3.3 Derivatives of Trigonometric Functions

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1. Overview

You need to memorize the derivatives of all the trigonometric functions. If you don't get them straight before we learn integration, it will be much harder to remember them correctly.

$$\begin{aligned}
(\sin x)' &= \cos x \\
(\cos x)' &= -\sin x \\
(\tan x)' &= \sec^2 x \\
(\sec x)' &= \sec x \tan x \\
(\csc x)' &= -\csc x \cot x \\
(\cot x)' &= -\csc^2 x
\end{aligned}$$

A couple of useful limits also appear in this section:

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$$

$$\lim_{\theta \to 0} \frac{\cos \theta - 1}{\theta} = 0$$

2. Examples

1.) Find the derivative of

$$g(x) = 4 \sec t + \tan t$$

We use the derivatives of sec and tan:

$$g'(x) = 4 \sec t \tan t + \sec^2 t$$

2.) Find the derivative of

$$y = \frac{1 + \sin x}{x + \cos x}$$

Since y is the quotient of two functions we first use the quotient rule:

$$y' = \frac{(1 + \sin x)'(x + \cos x) - (1 + \sin x)(x + \cos x)'}{(x + \cos x)^2}$$

Evaluating the derivatives we get:

$$y' = \frac{(\cos x)(x + \cos x) - (1 + \sin x)(1 - \sin x)}{(x + \cos x)^2}$$

Simplifying the numerator:

$$\begin{aligned}
y' &= \frac{(x \cos x + \cos^2 x) - (1 - \sin^2 x)}{(x + \cos x)^2} \\
&= \frac{x \cos x + \cos^2 x - 1 + \sin^2 x}{(x + \cos x)^2}
\end{aligned}$$

We now use the trig identity $\sin^2 + \cos^2 = 1$:

$$y' = \frac{x\cos x - 1 + 1}{(x + \cos x)^2} = \frac{x\cos x}{(x + \cos x)^2}$$

2.) Find the derivative of

$$y = x\sin x \cos x$$

Since y is a product of functions we'll use the product rule. We have to use it twice, actually, because y is a product of three functions. Applying it once, we get:

$$y' = (x)'(\sin x \cos x) + (x)(\sin x \cos x)'$$

And now applying it to the product $\sin x \cos x$, we get:

$$y' = (x)'(\sin x \cos x) + (x)((\sin x)'(\cos x) + (\sin x)(\cos x)')$$

Now taking derivatives, we get:

$$y' = \sin x \cos x + x((\cos x)(\cos x) + (\sin x)(-\sin x))$$

And simplifying:

$$y' = \sin x \cos x + x(\cos^2 x - \sin^2 x) = \sin x \cos x + x\cos^2 x - x\sin^2 x$$

3.) Find tangent to the curve at the point $(0, 1)$.

$$y = \frac{1}{\sin x + \cos x}$$

The slope of the tangent line will be the value of the derivative at $x = 0$. So the first thing we do is compute y'. We use the quotient rule:

$$y' = \frac{(1)'(\sin x + \cos x) - (1)(\sin x + \cos x)'}{(\sin x + \cos x)^2}$$

Computing derivatives:

$$y' = \frac{0 - (\cos x + (-\sin x))}{(\sin x + \cos x)^2}$$

And simplifying:

$$y' = \frac{\sin x - \cos x}{(\sin x + \cos x)^2}$$

So the slope of the tangent is:

$$m_{\tan} = y'(0) = \frac{\sin 0 - \cos 0}{(\sin 0 + \cos 0)^2} = \frac{0 - 1}{(0 + 1)^2} = -1$$

Using the point-slope form of the line, we can say that the tangent line is:

$$(y - 1) = (-1)(x - 0)$$

i.e. the tangent line is:

$$y = 1 - x$$

4.) Prove $\frac{d}{dx}\sec x = \sec x \tan x$

Remember that sec is defined to be $\frac{1}{\cos}$. So we can use the quotient rule to find the derivative of sec:

$$\begin{aligned}
\tfrac{d}{dx}\sec x &= \left(\frac{1}{\cos x}\right)' \\
&= \frac{(1)'(\cos x) - (1)(\cos x)'}{(\cos x)^2} \\
&= \frac{0 - (-\sin x)}{(\cos x)^2} \\
&= \frac{\sin x}{(\cos x)(\cos x)} \\
&= \left(\frac{\sin x}{\cos x}\right)\left(\frac{1}{\cos x}\right) \\
&= \tan x \sec x
\end{aligned}$$

