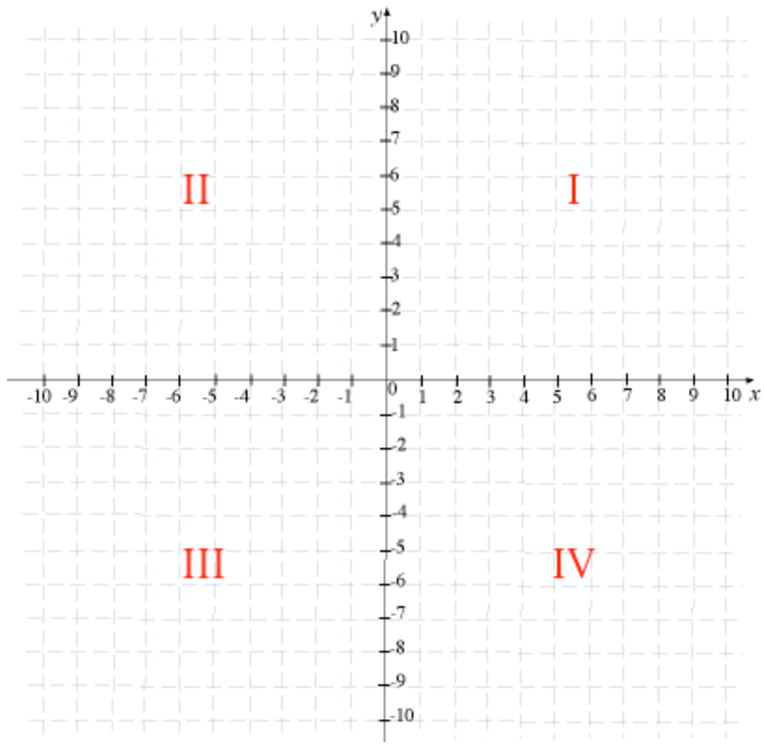
CHAPTER 1
Linear Equations

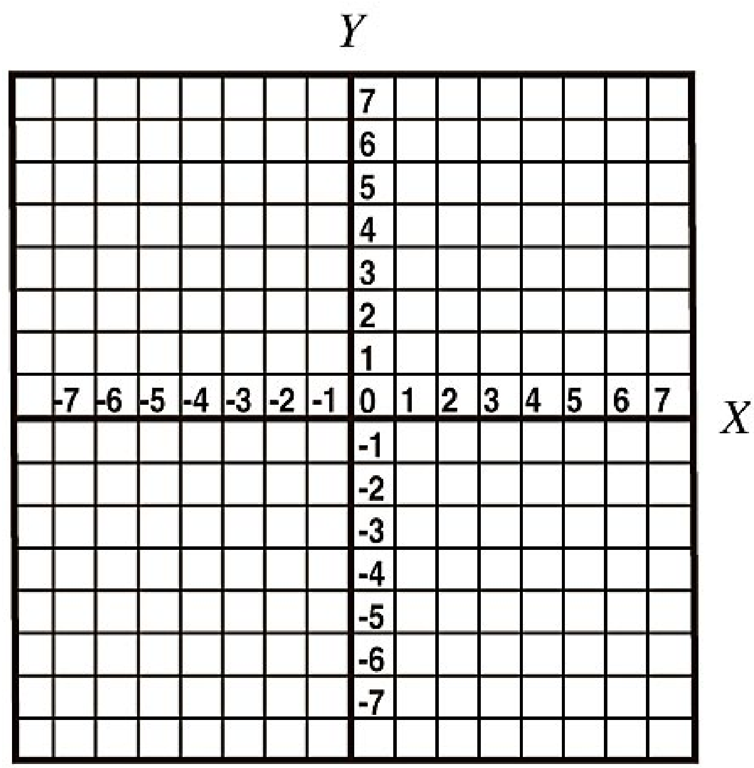
1.1. Lines

The rectangular coordinate system is also called the **Cartesian plane**. It is formed by two real number lines, the horizontal axis or x**-axis**, and the vertical axis or y**-axis**. The two axes intersect at a right angle at the **origin** divide the plane into four **quadrants**, numbered I, II, III, IV. Each point in the plane can be represented by a **coordinate pair**, (x, y).

Exercise 102. Plot the points $(1, 6), (-4, 5), (5, 0)$, and $(-3, -5)$ in the coordinate system below.

