DETERMINATION OF WEIGHTS FOR RELAXATION RECURRENT NEURAL NETWORKS

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Abstract - A theorem which establishes the solutions of a given optimization problem as stable points in the state space of single-layer relaxation-type recurrent neural networks is proposed. This theorem establishes the necessary conditions for the neural network to converge to a solution by proposing certain values for the constraint weight parameters of the network. Convergence performance of the discrete Hopfield network with the proposed bounds on constraint weight parameters is tested on a set of constraint satisfaction and optimization problems including the Traveling Salesman Problem, the Assignment Problem, the Weighted Matching Problem, the N-Queens Problem and the Graph Path Search Problem. Simulation and stability analysis results indicate that the set of solutions become a subset of the set of stable points in the state space as a result of the suggested bounds. For the cases of the Traveling Salesman, Assignment and Weighted Matching Problems, two sets are equal leading to convergence to a solution after each relaxation. Convergence to a solution after each relaxation is not guaranteed for the N-Queens and the Graph Path Search Problems since the solution set is a proper subset of the stable point set. Furthermore the simulation results indicate that the discrete Hopfield network converged to mostly average quality solutions as expected from a gradient-descent search algorithm. In conclusion, the suggested bounds on weight parameters guarantee that the discrete Hopfield network will locate a solution after each relaxation for a class of optimization problems of any size, although the solutions will be average quality rather than optimum.

Definitions

Def. 1. The *state space set* contains all 2^N *N*-bit binary vectors for an *N*-node network.

Def. 2. The *stable point set* includes the binary vectors which are stable points of the recurrent network dynamics for a given optimization problem.

Def. 3. The *solution set* contains those *N*-bit binary vectors for an *N*-node recurrent network which are solutions of an optimization problem.

Def. 4. A *relaxation* of the recurrent network is the total computation effort expended starting from an initial state until convergence to a final state.

Def. 5. A constraint is called *hard* if violating that constraint necessarily prevents the network from finding a solution.

Def. 6. A *soft* constraint is employed to map a cost measure associated with the quality of a solution as typically found in optimization problems.

1. Introduction

There is a large array of Artificial Neural Network (ANN) algorithms suitable for application to static optimization problems in the literature [6]. The Hopfield network (HN) and its derivatives are perhaps the most widely used ANN algorithms that address static optimization problems; they topologically belong to the class of single-layer, relaxation-type recurrent ANNs [8-11]. The HN derivatives rely on gain scheduling as in simulated annealing, network with nodes modeled by lossless integrators, network of nodes with unipolar activation functions, network with additive uncorrelated noise with zero mean and a variance gradually decreasing in time, mean field theory network, and mean field annealing network, among others. In practice, the HN and its derivatives offer a computationally simple way to address a class of optimization problems.

The HN and its derivatives are dynamic systems and as such can be studied from the perspective of a dynamic system. The time behavior of the HN dynamics minimizes a quadratic Lyapunov function. A Lyapunov function typically represents the set of constraints of a given problem for which a solution is being sought. The HN and its derivatives have been employed as fixed-point attractors to solve a large set of constraint satisfaction and optimization problems [2-5, 7, 12,19-24]. Their promise is to converge to the stable fixed-point located within the basin of attraction implied by the initial conditions of the network dynamics. It is a well-known deficiency of the HN and its derivatives that these algorithms do not scale well with increases in the size of the static optimization problem [1, 9, 14].

Performance of the HN is highly correlated to the values of a set of parameters, called the constraint weight parameters. When the HN is employed in a fixed-point attractor mode, the neural network design task incorporates establishing the solutions of a given optimization problem as local minimum points or equivalently stable points in the Lyapunov space or state space, respectively [17,18]. This task has been performed in an ad hoc manner at the beginning, since the original Hopfield algorithm did not propose a way to define the constraint weight parameters [8-11]. In most cases, constraint weight parameters were set using empirical guidelines. As a result, stability properties of solution equilibrium points could not be determined. Starting with Abe [26] and Aiyer et al. [25], a number of researchers attempted to address this problem. Abe employed the insight gained by studying the energy space and was able to develop bounds for constraint weight parameters for the Traveling Salesman Problem, which established the solutions as local minima in the Lyapunov space. Aiyer et al. were successful developing a good theoretical insight into the convergence properties of the Hopfield network algorithm and formulated means to define the values of those weight parameters associated with hard constraints but failed to do so the same for the soft constraint (minimum distance) of the Traveling Salesman Problem. Ali et al. [2] provided theoretical guidelines to determine the values of constraint weight parameters for the Shortest Path and Routing Problems noting that their guidelines were derived using a specific optimization problem rather then a generalized theorem.

This paper presents a general procedure in the form of a theorem that defines bounds on the constraint weight parameters of the Hopfield network to establish the solutions of a given optimization problem as stable equilibrium points in the state space of the network dynamics. An earlier work by the authors demonstrated that solutions of a constraint satisfaction or optimization problem are stable points in the state space of the neural networks dynamics for only certain values of the constraint weight parameters [13]. The implication is that a subset of solutions are not likely to be stable for heuristically determined parameter values. In that case, the neural network would relax to a stable point that is not a solution. In the state space of the Hopfield network and its derivatives, there are three sets of equilibrium points that are of interest: the set of stable points, the set of solution points and the set of stable non-solution points. Noting that if the set of stable points is equal to the set of solution points, the HNs and its derivatives will always relax to a solution. Similarly, if the set of stable points is a superset of the set of solutions, the neural network is not guaranteed to converge to a solution after each relaxation. The proposed theorem in this paper will guarantee the solution set to be a subset of the stable point set. The same theorem will define bounds on constraint weight parameters. The scope of the paper will be limited to studying the discrete Hopfield network (DHN) dynamics for a set of static optimization problems. However, results can easily be extended to other algorithms in the family of single-layer relaxation-type recurrent neural networks including the Boltzmann Machine and the Mean Field Annealing network. In the next section, a formal description of the discrete Hopfield network is presented. An analysis of the quadratic Lyapunov function for the discrete Hopfield network follows in Section 3. The theorem, which establishes bounds on constraint weight parameters, is introduced in Section 4. Simulation results, which include a comparative analysis of the bounds derived through the proposed theorem and other techniques in the literature, and conclusions are presented in Sections 5 and 6, respectively.

2. The Discrete Hopfield Network

The discrete Hopfield network (DHN) [8-11] is a nonlinear dynamic system with the following formal definition. Let s_i represent a node output where $s_i = 0,1$ for $i = 1, 2, \cdots, N$ and N is the number of network nodes. Then, the equation given by

$$E = -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} w_{ij} s_i s_j - \sum_{i=1}^{N} b_i s_i + \sum_{i=1}^{N} \Theta_i s_i \ , \quad i \neq j \tag{1}$$

is the Lyapunov function whose local minima are the final states of the network with node dynamics defined by

$$\begin{aligned} &s_i^{k+1} = 0 \text{ if } net_i^k < \Theta_i, \\ &s_i^{k+1} = 1 \text{ if } net_i^k > \Theta_i, \text{ and} \\ &s_i^{k+1} = s_i^k \text{ if } net_i^k = \Theta_i,\ i = 1, 2, \cdots, N, \end{aligned} \tag{2}$$

where k is a discrete time index,

$$net_i^k = \sum_{j=1}^{N} w_{ij} s_j^k + b_i \tag{3}$$

with $i \neq j$ and Θ_i is the threshold of node s_i. The weight term is defined by

$$w_{ij} = \sum_{\varphi=1}^{Z} g_\varphi \delta_{ij}^{\varphi} d_{ij}^{\varphi}$$

where Z is the number of constraints. Given the set of constraints $C_\varphi \in \{C_1, C_2, \ldots, C_Z\}$, $g_\varphi \in R^+$ if the hypotheses nodes s_i and s_j each represent for C_φ are mutually supporting and $g_\varphi \in R^-$ if the same hypotheses are mutually conflicting. The term δ_{ij}^{φ} is equal to 1 if the two hypotheses represented by nodes s_i and s_j are related under C_φ and is equal to 0 otherwise. The d_{ij}^{φ} term is equal to 1 for all i and j under a hard constraint and is a predefined cost for a soft constraint, which is typically associated with a cost term in optimization problems.

