ÜBUNGSBLATT 3

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Aufgabe 1: Path integral for free particle and harmonic oscillator (10 Punkte)

a) Free particle: The Lagrangian is given by $L = \frac{1}{2}m\Big(\frac{dx}{dt}\Big)^2$. The amplitude $U(x_f, t_f; x_i, t_i)$ traveling from x_i at t_i to x_f at t_f is given by the following path integral

$$\begin{aligned} U(x_f, t_f; x_i, t_i) &= \int \mathcal{D}x(t) \exp\Big[\frac{i}{\hbar}\int_{t_i}^{t_f} \frac{1}{2}m\Big(\frac{dx}{dt}\Big)^2 dt\Big] \\ &= \lim_{N\to\infty}\Big(\frac{m}{2\pi i\hbar\varepsilon}\Big)^{\frac{N+1}{2}} \int dx_1 \cdots dx_N \exp\Big[\frac{im}{2\hbar\varepsilon}\sum_{j=0}^{N}(x_{j+1}-x_j)^2\Big] \end{aligned}$$

where $x_{N+1} = x_f,\ x_0 = x_i$ and $\varepsilon = \frac{t_f - t_i}{N+1}$. Show that

$$U(x_f, t_f; x_i, t_i) = \sqrt{\frac{m}{2\pi i\hbar(t_f - t_i)}} \exp\Big[\frac{im}{2\hbar(t_f - t_i)}(x_f - x_i)^2\Big]. \tag{1}$$

The following integral formula may be useful

$$\int_{-\infty}^{\infty} du e^{-a(x-u)^2} e^{-b(u-y)^2} = \sqrt{\frac{\pi}{a+b}} \exp\Big[-\frac{ab}{a+b}(x-y)^2\Big].$$

b) Show that the resulting amplitude (1) can be written as the form

$$U(x_f, t_f; x_i, t_i) = f(t_f - t_i) \exp\Big(\frac{i}{\hbar}S_{cl}\Big) \tag{2}$$

where S_{cl} is a classical action for a given boundary condition and the function $f(t)$ is determined by the fact that U satisfies the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t_f}U(x_f, t_f; x_i, t_i) = HU(x_f, t_f; x_i, t_i).$$

c) Harmonic oscillator: The Lagrangian is given by $L = \frac{1}{2}m\Big(\frac{dx}{dt}\Big)^2 - \frac{1}{2}m\omega^2 x^2$. Indeed Eq.(2) is generally true as far as the action is at most quadratic in

$x(t)$. Using this fact, show that the amplitude for the harmonic oscillator is given by

$$U(x_f, t_f; x_i, t_i) = \sqrt{\frac{m\omega}{2\pi i\hbar \sin\omega(t_f - t_i)}} \exp\left[\frac{im\omega}{2\hbar \sin\omega(t_f - t_i)}\left\{(x_i^2 + x_f^2)\cos\omega(t_f - t_i) - 2x_i x_f\right\}\right].$$

Aufgabe 2: Scalar QED (Problem 9.1 in Peskin and Schroeder) (10 Punkte)

Consider the theory of a complex scalar field ϕ interacting with the electromagnetic field A_μ. The Lagrangian is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^*(D^\mu\phi) - m^2\phi^*\phi,$$

where $D_\mu = \partial_\mu + ieA_\mu$ is the gauge-covariant derivative.
a) Use the functional method to show that the propagator of the complex scalar field is the same as that of a real field:

 $$= \frac{i}{p^2 - m^2 + i\epsilon}.$$

Also derive the Feynman rules for the interactions between photons and scalar particles; you should find

$$= -ie(p+p')^\mu, \qquad\qquad = 2ie^2 g^{\mu\nu}.$$

b) Compute the contribution of the charged scalar to the photon vacuum polarization, using dimensional regularization. Note that there are two diagrams. To put the answer into the expected form,

$$\Pi^{\mu\nu}(q^2) = (g^{\mu\nu}q^2 - q^\mu q^\nu)\Pi(q^2),$$

it is useful to add the two diagrams at the beginning, putting both terms over a common denominator before introducing a Feynman parameter. Show that, for $-q^2 \gg m^2$, the charged boson contribution to $\Pi(q^2)$ is exactly 1/4 that of a virtual electron-positron pair.