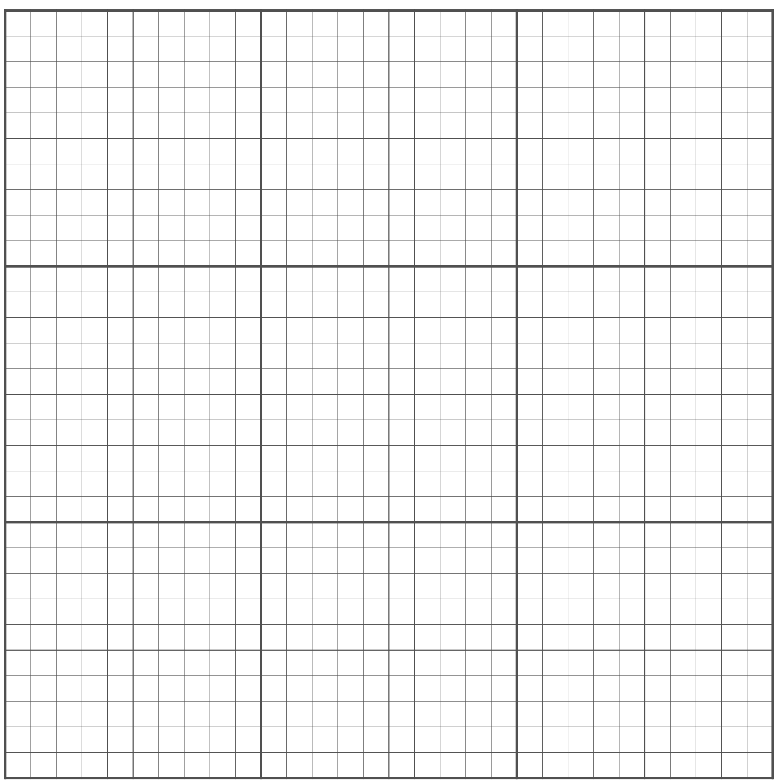


Exercise 103. Graph the equation $y = 2x + 3$ in the coordinate system below. Make a table first with coordinate pairs (x, y), then plot the points.



Exercise 104. Data on US export of goods and services between 1997 and 2000, in billions of dollars, is shown below. Plot the data in a coordinate system and connect the points with line segments.

Year	Dollars (billions)
1997	935.0
1998	932.7
1999	957.4
2000	1065.7



Definition of a Relation

A **relation** is a set of ordered pars (x, y). An example of a relation is the equation $x^2 + y^2 = 1$, whose graph is a circle with radius 1, centered at the origin.

Definition of a Function

A **function** is a rule f that assigns exactly one value $f(x)$, read "f of x" to each value of the input variable x. $y = f(x)$ is called the output variable.

Domain

The **domain** of a function $f(x)$ is the set of all allowed values of the input variable x.

Range

The **range** of a function $f(x)$ is the set of all values of the output variable $y = f(x)$.

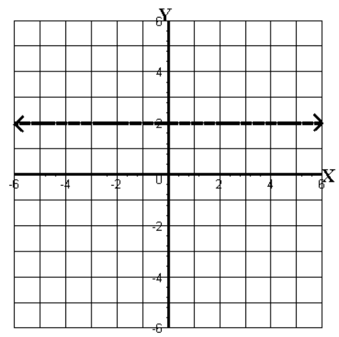
Independent and Dependent Variables

The input variable, usually called x is the **independent variable** as its value can be chosen freely within the domain. The output value, $y = f(x)$ is called the **dependent variable** as its value depends on the choice of x.

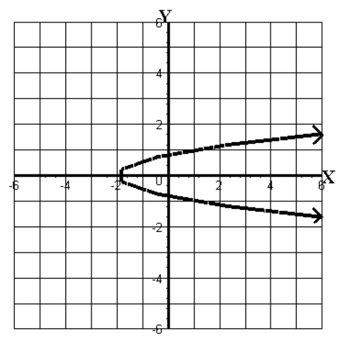
The Vertical Line Test

If a vertical line intersects the graph of an relation in more than one place, then the relation is a NOT a function.

Exercise 105. Find the domain and range of each relation. Determine whether the relation is also a function.



(a) domain ____________ range ____________
function? yes/no (circle one)



(b) domain ____________ range ____________
function? yes/no (circle one)

Function Notation $y = f(x)$

The function notation is used to emphasize the fact that the value of y depends on the value of x. If $f(x) = 3x - 5$, $f(4)$ means the the y-value when $x = 4$, i.e. $f(4) = 3(4) - 5 = 12 - 5 = 7$. Notice how x was replaced by (4) **with parentheses** in $f(x)$. Omitting the parenthesis often leads to errors when evaluating functions.