This generic description of the weight term, which will also be useful for the formulation and proof of the proposed theorem, facilitates a compact representation for both hard and soft constraints of a given optimization problem. Following section presents detailed discussion on the number of active nodes within the interaction topology of a given hard or soft constraint by employing this description of the weight term.

3. Observations on the Lyapunov Function

Towards formulating bounds on constraint weight parameters in the form a theorem, it is first necessary to analyze the Lyapunov function to deduce a set of observations. This analysis will define the number of active nodes that exist within the interaction topology of a certain type of constraint in a solution vector. Once equipped with this knowledge, the next step will be to use the active and inactive node update equations to derive bounds on constraint weight parameters in the form of a theorem.

Assume that nodes in a given neural network can be partitioned into sets: membership to these sets is solely dependent upon the set of constraints a node employs to interact with other nodes. A partition of nodes and its associated constraint set will be denoted by P_ϕ and S_ϕ, respectively. The subscript ϕ is an index over the set of partitions and associated constraint sets. The constraint set, S_ϕ, associated with the node partition set, P_ϕ, will be partitioned into four disjoint sets. These are the set of local inhibitory hard constraints (interaction is among a subset of the total network nodes and inhibitory), the set of local excitatory hard constraints (interaction is among a subset of the total network nodes and excitatory), the set of local inhibitory soft constraints with labels Γ_ϕ, Λ_ϕ, and Ψ_ϕ, respectively, and the set of global inhibitory hard constraints (interaction is among all nodes in the network and inhibitory). Specifically, analyses are conducted for the interaction topologies of a local inhibitory hard constraint, a local excitatory hard constraint, a local inhibitory soft constraint, and a global inhibitory hard constraint. Furthermore, following assumptions are made for the analysis:

1) The total number of active nodes within a solution array is known and represented by M.

2) Values of the various energy terms that form the resultant energy function for the DHN are known a priori for the set of solutions of a given problem, which is typically the case for a given problem.

The value of the quadratic energy term for a local inhibitory hard constraint, by definition, is equal to zero for a solution array: a good example for this is the constraint which imposes that at most one node can be active in a given row/column for a two-dimensional array as the network topology. The quadratic energy term given by

$$-\frac{1}{2} g_\alpha \sum_{i=1}^{N} \sum_{j=1}^{N} \delta_{ij}^{\alpha} s_i s_j \tag{6}$$

has zero as its minimum value when the following conditions are true:

$$\text{If } s_i = 0 \text{ then } \sum_{j=1}^{N} \delta_{ij}^{\alpha} s_j = n_i^{\alpha} \in \left[n_{\min}^{\alpha}, n_{\max}^{\alpha} \right] \text{ and} \tag{7}$$

$$\text{if } s_i = 1, \text{ then } \sum_{j=1}^{N} \delta_{ij}^{\alpha} s_j = 0,\ i = 1, 2, \cdots, N;\ i \neq j, \tag{8}$$

where $g_\alpha \in R^-$, n_i^α is the number of active nodes which exist within the interaction topology of C_α and with which an inactive node within a solution vector interacts, $n_{\min}^{\alpha}$ and $n_{\max}^{\alpha}$ are the lower and upper bounds, respectively, for n_i^α, and $C_\alpha \in \{C_1, C_2, \cdots, C_m\}$, the set of local inhibitory hard constraints. Two observations can be made for the local interaction topology of an inhibitory hard constraint:

1) Equation 7 indicates an inactive node interacts with n_i^α active nodes in the interval $\left[n_{\min}^{\alpha}, n_{\max}^{\alpha} \right]$, where the upper and lower limits for the interval are problem dependent.

2) Equation 8 shows an active node does not interact with any other active node.

The quadratic energy term, which maps a local excitatory hard constraint to the discrete Hopfield network, is given by

$$-\frac{1}{2} g_\beta \sum_{i=1}^{N} \sum_{j=1}^{N} \delta_{ij}^{\beta} s_i s_j \tag{9}$$

where $g_\beta \in R^+$ and $C_\beta \in \{C_1, C_2, \cdots, C_q\}$, the set of local excitatory hard constraints. Equation 9 is minimum when all nodes within the network topology are active. Typically, all nodes are not active for a solution vector, otherwise the solution is trivial. Therefore, only a subset, M, of all N nodes can be active in a solution vector. The value of this energy term for a solution vector with M nodes active is predefined and a negative real number that is problem dependent. Assume the non-weighted value of the energy term for a solution vector is given by

$$\sum_{i=1}^{N} \sum_{j=1}^{N} \delta_{ij}^{\beta} s_i s_j = L_\beta, \tag{10}$$

where $L_\beta \in Z^+$, the set of positive integers. Expanding the quadratic energy term yields

$$\sum_{i=1}^{N}\sum_{j=1}^{N}\delta_{ij}^{\beta}s_i s_j = s_1\sum_{j=1}^{N}\delta_{1j}^{\beta}s_j + s_2\sum_{j=1}^{N}\delta_{2j}^{\beta}s_j + \cdots + s_N\sum_{j=1}^{N}\delta_{Nj}^{\beta}s_j \;. \tag{11}$$

For a solution vector, M out of N nodes are active. Therefore, only M out of N s_i terms are nonzero. After an arbitrary reordering of the M terms that are nonzero, Equation 11 becomes

$$\sum_{i=1}^{N}\sum_{j=1}^{N}\delta_{ij}^{\beta}s_i s_j = \sum_{j=1}^{N}\delta_{1j}^{\beta}s_j + \sum_{j=1}^{N}\delta_{2j}^{\beta}s_j + \cdots + \sum_{j=1}^{N}\delta_{Mj}^{\beta}s_j \;. \tag{12}$$

where there are M summations. Each summation is the definition of the network input due to the excitatory hard constraint C_β for the associated active node. Assuming any active node interacts with an equal number of other active nodes, the summations have the same values, which will be denoted by ξ. Equation 12 can be simplified to

$$\sum_{i=1}^{N}\sum_{j=1}^{N}\delta_{ij}^{\beta}s_i s_j = M\xi, \tag{13}$$

where $\xi = \sum_{j=1}^{N}\delta_{ij}^{\beta}s_j$, for $s_i = 1$ and i = 1,2,...,N. Equations 10 and 13 can be combined to yield

$$\sum_{j=1}^{N}\delta_{\mathrm{ij}}^{\beta}s_j = \frac{L_\beta}{M} \text{ for } s_i = 1 \text{ and } i = 1,2,\ldots,N\,. \tag{14}$$

The summation in Equation 14 is the definition of the total non-weighted input to an active node from all other active nodes, which are within the local interaction topology of an excitatory hard constraint C_β. In other words, the summation is the count of the active nodes within the interaction topology of an active node for which the network input for constraint C_β is computed. Given values of the parameters L_β (non-weighted values of the quadratic energy term that map an excitatory hard constraint) and M (total number of active nodes in a solution) are known for a problem, the value of the summation, which is the network input to an active node for an excitatory hard constraint, can be computed using Equation 14.

In summary, the value of the quadratic energy term for a local excitatory hard constraint is defined by

$$-\frac{1}{2}g_{\beta}\sum_{i=1}^{N}\sum_{j=1}^{N}\delta_{ij}^{\beta}s_i s_j = -\frac{1}{2}g_{\beta}L_{\beta}$$

when the following conditions are true:

$$\text{If } s_i = 0, \text{ then } \sum_{j=1}^{N} \delta_{ij}^{\beta}s_j = n_i^{\beta} \in \left[n_{min}^{\beta},\ n_{max}^{\beta}\right] \text{ and} \tag{15}$$

$$\text{if } s_i = 1, \text{ then } \sum_{j=1}^{N} d_{ij}^{\beta}s_j = \frac{L_{\beta}}{M},\ i = 1,2,\ldots,N;\ i \neq j, \tag{16}$$

where $C_{\beta} \in \left\{C_1, C_2, \cdots, C_q\right\}$. The parameter n_i^{β} represents the number of active nodes with which an inactive node interacts. Equation 15 shows that an inactive node interacts with any number of active nodes in the interval $\left[n_{\min}^{\beta},\ n_{\max}^{\beta}\right]$ for a solution array and the bounds of the interval are problem dependent. Similarly, Equation 16 indicates an active node interacts with L_{β}/M other active nodes for a solution array.