5.) Compute the following limit:

$$\lim_{x\to 0}\frac{\sin 4x}{\sin 6x}$$

If we plug in 0, both the top and the bottom are zero, so we hope to use some trick to evaluate the limit. We would like to use the fact that:

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$$

So we divide top and bottom by x:

$$\lim_{x\to 0}\frac{\sin 4x}{\sin 6x} = \lim_{x\to 0}\frac{\left(\frac{\sin 4x}{x}\right)}{\left(\frac{\sin 6x}{x}\right)}$$

Looking at the top fraction, θ is $4x$, because that is what is inside the sine function. In order to have $\frac{\sin\theta}{\theta}$, we need to have $\sin 4x$ divided by $4x$, not x. So we multiply and divide by 4:

$$\frac{\sin 4x}{x} = 4\left(\frac{\sin 4x}{4x}\right)$$

We do the same thing for the bottom fraction:

$$\frac{\sin 6x}{x} = 6\left(\frac{\sin 6x}{6x}\right)$$

Putting it back together, we have:

$$\lim_{x\to 0}\frac{\left(\frac{\sin 4x}{x}\right)}{\left(\frac{\sin 6x}{x}\right)} = \lim_{x\to 0}\frac{4\left(\frac{\sin 4x}{4x}\right)}{6\left(\frac{\sin 6x}{6x}\right)}$$

Now we can take the limit as $x \to 0$:

$$\lim_{x\to 0}\frac{4\left(\frac{\sin 4x}{4x}\right)}{6\left(\frac{\sin 6x}{6x}\right)} = \frac{4\cdot 1}{6\cdot 1} = \frac{2}{3}$$

So we can conclude:
$$\lim_{x\to 0}\frac{\sin 4x}{\sin 6x}=\frac{2}{3}$$

Note: After having done several examples like this one you might be tempted to do the following:
$$\frac{\sin 4x}{\sin 6x}=\frac{4\sin x}{6\sin x}=\frac{4}{6}=\frac{2}{3}$$
Do NOT do that!! This is very wrong. You cannot factor the 4 out of $\sin 4x$!

5.) Compute the following limit:
$$\lim_{x\to 0}\frac{\cos x-1}{\sin x}$$

If we plug in 0, both the top and the bottom are zero, so we hope to use some trick to evaluate the limit. We would like to use the fact that:
$$\lim_{x\to 0}\frac{\sin x}{x}=1 \quad \text{and} \quad \lim_{x\to 0}\frac{\cos x-1}{x}=0$$
So we divide top and bottom by x:
$$\lim_{x\to 0}\frac{\cos x-1}{\sin x}=\lim_{x\to 0}\frac{\left(\frac{\cos x-1}{x}\right)}{\left(\frac{\sin x}{x}\right)}$$
Now we can take the limit as $x\to 0$:
$$\lim_{x\to 0}\frac{\left(\frac{\cos x-1}{x}\right)}{\left(\frac{\sin x}{x}\right)}=\frac{0}{1}=0$$
So we can conclude:
$$\lim_{x\to 0}\frac{\cos x-1}{\sin x}=0$$

6.) Compute the following limit:
$$\lim_{t\to 0}\frac{\sin^2 3t}{t^2}$$

Again, if we plug in $t=0$ we will get zero over zero, so we must use a trick. Notice that in this case the top and bottom are both squared:
$$\frac{\sin^2 3t}{t^2}=\frac{(\sin 3t)^2}{t^2}=\left(\frac{\sin 3t}{t}\right)^2$$
Now we want to have $\sin\theta$ over θ, so we divide and multiply by 3:
$$\left(\frac{\sin 3t}{t}\right)^2=\left(3\frac{\sin 3t}{3t}\right)^2=9\cdot\left(\frac{\sin 3t}{3t}\right)^2$$
Now we can take the limit:
$$\lim_{t\to 0}9\cdot\left(\frac{\sin 3t}{3t}\right)^2=9\cdot(1^2)=9$$
So we can conclude:
$$\lim_{t\to 0}\frac{\sin^2 3t}{t^2}=9$$