Exercise 106. If $f(x) = x^2 - 3$, find,

(a) $f(1)$

(b) $f(a)$

(c) $f(t+1)$

Exercise 107. Let x be the number of miles, in thousands, that a particular tire is driven, and $f(x) = 2(1 - x/40)$ be the depth of the tread in centimeters. Find the domain of this function and sketch its graph throughout the domain.

Exercise 108. Let p be the price, in hundreds of dollars, of a particular brand of stereo and $S(p) = 30(16 - p^2/4)$ be the number sold each year by Andrew's Department Store. Find the domain of this function and sketch its graph throughout the domain.

Linear equations in two variables are graphed as straight lines, hence the name linear. There are different forms of writing a linear equation. One of the most common forms is the slope-intercept form listed below.

Slope-Intercept Form

$$y = mx + b \tag{1.1}$$

where m is the slope of the line and $(0, b)$ is the y-intercept, i.e. the point where the line crosses the y-axis. The slope measures the steepness of the line. For a line going through the points (x_1, y_1) and (x_2, y_2), the slope is:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x} = \frac{\text{rise}}{\text{run}}$$

The general or standard form of a linear equation is listed next:

The General Form of an Equation of a Line

$$Ax + By = C, \tag{1.2}$$

where A, B, and C are constants and A and B not both zero.

The point-slope form of a linear equation is used when one point and the slope of a line is given. It is listed next:

The Point-Slope Form of a Line

$$y - y_1 = m(x - x_1), \tag{1.3}$$

where (x_1, y_1) is a point on the line, and m is its slope.

The Horizontal Line

A horizontal line has a zero slope. It is on the form:

$$y = a, \tag{1.4}$$

where a is any number.

The Vertical Line

The slope of a vertical line is undefined. It is on the form:

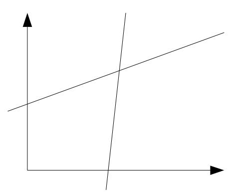
$$x = a, \tag{1.5}$$

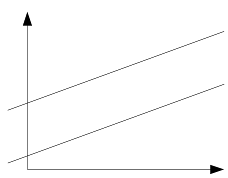
where a is any number.

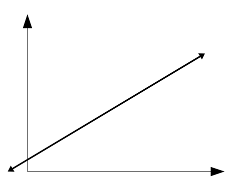
1.2. Pairs of Lines

There are only three possibilities when graphing two linear equations:

1. The lines **intersect** in exactly one point.
2. The lines do not intersect at all, i.e. they are **parallel**.
3. The two lines are **coincident**, i.e. they are actually the same line and have the same graph.


Intersecting Lines


Parallel Lines


Coincident Lines

Parallel Lines
Two lines are parallel if they have the **same slope** and no points in common

Perpendicular Lines
Two lines, $y = m_1x + b_1$ and $y = m_2x + b_2$ are perpendicular if their slopes are negative reciprocals, i.e. $m_1 \cdot m_2 = -1$.

Exercise 109. A line is going through the points $(2, 3)$ and $(6, 3)$.

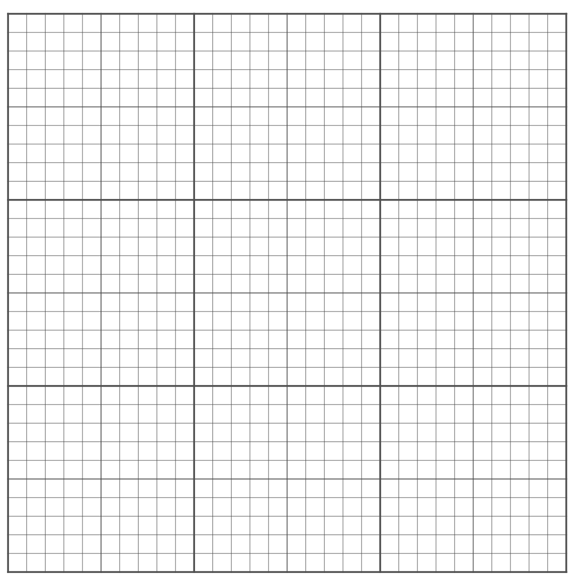
(a) Find the slope.

(b) Find the equation of the line on slope-intercept form.

(c) Find the x-intercept.

(d) Find the y-intercept.

(e) Graph the equation below.



Exercise 110. Find the equation of a line perpendicular to $2y - 3x = 6$, going through the point $(2, 6)$, on general form.

Exercise 111. Find the equation of a vertical line, going through the point $(-7, 3)$.

Exercise 112. Find the equation of a horizontal line, going through the point $(-1, 5)$.

1.3. Applications to Business and Economics

In this section we will study linear models of real life situations.