The definition of the quadratic energy term, which maps a local inhibitory soft constraint, is given by

$$-\frac{1}{2}g_{\eta}\sum_{i=1}^{N}\sum_{j=1}^{N}\delta_{ij}^{\eta}d_{ij}^{\eta}s_i s_j, \tag{17}$$

where $g_{\eta} \in R^{-}$ and $C_{\eta} \in \left\{C_1, C_2, \cdots, C_p\right\}$, the set of local inhibitory soft constraints. Equation 17 is used to compute the associated cost of the solution found by the neural network. At least two nodes must be active within the local interaction topology of a local inhibitory soft constraint for a partial cost to occur. Therefore, an active node interacts with at least one other active node within the interaction topology of an inhibitory soft constraint. Assuming any active node interacts with the same number of other active nodes, then L_{η}^{1}, where $L_{\eta}^{1} \geq 1$, can be used to represent the number of active nodes with which an active node interacts. Let $k = 1, 2, \cdots, L_{\eta}^{1}$, then

$$\sum_{j=1}^{N}\delta_{ij}^{\eta}d_{ij}^{\eta}s_j = \sum_{k=1}^{L_{\eta}^{1}} d_k^{\eta}, \tag{18}$$

for $i = 1,2,\cdots,N;\ i \neq j;$ and $C_{\eta} \in \left\{C_1, C_2, \cdots, C_p\right\}$, where d_k represents one of the d_{ij}'s for each k.

The interaction of an inactive node with other active nodes does not contribute any partial cost to the total cost given by the quadratic energy term. Thus, it is not possible to determine the number of active nodes with which an inactive node interacts. Then L_η^0 , where $L_\eta^0 \in \{0, n_{\max}^\eta\}$, will be used to represent the number of active nodes with which an inactive node interacts within the local interaction topology of an inhibitory soft constraint. Assume k is the index for the set of cost terms, where $k \in \{0, L_\eta^0\}$; then the sum of all elements in the set of cost terms can be written as

$$\sum_{j=1}^{N} \delta_{ij}^{\eta} d_{ij}^{\eta} s_j = \sum_{k=0}^{L_\eta^0} d_k^{\eta}, \tag{19}$$

for $i = 1,2,\cdots,N;\ i \neq j;$ and $\forall C_\eta \in \{C_1, C_2, \cdots, C_p\}$. The lower bound for L_η^0 is equal to zero. Therefore, it is necessary to define

$$\sum_{k=0}^{L_\eta^0} d_k^{\eta} = 0, \text{ for } L_\eta^0 = 0.$$

The bounds of the interval $[0, n_{\max}^\eta]$ and the value of parameters L_η^0 and L_η^1 are problem dependent.

There are M active nodes in a solution vector. Since any node interacts with all the other nodes in the network within the global interaction topology of an inhibitory hard constraint C_γ, an inactive node interacts with M other active nodes. An active node interacts with M-1 active nodes, since self-feedback is zero for the discrete Hopfield network. These statements can be formalized as follows:

$$\text{If } s_i = 0, \text{ then } \sum_{j=1}^{N} \delta_{ij}^{\gamma} s_j = M \text{ and} \tag{20}$$

$$\text{if } s_i = 1, \text{ then } \sum_{j=1}^{N} \delta_{ij}^{\gamma} s_j = M - 1,\ i = 1,2,\cdots,N;\ i \neq j. \tag{21}$$

Observations made based on the energy terms are employed to facilitate the proof for the theorem presented in the next section.

4. Bounds on Constraint Weight Parameter Magnitudes

Bounds on constraint weight parameter magnitudes, which guarantee that the solutions of constraint satisfaction or optimization problems are the stable points of the discrete Hopfield network dynamics, are formulated as a theorem next.

Theorem. The solutions for an optimization or constraint satisfaction problem are stable points of the discrete Hopfield network dynamics if, and only if, the following set of inequalities on constraint weight parameter magnitudes hold:

$$\sum_{\alpha=1}^{|\Gamma_\phi|} n_{\min}^{\alpha}\left|g_\alpha\right| + \sum_{\eta=1}^{|\Psi_\phi|}\left|g_\eta\right| L_\eta^0 d_{\min}^{\eta} + M\left|g_\gamma\right| + \Theta_{\min}^{\phi} \geq \sum_{\beta=1}^{|\Lambda_\phi|} n_{\max}^{\beta} g_\beta + b_{\max}^{\phi} \tag{22}$$

and

$$\sum_{\eta=1}^{|\Psi_\phi|}\left|g_\eta\right| L_\eta^1 d_{\max}^{\eta} + (M-1)\left|g_\gamma\right| + \Theta_{\max}^{\phi} \leq \sum_{\beta=1}^{|\Lambda_\phi|} \frac{L_\beta}{M} g_\beta + b_{\min}^{\phi} , \tag{23}$$

$\forall \mathsf{P}_\phi \in \left\{\mathsf{P}_1, \mathsf{P}_2, \cdots, \mathsf{P}_\omega\right\}$, the set of node partition sets, and $\forall \mathsf{S}_\phi \in \left\{\mathsf{S}_1, \mathsf{S}_2, \cdots, \mathsf{S}_\omega\right\}$, the set of associated constraint sets, with

a) $\varnothing \subseteq \Gamma_\phi \subseteq \left\{C_1, C_2, \cdots, C_m\right\}$,

b) $\varnothing \subseteq \Lambda_\phi \subseteq \left\{C_1, C_2, \cdots, C_q\right\}$,

c) $\varnothing \subseteq \Psi_\phi \subseteq \left\{C_1, C_2, \cdots, C_p\right\}$,

d) $\mathsf{S}_\phi \equiv \Gamma_\phi \cup \Lambda_\phi \cup \Psi_\phi \cup C_\gamma$,

e) $b_{\min}^{\phi} \leq b_i \leq b_{\max}^{\phi}$,

f) $\Theta_{\min}^{\phi} \leq \Theta_i \leq \Theta_{\max}^{\phi}$, and

g) $d_{\min}^{\eta} \leq d_{ij}^{\eta} \leq d_{\max}^{\eta}$,

for $i, j = 1, 2, \cdots, N,\ i \neq j$, and

where

1) α, β, and η sums over $\tau_\phi \left(0 \le \tau_\phi \le m \text{ and } \tau_\phi = \left|\Gamma_\phi\right|\right)$ local inhibitory hard constraints, $\zeta_\phi \left(0 \le \zeta_\phi \le q \text{ and } \zeta_\phi = \left|\Lambda_\phi\right|\right)$ local excitatory hard constraints, and $\chi_\phi \left(0 \le \chi_\phi \le p \text{ and } \chi_\phi = \left|\Psi_\phi\right|\right)$ local inhibitory soft constraints within the sets Γ_ϕ, Λ_ϕ, and Ψ_ϕ respectively;

2) g_γ is the gain parameter associated with the global inhibitory hard constraint, C_γ, which enforces M nodes to be active for an N-node network, $M < N$, and M is the total number of active nodes in a solution vector;

3) L_η^0 and L_η^1 are the number of active nodes with which an inactive and active node interact under a local inhibitory soft constraint C_η for a solution vector;

4) d_{ij}^η is the cost due to two nodes s_i and s_j being simultaneously active in a solution vector for a local inhibitory soft constraint C_η with $d_{\min}^\eta = \min\left\{d_{ij}^\eta\right\}$ and $d_{\max}^\eta = \max\left\{d_{ij}^\eta\right\}$;

5) $n_{\min}^\alpha$ is the minimum number of active nodes that exist within the interaction topology of a local inhibitory hard constraint C_α, with which an inactive node within a solution vector interacts;

6) $n_{\max}^\beta$ represents the maximum number of active nodes which exist within the interaction topology of a local excitatory hard constraint C_β for an inactive node in a solution vector; and

7) L_β is the non-weighted value of the quadratic energy term that maps a local excitatory hard constraint C_β for a solution vector.

