \cos(t), \qquad y\left(\frac{\pi}{2}\right) = 0.$$

(a) $y(t) = \ln\left(\dfrac{1}{\sqrt{3 - 2\sin(t)}}\right)$

(b) $y(t) = \ln\left(\sqrt{3 - 2\sin(t)}\right)$

(c) $y(t) = \sqrt{\ln(3 - 2\sin(t))}$

(d) $y(t) = \ln\left((3 - 2\sin(t))^2\right)$

4. (6 points) Suppose the Laplace transform of $f(t)$ is $F(s) = \dfrac{16}{s^2 + 3}$. What is the Laplace transform $\mathcal{L}\{te^t f(t)\}$?

(a) $\dfrac{16}{s^2(s - 1)(s^2 + 3)}$

(b) $\dfrac{16}{(s - 1)^2(s^2 + 3)}$

(c) $\dfrac{32(s - 1)}{((s - 1)^2 + 3)^2}$

(d) $-\dfrac{32s}{((s - 1)^2 + 3)^2}$

5. (6 points) Find the inverse Laplace transform of

$$F(s) = \frac{se^{-2s}}{s^2 + 8s + 25}.$$

(a) $f(t) = u_2(t)\left(e^{-4t+8}\cos(3t-6) - \dfrac{4}{3}e^{-4t+8}\sin(3t-6)\right)$

(b) $f(t) = u_2(t)\left(e^{4t-8}\cos(3t-6) + \dfrac{4}{9}e^{4t-8}\sin(3t-6)\right)$

(c) $f(t) = \delta(t-2)e^{4t}\cos(3t)$

(d) $f(t) = u_2(t)e^{-4t+8}\cos(3t-6)$

6. (6 points) Which of the following systems of linear equations has the property that all of its solutions converge to its critical point at $(0,0)$ as $t \to +\infty$?

(a) $\mathbf{x}' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\mathbf{x}$

(b) $\mathbf{x}' = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}\mathbf{x}$

(c) $\mathbf{x}' = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}\mathbf{x}$

(d) $\mathbf{x}' = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\mathbf{x}$

7. (6 points) Given that the point $(-1, 2)$ is a critical point of the nonlinear system of equations

$$\begin{array}{rcl} x' & = & xy + y \\ y' & = & 2x^2 + y^2 - 3y \end{array}.$$

Which of the following statements is **true**?

(a) This critical point is asymptotically stable.

(b) This critical point is (neutrally) stable.

(c) This critical point is a node.

(d) This critical point is a saddle point.

8. (6 points) Consider the two linear partial differential equations.

$$\begin{array}{ll} (I) & u_{tt} + x u_{xt} = u_x \\ (II) & u_x + t u_{xx} + 5 u_{xt} = 0 \end{array}$$

Use the substitution $u(x, t) = X(x)T(t)$, where $u(x, t)$ is not the trivial solution, and attempt to separate each equation into two ordinary differential equations. Which statement below is true?

(a) Neither equation is separable.

(b) Only (I) is separable.

(c) Only (II) is separable.

(d) Both equations are separable.

9. (6 points) Find the steady-state solution, $v(x)$, of the heat conduction problem with nonhomogeneous boundary conditions:

$$10u_{xx} = u_t, \qquad 0 < x < 5, \qquad t > 0$$
$$u_x(0,t) = 1, \qquad u(5,t) = 10.$$

(a) $v(x) = \dfrac{9}{5}x + 1$

(b) $v(x) = x + 5$

(c) $v(x) = 10x + 1$

(d) $v(x) = 10$

10. (6 points) Each graph below shows a single period of a certain periodic function. Which function will have a Fourier series consisting only of sine terms?

(a)

(b)

(c)

(d)
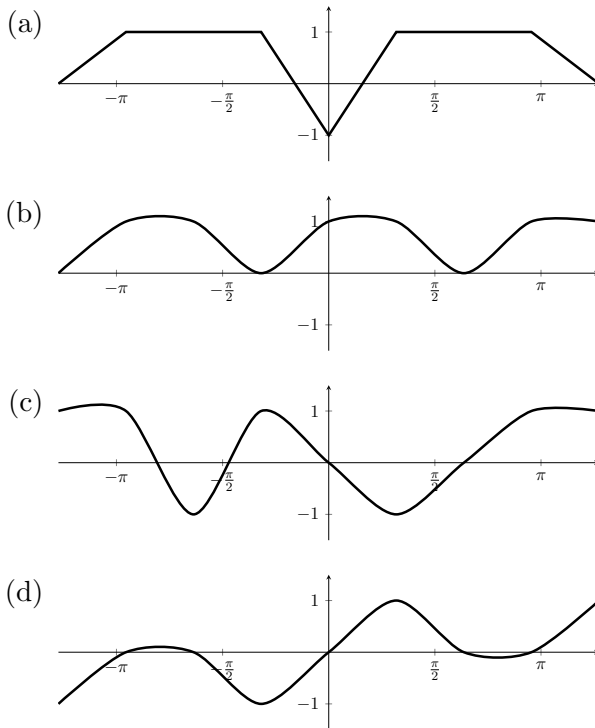
11. (15 points) True or false:

   (a) (3 points) $t^2 y' = e^t \sin(5t) y$ is a first order equation that is linear and separable.

   (b) (3 points) A suitable integrating factor that could be used to solve the linear equation

   $$7y' + \frac{1}{t}y = \ln(3t)$$

   is $\mu(t) = \frac{1}{7}t$.

   (c) (3 points) $y^2 e^{x-1} + 3x^2 + \left(2ye^{x-1} + x\right) y' = 0$ is an exact equation.