Linear Cost Functions

A linear cost function has the form

$$C(x) = mx + b \tag{1.6}$$

where x is the **number of items**, m, the slope is the **marginal cost** or **direct cost per item**, mx is the **variable cost** and b is the **fixed cost**.

Exercise 113. For the cost function, $C(x) = 1.6x + 1600$, find

(a) the marginal cost.

(b) the fixed cost.

(c) the total cost for 20 items.

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Linear Revenue Functions

A linear revenue function has the form

$$R(x) = mx \tag{1.7}$$

where x is the **number of items sold** and m, the slope is the **price per item**.

Suppose the cost function is $C(x)$ and the revenue function is $R(x)$. If the cost is higher than the revenue, a business is running at a loss. If, on the other hand, the revenue is higher than the cost, the business is making a profit. The point where the cost equals the revenue is called the break-even point.

The Break-Even Point
For a cost function, $C(x)$ and a revenue function, $R(x)$, the number n_0 for which $R(n_0) = C(n_0)$ is known as the **break-even quantity**, and the point of intersection of the graphs $y = C(x)$ and $y = R(x)$, $(n_0, C(n_0)) = (n_0, R(n_0))$ is known as the **break-even point**, i.e.

$$R(x) = C(x) \tag{1.8}$$

when $x = n_0$.

The profit is is the difference between the revenue and the cost.

Profit
Profit is defined as revenue minus cost,

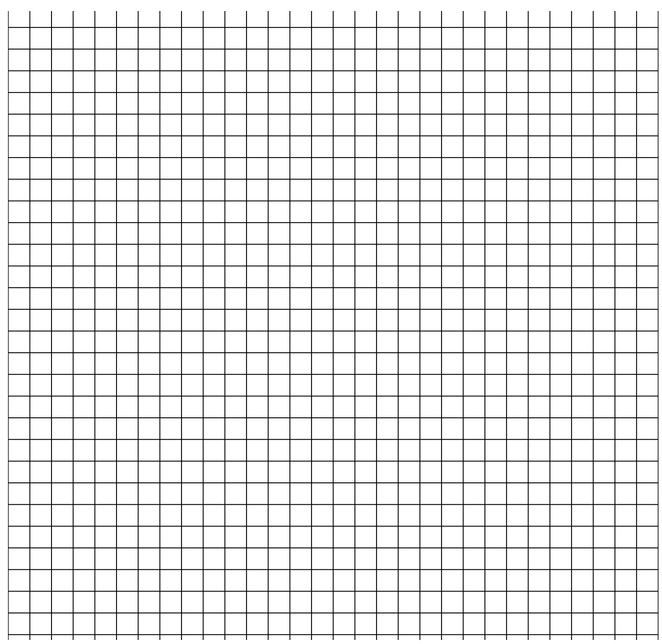
$$P(x) = R(x) - C(x) \tag{1.9}$$

Exercise 114. A rental firm has purchased a new ditchdigger for $\$36,000$ and expects that maintenance costs will average 75 cents per actual hour of use of the machine. The company plans to rent the ditchdigger for $200 per day plus $5 per hour of actual use. Assume that the ditchdigger will be rented for 150 days during the first year. Let x be the number of hours of actual use the ditchdigger gets during during that first year.

(a) What is the cost and revenue functions for purchasing and renting the machine in terms of x?

(b) What is the profit function in terms of x? How many hours must the machine be used to break even during the first year?

(c) Graph the cost, revenue and profit functions on the same set of axes.



Supply and Demand

The supply of a given item is a function of its price. As the price increases, more producers will interested in producing an item. The opposite is true if the price decreases. Demand on the other hand will increase if the price is lowered, and decrease if an item becomes more expensive. The market reaches an equilibrium when supply equals demand. Here we will model supply and demand by linear functions.

The Equilibrium Point

If $D(p)$ is the demand function, and $S(p)$ is the supply function for a given commodity as a function of its price p, then the **equilibrium price** is the price p_0 for which

$$D(p_0) = S(p_0). \tag{1.10}$$

The **equilibrium quantity** is the value of $D(p_0) = S(p_0)$. The equilibrium point is the ordered pair $(p_0, D(p_0)) = (p_0, S(p_0))$.

Exercise 115. The Sun Software Company produces tutorial software for finite mathematics. The supply and demand functions for this software are $S(p) = p - 12$ and $D(p) = 50 - 0.5p$, where p is in dollars, and $S(p)$ and $D(p)$ are measured in hundreds of units.

(a) Find the equilibrium point for the software.

(b) Find the equilibrium quantity for the software.

(c) Graph the supply and demand functions on the same set of axes and label the equilibrium point as well as the shortage and surplus regions.?