Proof: First, proof of necessity is presented. Assume $\bar{s} = \left[s_1 s_2 \cdots s_N\right]^T$ represents a solution and is a stable point. Then an element of $\bar{s}$, s_i, is in one of two states, either 1 or 0, $\forall i$. Stability of the node s_i requires [1, 14]

$$\lambda_i = \left(1 - 2s_i\right)\left(net_i - \Theta_i\right) \le 0 . \tag{24}$$

where λ_i is called the eigenvalue of the node s_i and

$$net_i = \sum_{j=1}^{N} w_{ij} s_j + b_i \quad \text{for } i = 1, 2, \ldots, N;\ i \ne j.$$

Equivalently, output of node s_i is stable when the following conditions are satisfied:

$$\text{If } s_i = 0,\ net_i = \sum_{j=1}^{N} w_{ij} s_j + b_i \le \Theta_i \ \text{ else} \tag{25}$$

$$\text{if } s_i = 1,\ net_i = \sum_{j=1}^{N} w_{ij} s_j + b_i \geq \Theta_i,\ i\ =\ 1,2,\ldots,N;\ i\ \neq\ j. \tag{26}$$

For a given constraint satisfaction or optimization problem, an element of the weight matrix, w_{ij}, the weight between nodes s_i and s_j, is defined by

$$w_{ij} = \sum_{\varphi} g_\varphi \delta_{ij}^{\varphi} d_{ij}^{\varphi}, \tag{27}$$

$\varphi = 1,2,\ldots,Z$, an index over the set of Z constraints and $C_\varphi \in \{C_1, C_2, \ldots, C_Z\}$, the set of constraints, where $g_\varphi \in R^+$ if the hypotheses both nodes represent for C_φ are mutually supporting and $g_\varphi \in R^-$ if the hypotheses are mutually conflicting. The term δ_{ij}^{φ} is equal to 1 if the two hypotheses represented by nodes s_i and s_j are related under C_φ and is equal to 0 otherwise. The d_{ij}^{φ} term is equal to 1 for all i and j for a hard constraint and is a predefined cost for a soft constraint.

Using the definition of the weight entry in terms of the constraint weight parameters, given by Equation 27, as follows can expand the summation:

$$\sum_{j=1}^{N} w_{ij} s_j = \sum_{j=1}^{N} \sum_{\varphi=1}^{m+q+p+1} g_\varphi \delta_{ij}^{\varphi} d_{ij}^{\varphi} s_j, \tag{28}$$

where the constraint satisfaction or the optimization problem has m local inhibitory hard constraints, q local excitatory hard constraints, p local inhibitory soft constraints, and one global inhibitory hard constraint $(Z = m + q + p + 1)$. Interchanging the order of summations in Equation 28 gives

$$\sum_{j=1}^{N} w_{ij} s_j = \sum_{\varphi=1}^{m+q+p+1} g_\varphi \sum_{j=1}^{N} \delta_{ij}^{\varphi} d_{ij}^{\varphi} s_j, \tag{29}$$

The summation with index j in Equation 29 can be broken up into four summation terms to yield

$$\sum_{\varphi=1}^{m+q+p+1} g_\varphi \sum_{j=1}^{N} d_{ij}^{\varphi} \delta_{ij}^{\varphi} s_j = \sum_{\alpha=1}^{m} g_\alpha \sum_{j=1}^{N} \delta_{ij}^{\alpha} s_j + g_\gamma \sum_{j=1}^{N} \delta_{ij}^{\gamma} s_j + \sum_{\beta=1}^{q} g_\beta \sum_{j=1}^{N} \delta_{ij}^{\beta} s_j + \sum_{\eta=1}^{p} g_\eta \sum_{j=1}^{N} d_{ij}^{\eta} \delta_{ij}^{\eta} s_j \tag{30}$$

where g_α, g_γ, $g_\eta \in R^-$ and $g_\beta \in R^+$. All four summations with index j in Equation 30 can be computed for both inactive and active nodes in a solution using Equations 7, 8, 15, 16, 18, 19, 20, and 21.

Consider a node, s_i, which belongs to the node partition set P_ϕ, where ϕ is the index over all partition sets on network nodes, with associated constraint set $\mathsf{S}_\phi \left(\mathsf{S}_\phi \equiv \Gamma_\phi \cup \Lambda_\phi \cup \Psi_\phi \cup C_\gamma\right)$ within a solution vector. Assume node s_i belongs to the interaction topologies of τ_ϕ ($0 \le \tau_\phi \le m$ and $\tau_\phi = \left|\Gamma_\phi\right|$) local inhibitory hard constraints, ζ_ϕ ($0 \le \zeta_\phi \le q$ and $\zeta_\phi = \left|\Lambda_\phi\right|$) local excitatory hard constraints, and χ_ϕ ($0 \le \chi_\phi \le p$ and $\chi_\phi = \left|\Psi_\phi\right|$) local inhibitory soft constraints, and one global inhibitory hard constraint, which constitute the constraint set S_ϕ associated with the partition set, P_ϕ.

Let the node be inactive; $s_i = 0$. The network input for an inactive node due to a local inhibitory hard constraint is given by Equation 7,

$$\sum_{j=1}^{N} \delta_{ij}^{\alpha} s_j = n_i^{\alpha} \in [n_{\min}^{\alpha}, n_{\max}^{\alpha}],\ \forall s_i \in \mathsf{P}_\phi \text{ and } \forall \alpha \ni C_\alpha \in \Gamma_\phi \subseteq \left\{C_1, C_2, \cdots, C_m\right\}. \tag{31}$$

Equation 15 indicates the network input for an inactive node due to a local excitatory hard constraint is defined by

$$\sum_{j=1}^{N} \delta_{ij}^{\beta} s_j = n_i^{\beta} \in \left[n_{\min}^{\beta}, n_{\max}^{\beta}\right],\ \forall s_i \in \mathsf{P}_\phi \text{ and } \forall \beta \ni C_\beta \in \Lambda_\phi \subseteq \left\{C_1, C_2, \cdots, C_q\right\}. \tag{32}$$

Equation 19 shows the network input for an inactive node due a local inhibitory soft constraint is given by

$$\sum_{j=1}^{N} \delta_{ij}^{\eta} d_{ij}^{\eta} s_j = \sum_{k=1}^{L_\eta^0} d_k^{\eta},\ \forall s_i \in \mathsf{P}_\phi \text{ and } \forall \eta \ni C_\eta \in \Psi_\phi \subseteq \left\{C_1, C_2, \cdots, C_p\right\}. \tag{33}$$

The network input for an inactive node due to the global inhibitory hard constraint C_γ, Equation 20, is given by

$$\sum_{j=1}^{N} \delta_{ij}^{\gamma} s_j = M,\ \forall s_i \in \mathsf{P}_\phi. \tag{34}$$

Substitution of Equations 31, 32, 33, and 34 in Equation 30 yields,

$$\sum_{j=1}^{N} w_{ij} s_j = \sum_{\alpha=1}^{\tau_\phi} g_\alpha n_i^\alpha + \sum_{\beta=1}^{\zeta_\phi} g_\beta n_i^\beta + \sum_{\eta=1}^{\chi_\phi} g_\eta \sum_{k=1}^{L_\eta^0} d_k^\eta + g_\gamma M, \tag{35}$$

$\forall s_i \in \mathsf{P}_\phi$ and $\forall \phi$.

