Math 540A Fall 2008 Test #2, Take 1

Each question except for #1, 2, and 3 is worth 20 points.

No notes or books, no calculators, no loitering, no smoking, no walking on the grass, no feeding the animals, no parking, no stopping or standing, no dumping, no running, no fishing, no vacancy.

#1) What is your name?

#2) What is your email address? (I'll email you your score.)

#3) Please turn the ringer off on your cell phone.

#4) Let A, B be abelian groups. Prove that $A \times B$ is abelian.
Let $(a_1, b_1), (a_2, b_2) \in A \times B$. Then $(a_1, b_1) \cdot (a_2, b_2) = (a_1 \cdot a_2, b_1 \cdot b_2) = (a_2 \cdot a_1, b_2 \cdot b_1) = (a_2, b_2) \cdot (a_1, b_1)$. Therefore $A \times B$ is abelian.

#5) Let G be an group of order $175 = 5^2 \cdot 7$. Prove that G is abelian.
Let n_5 be the number of Sylow 5-subgroups, and let n_7 be the number of Sylow 7-subgroups. By Sylow's theorem, we have that $n_5 \equiv 1 (\text{mod} 5)$ and $n_5 | 7$ and $n_7 \equiv 1 (\text{mod} 7)$ and $n_7 | 25$. Hence $n_5 = n_7 = 1$. Let H be the unique Sylow 5-subgroup, and let K be the unique Sylow 7-subgroup. Then $H, K \trianglelefteq G$ and $|H| = 25$ and $|K| = 7$. Since 25 and 7 are coprime, we have $H \cap K = 1$. Hence $HK \cong H \times K$. Since K has prime order, K is abelian. Since the order of H is the square of a prime, H is abelian. Hence $H \times K$ is abelian. But $|HK| = \dfrac{|H| \cdot |K|}{|H \cap K|} = 175 = |G|$. So $G = HK \cong H \times K$. Therefore G is abelian.

#6) Let H be a proper subgroup of a finite group G and for each g in G,
let $H^g = \{g^{-1}hg \mid h \in H\}$. Let $K = \bigcup_{g \in G} H^g$. Prove that $K \neq G$.
Let $A = \{H^g \mid g \in G\}$. (Note that $H^g = g^{-1}Hg$. The appearance of conjugates suggests that we should be thinking about the action of G on A by conjugation.) Then $|A| = |G : N_G(H)|$. (That is, the number of conjugates of H equals the index of the normalizer of H.) Now, $H \leq N_G(H)$, so $|G : N_G(H)| \leq |G : H|$, so $|H| \cdot |A| = |H| \cdot |G : N_G(H)| \leq |H| \cdot |G : H| = |G|$. Now, all conjugates of H have the same order. So $|K| < |H| \cdot |A|$. (Justification: If the conjugates of H were all disjoint sets, then we would have $|K| = |H| \cdot |A|$. But there is some overlap, since the conjugates of H all contain the identity element.) So $|K| < |G|$. Hence $|K| \neq |G|$.

#7) Let G be a finite group with $|G| > 1$, and let $\mathrm{Inn}(G)$ be the group of inner automorphisms of G. Show that if G is isomorphic to $\mathrm{Inn}(G)$, then $|G|$ has at least two distinct prime factors. (Hint: Do a proof by contradiction.)
Assume temporarily that $|G|$ has exactly one prime factor p. Then G is a p-group. So $Z(G) \neq 1$. So $|G/Z(G)| < |G|$. But $G/Z(G) \cong \mathrm{Inn}(G)$, so $|G/Z(G)| = |\mathrm{Inn}(G)| = |G|$. Contradiction.

#8) Let G be a group containing more than one element. Let G' be the commutator subgroup of G. Prove that if G is solvable, then $G \neq G'$.
By definition of solvable, there exist distinct subgroups $G_1, \ldots, G_n$ such that $1 = G_1 \trianglelefteq \cdots \trianglelefteq G_n = G$ and G_{i+1}/G_i is abelian for $i = 1, \ldots, n-1$. In particular, we have that G/G_{n-1} is abelian and $G_{n-1} \neq G$. The fact that G/G_{n-1} is abelian implies that $G' \leq G_{n-1}$. Hence $G \neq G'$.