   (d) (3 points) Given that $y = t^2 - t$ is a solution of $y'' + y = g(t)$. Then $y_1 = 5t^2 - 5t$ is also a solution of the same equation.

   (e) (3 points) Using the formula $u(x,t) = X(x)T(t)$, the boundary conditions $u(0,t) = 0$ and $u_x(2\pi, t) = 10$ can be rewritten as $X(0) = 0$ and $X'(2\pi) = 10$.

12. (16 points) Consider the two-point boundary value problem

$$X'' + \lambda X = 0, \qquad X(0) = 0, \qquad X'(\pi) = 0.$$

(a) (12 points) Find all **positive** eigenvalues and corresponding eigenfunctions of the boundary value problem.

(b) (4 points) Is $\lambda = 0$ an eigenvalue of this problem? If yes, find its corresponding eigenfunction. If no, briefly explain why it is not an eigenvalue.

13. (16 points) Let $f(x) = \begin{cases} x, & 0 \le x < 1 \\ 1, & 1 < x < 2 \end{cases}$.

(a) (4 points) Consider the **odd** periodic extension, of period $T = 4$, of $f(x)$. Sketch 3 periods, on the interval $-6 < x < 6$, of this odd periodic extension.

(b) (3 points) Find $\dfrac{a_0}{2}$, the constant term of the Fourier series of the periodic function described in (a).

(c) (5 points) Which of the integrals below can be used to find the Fourier sine coefficients of the odd periodic extension in (a)?

(i) $b_n = \dfrac{1}{2}\left(\displaystyle\int_{-2}^{-1} \sin\dfrac{n\pi x}{2}\,dx + \int_{-1}^{1} x\sin\dfrac{n\pi x}{2}\,dx + \int_{1}^{2}\sin\dfrac{n\pi x}{2}\,dx\right)$

(ii) $b_n = \left(\displaystyle\int_{-2}^{-1}\sin\dfrac{n\pi x}{2}\,dx + \int_{-1}^{1} x\sin\dfrac{n\pi x}{2}\,dx + \int_{1}^{2}\sin\dfrac{n\pi x}{2}\,dx\right)$

(iii) $b_n = \left(\displaystyle\int_{0}^{1} x\sin\dfrac{n\pi x}{2}\,dx + \int_{1}^{2}\sin\dfrac{n\pi x}{2}\,dx\right)$

(iv) $b_n = \dfrac{1}{2}\left(\displaystyle\int_{0}^{1} x\sin\dfrac{n\pi x}{2}\,dx + \int_{1}^{2}\sin\dfrac{n\pi x}{2}\,dx\right)$

(d) (4 points) To what value does the Fourier series of this odd periodic extension converge at $x = -6$? At $x = \dfrac{7}{2}$?

14. (12 points) Consider the following two heat conduction initial-boundary value problems

I.
$$5u_{xx} = u_t, \qquad 0 < x < 2, \qquad t > 0$$
$$u(0, t) = 10, \qquad u(2, t) = 20,$$
$$u(x, 0) = 30.$$

II.
$$5u_{xx} = u_t, \qquad 0 < x < 2, \qquad t > 0$$
$$u_x(0, t) = 0, \qquad u_x(2, t) = 0,$$
$$u(x, 0) = 30.$$

(a) (3 points) Which problem (**I** or **II**) models the temperature distribution of a rod with both ends kept at certain constant temperatures?

(b) (3 points) What is the physical meaning of the boundary conditions of the other heat conduction problem?

(c) (6 points) Suppose $u_1(x, t)$ is the solution of problem **I** and $u_2(x, t)$ is the solution of problem **II**. Compare the temperatures at the middle of the rod (that is, at $x = 1$) as $t \to \infty$.

(i) $\lim\limits_{t \to \infty} u_1(1, t) > \lim\limits_{t \to \infty} u_2(1, t)$

(ii) $\lim\limits_{t \to \infty} u_1(1, t) < \lim\limits_{t \to \infty} u_2(1, t)$

(iii) $\lim\limits_{t \to \infty} u_1(1, t) = \lim\limits_{t \to \infty} u_2(1, t)$

(iv) There is not enough available information to compare them.

15. (16 points) Suppose the displacement $u(x,t)$ of a piece of flexible string that has both ends firmly fixed in places is given by the initial-boundary value problem

$$16u_{xx} = u_{tt}, \qquad 0 < x < 4, \qquad t > 0$$
$$u(0,t) = 0, \qquad u(4,t) = 0,$$
$$u(x,0) = f(x) = \sqrt{5}\sin(\pi x) - 2\sqrt{7}\sin(\tfrac{5}{2}\pi x),$$
$$u_t(x,0) = 0.$$

(a) (12 points) State the general form of its solution. Then find the particular solution of the initial-boundary value problem.

(b) (2 points) TRUE or FALSE: As $t \to +\infty$, the solution $u(x,t)$ will go to 0 for all x.

(c) (2 points) TRUE or FALSE: If $f(x) = 0$, then $u(x,t) = 0$ is the unique solution.

16. (15 points) Suppose the temperature distribution function $u(x,t)$ of a rod that has both ends kept at constant zero degrees temperature is given by the initial-boundary value problem

$$
\begin{aligned}
u_{xx} &= u_t, & 0 < x < 40, & \quad t > 0, \\
u(0,t) &= 0, & u(40,t) = 0, & \quad t > 0, \\
u(x,0) &= 50, & 0 < x < 40.
\end{aligned}
$$

(a) (13 points) State the general form of its solution. Then find the particular solution of the initial-boundary value problem.

(b) (2 points) What is $\lim\limits_{t\to\infty} u(5,t)$?