Substitution of Equation 35 in Equation 25 produces

$$\sum_{\alpha=1}^{\tau_\phi} -\left|g_\alpha\right| n_i^\alpha + \sum_{\beta=1}^{\zeta_\phi} g_\beta n_i^\beta + \sum_{\eta=1}^{\chi_\phi} -\left|g_\eta\right| \sum_{k=1}^{L_\eta^0} d_k^\eta - \left|g_\gamma\right| M + b_i \le \Theta_i .$$

Rearranging the terms in the above equation gives

$$\sum_{\alpha=1}^{\tau_\phi} \left|g_\alpha\right| n_i^\alpha + \sum_{\eta=1}^{\chi_\phi} \left|g_\eta\right| \sum_{k=1}^{L_\eta^0} d_k^\eta + \left|g_\gamma\right| M + \Theta_i \ge \sum_{\beta=1}^{\zeta_\phi} g_\beta n_i^\beta + b_i . \tag{36}$$

Let the following hold,

1) $b_{\max}^\phi \ge b_i$,
2) $\Theta_{\min}^\phi \le \Theta_i$,
3) $n_{\max}^\beta \ge n_i^\beta$,
4) $n_{\min}^\alpha \le n_i^\alpha$, and
5) $d_{\min}^\eta = \min\left\{d_{ij}^\eta\right\}$,

$\forall s_i \in \mathsf{P}_\phi$, $\forall \alpha \ni C_\alpha \in \Gamma_\phi$, $\forall \beta \ni C_\beta \in \Lambda_\phi$, $\forall \eta \ni C_\eta \in \Psi_\phi$, and $\forall \phi$. Then Equation 36 is satisfied if

$$\sum_{\alpha=1}^{\tau_\phi} \left|g_\alpha\right| n_{\min}^\alpha + \sum_{\eta=1}^{\chi_\phi} \left|g_\eta\right| L_\eta^0 d_{\min}^\eta + \left|g_\gamma\right| M + \Theta_{\min}^\phi \ge \sum_{\beta=1}^{\zeta_\phi} g_\beta n_{\max}^\beta + b_{\max}^\phi$$

is established $\forall s_i \in \mathsf{P}_\phi$ with $\mathsf{S}_\phi \equiv \Gamma_\phi \cup \Lambda_\phi \cup \Psi_\phi \cup C_\gamma$ and $\forall \phi$.

Let the node be active; $s_i = 1$. Substitution of Equations 8, 16, 18, and 21 in Equation 30 yields

$$\sum_{j=1}^{N} w_{ij} s_j = \sum_{\beta=1}^{\zeta_\phi} \frac{L_\beta}{M} g_\beta - \sum_{\eta=1}^{\chi_\phi} \left| g_\eta \right| \sum_{k=1}^{L_\eta^1} d_k^\eta - (M-1)\left| g_\gamma \right|, \tag{37}$$

$\forall s_i \in \mathsf{P}_\phi$ with $\mathsf{S}_\phi \equiv \Gamma_\phi \cup \Lambda_\phi \cup \Psi_\phi \cup C_\gamma$ and $\forall \phi$. Substitution of Equation 37 in Equation 26 gives

$$\sum_{\eta=1}^{\chi_\phi} \left| g_\eta \right| \sum_{k=1}^{L_\eta^1} d_k^\eta + (M-1)\left| g_\gamma \right| + \Theta_i \le \sum_{\beta=1}^{\zeta_\phi} \frac{L_\beta}{M} g_\beta + b_i \,. \tag{38}$$

Equation 38 is satisfied for the following bound:

$$\sum_{\eta=1}^{\chi_\phi} \left| g_\eta \right| L_\eta^1 d_{\max}^\eta + (M-1)\left| g_\gamma \right| + \Theta_{\max}^\phi \le \sum_{\beta=1}^{\zeta_\phi} \frac{L_\beta}{M} g_\beta + b_{\min}^\phi \,,$$

where

1) $b_{\min}^\phi \le b_i$,

2) $\Theta_{\max}^\phi \ge \Theta_i$, and

3) $d_{\max}^\eta = \max\left\{ d_{ij}^\eta \right\}$,

$\forall s_i \in \mathsf{P}_\phi$ with $\mathsf{S}_\phi \equiv \Gamma_\phi \cup \Lambda_\phi \cup \Psi_\phi \cup C_\gamma$ and $\forall \phi$. This concludes the proof of necessity.

The proof of sufficiency is presented next. Assume $\bar{s} = \left[s_1 s_2 \cdots s_N \right]^T$ represents a solution vector and the set of inequalities on constraint weight parameter magnitudes given by the theorem hold. Consider a node, s_i, which belongs to the node partition set P_ϕ with associated constraint set $\mathsf{S}_\phi \left(\mathsf{S}_\phi \equiv \Gamma_\phi \cup \Lambda_\phi \cup \Psi_\phi \cup C_\gamma \right)$ within a solution vector. Let the node be inactive; $s_i = 0$. Equations 25 and 35 show the input for an inactive node is given by

$$net_i = \sum_{\alpha=1}^{\tau_\phi} g_\alpha n_i^\alpha + \sum_{\beta=1}^{\chi_\phi} g_\beta n_i^\beta + \sum_{\eta=1}^{\zeta_\phi} g_\eta \sum_{k=1}^{L_\eta^0} d_k^\eta + g_\gamma M + b_i \,.$$

If the bounds on constraint weight parameters hold, then $net_i \le \Theta_i$ and $\lambda_i = net_i + \Theta_i \le 0$, which in turn implies the inactive node is stable. Next, consider the input for an active node; $s_i = 1$. Equations 26 and 37 indicate the input for an active node is given by

$$net_i = \sum_{\beta=1}^{\chi_\phi} \frac{L_\beta}{M} g_\beta + \sum_{\eta=1}^{\zeta_\phi} g_\eta \sum_{k=1}^{L_\eta^1} d_k^\eta + g_\gamma (M-1) + b_i \,.$$

Given that the bounds on the constraint weight parameters hold, then $net_i \ge \Theta_i$ and thus, $\lambda_i = net_i + \Theta_i \le 0$, which in turn implies the active node is stable. Hence, all nodes within a solution vector are stable for the theorem bounds on the constraint weight parameters. This completes the proof of sufficiency.

Q.E.D.

5. Simulation Results

A simulation study was performed to evaluate the performance of the discrete Hopfield network with the proposed bounds. Specifically, two fundamental issues were observed: an assessment that the suggested procedure establishes the complete solution set as stable points and convergence characteristics of the DHN given that all solutions are stable points. A set of constraint satisfaction and optimization problems including the Traveling Salesman Problem (TSP), the Graph Path Search Problem (GPSP), the Assignment Problem (AP), the N-Queens Problem (NQP), and the Weighted Matching Problem (WMP) were used in the study [1, 8, 15, 16].

Initial test case involved the determination of the set ordering relationship between the solution set and the stable point set for the optimization problems considered. A simulation-based relaxation analysis was performed to observe this relationship. The DHN converged to a solution (in general, an average quality one) after each relaxation for the TSP, the AP and the WMP. The convergence rate was less than 100% for the NQP and the GPSP. In other words, simulation-based results indicated that the solution set contained the stable point set for the TSP, the AP and the WMP. Similarly, the solution set was a subset of the stable point set for the NQP and the GPSP. Exhaustive stability analysis of all equilibrium points for small problem sizes were also performed to confirm simulation-based results: this was done linearizing the DHN dynamics at each equilibrium point and checking the stability of that equilibrium point. Exhaustive stability analysis confirmed that the sets S and T were equal in the cases of the TSP, the AP and the WMP. Additionally, set S was a subset of the set T for the cases of the NQP and the GPSP. Results of this analysis are presented in Table 1: symbols S and T represent the solution set and the stable point set, respectively. Solutions located by the DHN had average quality in general as expected from a deterministic gradient descent search algorithm.

	Optimization Problem				
	TSP	**AP**	**WMP**	**NQP**	**GPSP**
Set Relationship	$S \equiv T$	$S \equiv T$	$S \equiv T$	$S \subset T$	$S \subset T$

Table 1. Set Ordering Relationships between the Solution Set (S) and the Stable Point Set (T)

A second test case was designed to observe the scaling properties of the DHN for large problem sizes. Instances of optimization problems were tested for the range 10 to 100 cities/graph nodes for the TSP, the AP and the WMP. In all test cases, the DHN converged to a solution after each relaxation. The bounds suggested by the theorem scaled with the increase in the size of these problems. Exhaustive stability analysis conducted for problem sizes 4 to 10 cities/nodes indicated that the solution set was again equal to the stable point set in all test cases for those problems, which confirmed the results of the simulation-based study. However, the convergence rate for the network was less than 100% for the NQP and the GPSP. Similar testing on the NQP and GPSP indicated that the convergence rate of the DHN deteriorated as the problem size increased. Exhaustive stability analysis demonstrated that the solution set was a subset of the stable point set for those problems. Additionally, the same study also showed that the cardinality of the stable point set increased at a much faster rate than the increase in the cardinality of the solution set as the problem size increased for the NQP and the GPSP. This development led to ever decreasing convergence rates for those two problems as the problem size increased.

In literature, Ali et al. [2] also addressed the Shortest Path (SP) problem (with weighted edges), which is similar to our GPSP formulation, using a set of bounds on constraint parameters derived through alternate theoretical considerations. They employed a linear formulation of the energy function to address the SP problem and obtained 100% convergence to optimal solutions for a five vertex graph. The same authors were able to derive a set of inequalities establishing bounds on the constraint weight parameters. Furthermore they also reported 100% convergence to optimal solutions for the Optimum Routing problem where the graph model of the problem required 15 vertices. The energy function formulation required five terms/constraint weight parameters and led to following set of inequalities (using the notation by Ali et al.):

$$\mu_1(C_{xi})_{\max} < 2\mu_3,\ \mu_2 = \mu_5,\ 2\mu_3 > \mu_4, \text{ and } \mu_5 >> \mu_1(C_{xi})_{\max}$$

The bounds on the constraint weight parameters derived by the Theorem proposed in this paper when applied to the SP problem of Ali and Komoun are given by

$$\mu_1(C_{xi})_{\min} + \mu_2 + \mu_4 > 2\mu_3 \text{ and } 4\mu_3 + \mu_4 > \mu_1(C_{xi})_{\max}.$$

The constraint weight parameter μ_5 is not defined as a result of applying the Theorem on the SP problem but it can easily be assessed as being equal to μ_2 as a reasonable choice of value. The two inequalities generated by the Theorem collectively imply that the first inequality generated by Ali and Kamoun ($\mu_1(C_{xi})_{\max} < 2\mu_3$) and are also compatible with the values employed by Ali et al. for the 5-vertex SP problem, which are $\mu_1 = 950,\ \mu_2 = 2500,\ \mu_3 = 1500,\ \mu_4 = 475, \text{ and } \mu_5 = 2500$. The theoretical approach proposed by Ali and Kamoun results in bounds on constraint weight parameters, which are correlated to the bounds derived using the proposed Theorem for the SP problem. However, further study of applicability of the method proposed by Ali et al. to any optimization problem (in a general sense) as well as application of the method to a set of larger size problems remain to be done to truly assess its merits and computational promise.

A third test case was used to observe the convergence rate and the quality of solutions for different values of constraint weight parameters. The variation in the values of these parameters did not have a noticeable effect on the convergence rate and the quality of solutions for the TSP, the AP and the WMP. However, it was observed that the performance of the neural network algorithm was moderately effected by variations in those parameter values for the NQP and the GPSP.

A comprehensive case study on the TSP is presented in the next section. Employment of the theorem to derive the bounds on constraint weight parameters is demonstrated on the TSP. Using the bounds derived, a simulation-based study is introduced.

5.1 Traveling Salesman Problem

Given a list of M cities, the Traveling Salesman Problem (TSP) involves a visit to each city once and only once such that the total travel distance is minimum. There are two constraints to satisfy: each city must be visited once, and only once, and the total travel distance should be minimum. Consider an N (= $M \times M$) node square array as the network topology for this problem, where rows and columns represent the cities and the visiting order, respectively. Any permutation matrix is a valid solution. The network dynamics can be forced to converge to a permutation matrix by implementing row, column and global inhibitions.

Consider the following energy function proposed by Hopfield to map this problem to the network topology [9],

$$E(s) = -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} g_r \delta_{ij}^{r} s_i s_j - \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} g_c \delta_{ij}^{c} s_i s_j - \frac{1}{2} g_\gamma \left(\sum_{i=1}^{N} s_i - M\right)^2 - \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} g_\eta d_{ij} \delta_{ij}^{\eta} s_i s_j, \tag{39}$$

where

$\delta_{ij}^{r} = 1$ if $row(i) = row(j)$ or $\delta_{ij}^{r} = 0$, otherwise;

$\delta_{ij}^{c} = 1$ if $col(i) = col(j)$ or $\delta_{ij}^{c} = 0$, otherwise;

$\delta_{ij}^{\eta} = 1$ if $\left\{ \left| col(i) - col(j) \right| = 1 \wedge \left[row(i) \neq row(j) \right] \right\}$ or

$\delta_{ij}^{\eta} = 0$, otherwise;

$i \neq j$; $g_r, g_c, g_\eta \in R^-$, d_{ij} is the distance between cities *row*(i) and *row*(j), and superscripts/subscripts r, c, γ and η stand for the row, column, global and distance inhibitions, respectively. Functions *row*(i) and *row*(j) return the row and column location of nodes i and j, respectively. The row and column inhibition energy terms are minimum if, at most, one node is active for each row and column. The energy term associated with the constraint weight parameter g_γ is minimum if exactly M nodes are active within the overall network topology. The energy term for the distance constraint is minimum if the solution has the minimum total distance. Comparison of this energy function with the generic energy function yields the following definitions for the weight matrix entries, the external bias terms, and the thresholds:

$$w_{ij} = g_r \delta_{ij}^{r} + g_c \delta_{ij}^{c} + g_\eta \delta_{ij}^{\eta} d_{ij} + g_\gamma,$$

$$b_i = \frac{1}{2}\left(1 - 2M\right) g_\gamma \text{ and } \Theta_i = 0 \text{ for } i, j = 1, 2, \cdots, N;\ i \neq j \tag{40}$$

Dynamics of nodes in a solution array for the discrete Hopfield network will obey the following inequalities:

$$\text{An inactive node: } net_i < \Theta_i \text{ and,} \tag{41}$$

$$\text{An active node: } net_i > \Theta_i \tag{42}$$

where $net_i = \sum_{j=1}^{N} w_{ij} s_j + b_i$, for $i = 1, 2, \cdots, N$.

The TSP problem has two local inhibitory hard constraints (row and column inhibitions), one local inhibitory soft constraint (distance inhibition), and one global inhibitory hard constraint (global inhibition). All nodes in the network employ the same set of constraints to interact with other nodes in the network: there is only one partition in the network hence the index ϕ can be dropped from the set symbols. Then, following observations can be made; $\Gamma \equiv \{C_r, C_c\}$, $\Lambda \equiv \varnothing$, $\Psi \equiv \{C_\eta\}$, and $\mathsf{S} \equiv \{C_r, C_c, C_\eta, C_\gamma\}$, where C_α is the constraint identified by subscript α.

Consider the solution for the 6-city TSP given in Figure 1. The set of nodes covered by dotted lines indicate those nodes which interact with each other under a particular type of constraint. The inactive node at row 4 and column 3 interacts with the active nodes located at a) row 2 and column 3 due to the column inhibition, b) row 4 and column 4 due to the row inhibition, c) row 1 and column 2 and row 4 and column 4 due to the distance inhibition, and d) row 1 and column 2, row 2 and column 3, row 3 and column 1, row 4 and column 4, row 5 and column 6, and row 6 and column 5 due to the global inhibition. Similarly, for the same solution of the problem the active node at row 4 and column 4 in Figure 2 does not interact with any other active nodes under the constraints which enforce, at most, one node active in each row and column. This active node interacts with active nodes located at a) row 2 and column 3 and row 6 and column 5 due to the minimum distance constraint and b) row 1 and column 2, row 2 and column 3, row 3 and column 1, row 5 and column 6, and row 6 and column 5 under the global inhibition interaction.

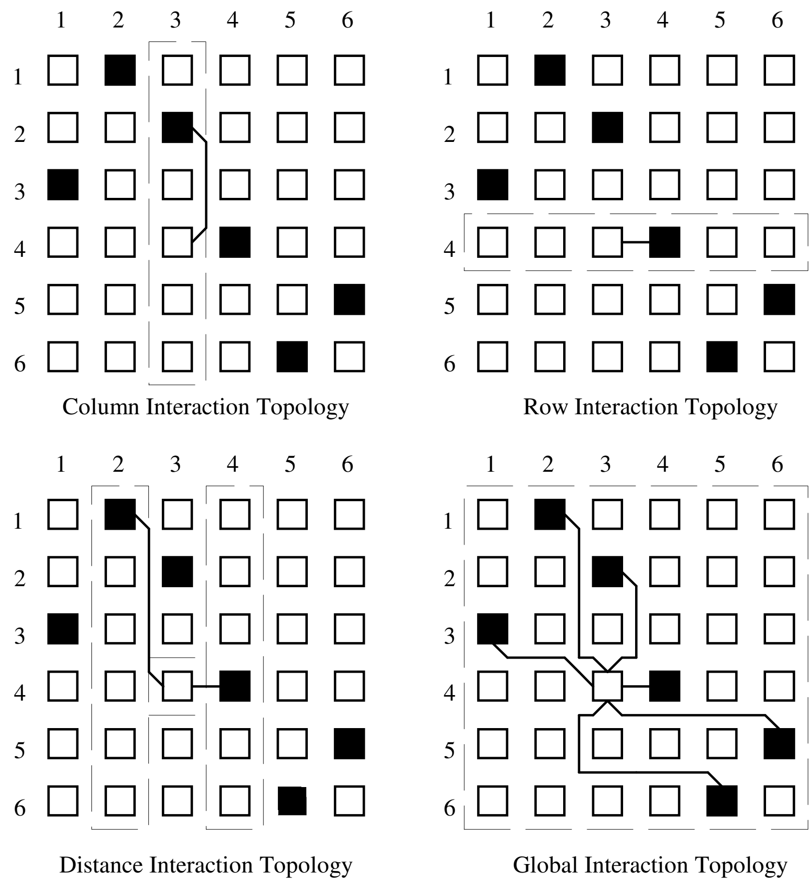

Figure 1. Column, row, distance and global inhibition interaction topologies of the inactive node located at row 4 and column 3 in a solution of the 6-city TSP.

Observations made on the 6-city instance of the TSP will next be generalized. Consider the input to an inactive node within a solution. There are M rows and M columns and a total of M nodes must be active such that each row and column has, at most, one node active. Therefore, each row and column must have exactly one node active for a solution. An inactive node receives inputs from two active nodes, one of which is in its row and the other is in its column, under the row and the column inhibition constraints, $n_{\min}^{c} = n_{\max}^{r} = 1$. A total of M active nodes collectively contribute Mg_γ to the input of an inactive node due to global inhibitory interaction. A node at column c of the network topology interacts with the nodes in the columns $|c-1|_M$ and $|c+1|_M$ within the local interaction topology of the inhibitory soft constraint. An inactive node receives inputs from two active nodes in these columns under the inhibitory soft constraint, since each of the two columns has exactly one node active for a solution vector, $L_\eta^0 = 2$.

Equation 40 shows that $\Theta_{\min} = \Theta_i = 0$ and $b_{\max} = b_i = 0.5(1-2M)g_\gamma$ for all $i = 1,2,\ldots,N$. As a result, the inequality given by Equation 41 for an inactive node takes the form of

$$\left|g_r\right| + \left|g_c\right| + 2\left|g_\eta\right| d_{\min} + \frac{1}{2}\left|g_\gamma\right| \geq 0. \tag{41}$$

This inequality is satisfied for any values of the constraint weight parameter magnitudes.

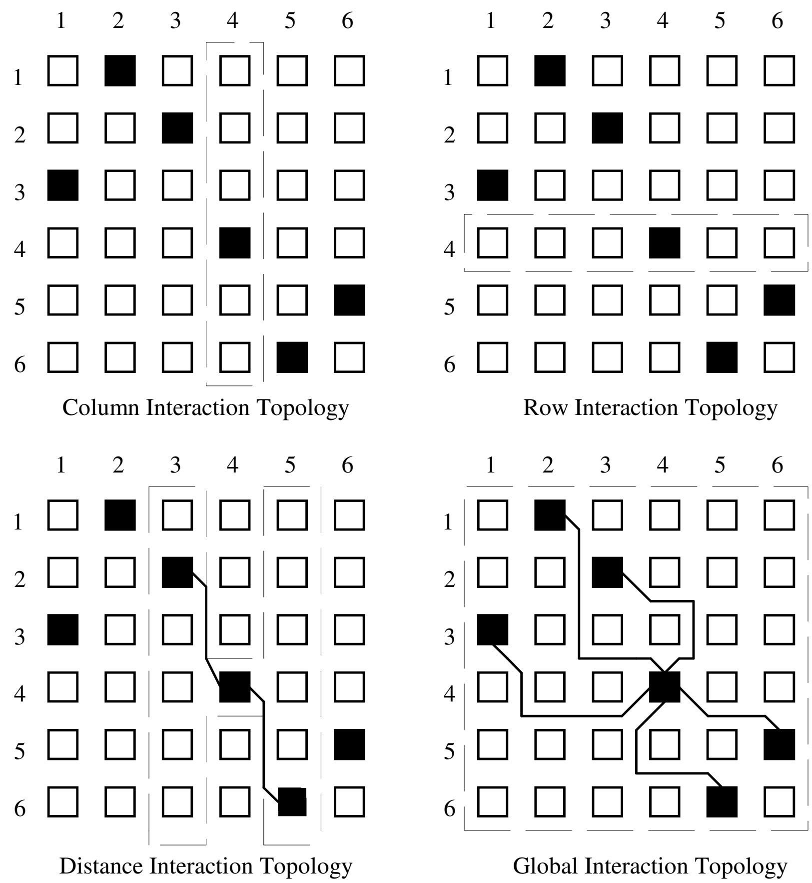

Figure 2. Column, row, distance and global inhibition interaction topologies of the active node located at row 4 and column 4 in a solution of the 6-city TSP.

The input for an active node does not include the constraint weight parameters associated with the row and column inhibitions because, at most, one node can be active within any row and column. Since a total of M nodes are active within a solution, M-1 other active nodes contribute $(M-1)g_\gamma$ to the input of the active node. Two other active nodes, one node in the previous and the other node in the next column with respect to the column of the active node within the interaction topology of the inhibitory soft constraint, also contribute to the input of the active node,

$L_\eta^0 = 1$. Equation 40 shows that $\Theta_{\max} = \Theta_i = 0$ and $b_{\min} = b_i = 0.5(1-2M)g_\gamma$ for all $i = 1, 2, \ldots, N$. Then the inequality for the active node becomes

$$4|g_\eta| d_{\max} \le |g_\gamma|. \tag{42}$$

This inequality is the only one that needs to be satisfied for the solutions to be the stable equilibrium points of the network dynamics. The bound given by Equation 42 will be comparatively evaluated through a number of previously reported studies on the TSP using the Hopfield network in the following paragraphs.

Hopfield, in his seminal paper [8], employed the continuous node dynamics with self-feedback for the nodes of the network to solve the TSP. Successful results were reported for a problem size of 10 cities, but not for the 30 cities. The constraint weight parameter values for his experiment were specified at or near $g_r = -500$, $g_c = -500$, $g_\gamma = -200$, and $g_\eta = -200$, and M = 15. Using these values of constraint weight parameters in the inequality of Equation 42, the following bound on the distance terms are obtained

$$0 < d_{ij} < 0.1, \text{ for } i, j = 1, 2, \ldots, M,$$

which gives the condition for solutions to be stable points of the network dynamics while noting that analysis of Hopfield's work is approximate since the Theorem bounds are derived by assuming that the network nodes have no self feedback, $w_{ii} = 0$. Therefore, any permutation of the cities, where at least one inter-city distance is greater than 0.1, is not a stable point of the network dynamics. This indicates that the Hopfield network dynamics are biased for the tours with shorter lengths. The trade-off is that some non-solution points are also stable points of the network dynamics. In summary, the network is not expected to converge to a solution after each relaxation for the constraint weight parameter values employed by Hopfield, but the tour length is significantly short when the network converges to a solution. Accordingly, Hopfield reported short tour distances for the solutions found by their network and did not mention instances of convergence to non-solution points for the 10-city TSP. He stated that the network often failed to converge to a permutation array for the 30-city TSP because the stable equilibrium point set included the interior of the unit hypercube due to the nonzero diagonal terms in the weight matrix.

Abe [26] used an alternate formulation of the energy function and the memoryless node dynamics with no self-feedback to address the TSP. He demonstrated that if the following conditions on the constraint weight parameters held:

1) $g_r = g_c$,

2) $b = -g_r$, and

3) $g_r < 2g_\eta d_{\max}$,

then all solutions of the problem became stable points of the Hopfield network dynamics. Abe actually performed the analysis in the Lyapunov/energy space to establish solutions as local energy minima. Therefore, these three conditions on constraint weight parameters must satisfy the Theorem bounds derived for Abe's formulation of the energy function. The nontrivial bound on the constraint weight parameters derived using the proposed Theorem for Abe's formulation of the energy function is given by

$$4|g_\eta| d_{\max} \le |g_\gamma| + |g_c| .$$

Substitution of the equality given by condition 1 in the above bound (derived through the proposed Theorem) yields

$$|g_r| \ge 2|g_\eta| d_{\max} ,$$

which implies the condition 3 given by Abe on the constraint weight parameters. Hence, all of the solutions are stable points of the Hopfield network dynamics for the set of three conditions proposed by Abe. This analysis establishes that Abe's work and the Theorem proposed in this paper lead to equivalent set of conditions on the constraint weight parameters through two different procedures.

Aiyer et al. [25] employed a theoretical approach to define the constraint weight parameters to guide the Hopfield network towards solutions of the TSP. More specifically, values for all constraint weight parameters except the one that enforces the minimum distance constraint were specified using insight developed through theoretical analysis. Authors employed the memoryless node dynamics with self-feedback and incorporated a number of modifications to the dynamics of the standard continuous Hopfield network. They reported that the Hopfield network converged to a solution after every iteration for 10, 30, 40, and 50 city unit-square TSPs. Quality of solutions determined by their network was reported to be as good as those found by the nearest neighbor algorithm. Authors performed an analysis to compare their findings with those of Abe's and determined that weight matrix entries for Abe's formulation could be obtained using the weight matrix definition that Aiyer et al., formulated after setting $A1 = 0$ and $C = 2(A \times N - N) / N^2$. Authors furthermore observed a discrepancy between the external bias term definitions (A for Abe's work and $2(A \times N - A)/N$ for Aiyer et. al.) and could not account for it. The bounds obtained through the Theorem proposed in this paper and those obtained through Abe's procedure are equivalent. In that sense, our work has the same discrepancy Abe's work had with that of Aiyer et. al. noting that our work and Abe's work define all constraint weight parameters through theoretical means while Aiyer et. al. needed to resort to empirical means to define the constraint weight parameter enforcing the minimum distance/cost constraint.

In our simulation study the TSP was mapped to discrete Hopfield network dynamics for the problem sizes of 10, 30, 50, and 100 cities and for three distinct operating points. The distances between cities are random variables uniform in the interval [0, 1]. A total of 100 relaxations were realized for each problem size and operating point pair. The operating points employed in the experiments were chosen at the limiting points of the admissible subspace of the constraint weight parameter space, which is defined by Equation 42. Specifically, small and large values for the difference given by

$$\left|g_\gamma\right| - 4\left|g_\eta\right| d_{\max}$$

were used as the test values for the operating points that are given in Table 2. The network converged to a solution after each relaxation in all test cases. In other words, 100% convergence was realized for all test cases indicating that the stable point set was equal to the solution set for the TSP.

Operating Points	g_r	g_c	g_η	g_γ
1	-1.0	-1.0	-0.1	-1.0
2	-100.0	-100.0	-0.1	-1.0
3	-1.0	-1.0	-0.1	-100.0

Table 2. Operating Point Definitions.

The quality of the solutions, the measure of which is the Normalized Total Distance (NTD), was observed for various problem sizes. The NTD is computed by dividing the total distance of a solution by the number of cities. The problem sizes used in the tests are 10, 30, and 50 cities. A total of 100 relaxations were performed for each problem size. The frequency distributions for the NTD are plotted in Figure 3. Simulation results in Figure 3 indicate that the quality of solutions located by the network tends to average and that the frequency distribution of the distance becomes centered around the average value of 0.5 with less spread as the problem size increases. As the problem size increases, the quality of solutions located by the Hopfield network becomes more average. In other words, the optimization property of the Hopfield network, which is not good for even small size TSPs, does not scale well with the problem size.

Recognizing this deficiency of the Hopfield network for optimization problems, numerous researchers presented extensions [22], enhancements [23] and modifications [24] to the original Hopfield network algorithm to improve its optimization performance. Significant performance improvements in terms of being able to locate good quality solutions for relatively large-scale TSPs are reported in these studies.

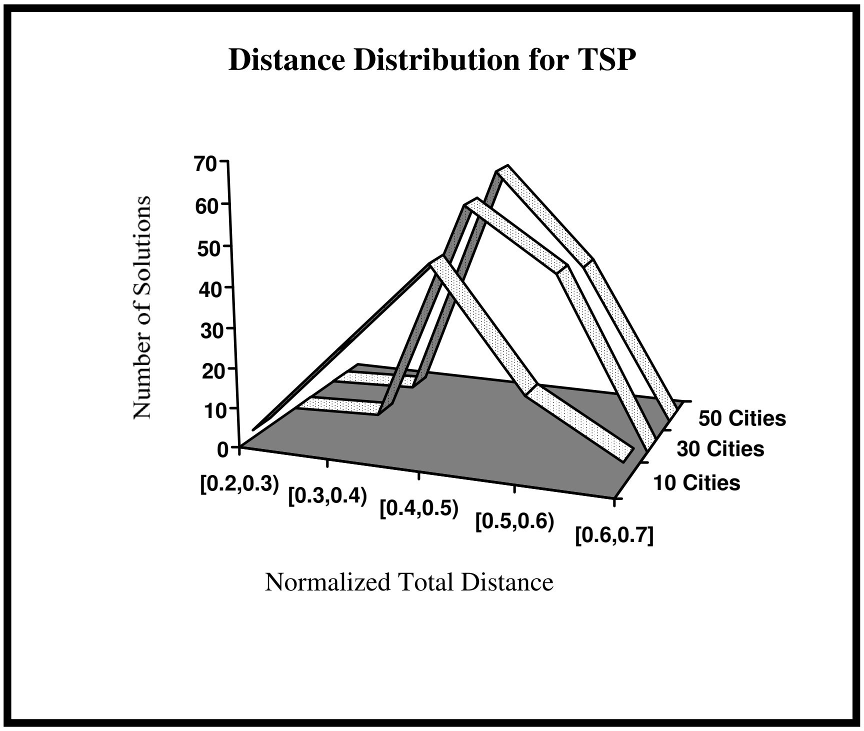

Figure 3. Normalized Distance Distribution for the TSP

6. Conclusions

Bounds on the values of the weight parameters for the single-layer, relaxation-type recurrent neural networks are proposed. These bounds, which are derived through a theorem, establish the solutions of a constraint satisfaction/optimization problem as stable points in the state space of the neural network dynamics. Weight values defined by this theorem guarantees the neural network, the discrete Hopfield network, to converge to an average quality solution after each relaxation for a class of optimization problems, which includes the Traveling Salesman Problem, the Assignment Problem and the Weighted Matching Problem. However, the same bounds also induce the stability of additional non-solution points as well as solution points for another class of optimization problems, which includes the *N*-Queens Problem and the Graph Path Search Problem. Additionally, the number of stable non-solution points grows drastically as the problem size increases for the latter class of problems (at a rate much faster than the growth in the number of solutions), which leads to unacceptable convergence performance for the discrete Hopfield network.

In conclusion, the suggested procedure guarantees a 100% convergence rate and scales (in terms of maintaining the 100% convergence rate) with the problem size for a subset of optimization problems including the Traveling Salesman Problem, the Weighted Matching Problem and the Assignment Problem. The quality of solutions found by the discrete Hopfield network using the proposed bounds on the constraint weight parameters is average as expected from a gradient-descent based search algorithm. Suggested bounds can easily be adapted to the Boltzmann Machine and the Mean Field Annealing algorithms to locate high quality solutions at the expense of significantly increased computational cost.

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