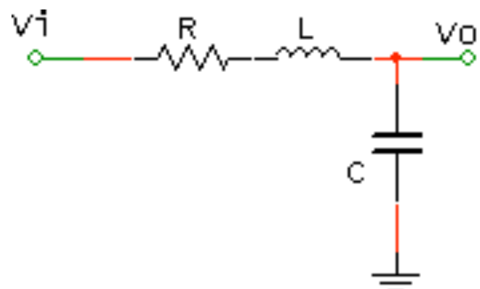
Lab 5 – Second Order Transient Response of Circuits

Lab Performed on November 5, 2008 by Nicole Kato, Ryan Carmichael, and Ti Wu

Report by Ryan Carmichael and Nicole Kato



E11 Laboratory Report – Submitted November 24, 2008

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Abstract:

In this lab, a voltage source, capacitor, inductor, and resistor were connected in series to form an RLC second-order circuit. By measuring the capacitor voltage (V_o), values were obtained for the frequency of oscillation (ω_d), the time constant of the decay envelope (α), the damping ratio (ζ), and the resonant frequency (ω_o) for various resistances. An op amp circuit was also created to achieve an output similar to the 1000Ω RLC circuit, without using inductors, which are the least ideal element in the RCL circuit. With the exceptions of α and ζ for the 100Ω RLC circuit, the experimental results closely followed the theory, with the op amp circuit having less error than its equivalent 1000Ω RLC circuit.

Introduction:

Capacitors and inductors are two of the three passive elements used in circuit design. These two passive elements are not able to dissipate or generate energy, but can return stored energy into a circuit. In the previous lab, it was seen that if one of these passive elements were in a circuit it would form a first order circuit because a first order differential equation would be required to solve for a voltage or current. If we have a capacitor, inductor, and resistor placed in series or parallel with either a voltage or current source we will form a RLC or second-order circuit. It is called a second-order circuit because of the second-order differential required to solve for the voltage or current. One of the most common uses for a second order circuit is tuning a radio frequency, such as an AM/FM radio. Ideal capacitors do not have inductance or resistance, and cannot dissipate energy, and ideal inductors do not have capacitance or resistance, and cannot dissipate energy, even though in reality this is rarely the case.

Theory:

In circuit 1, shown right, we can solve for the output voltage (V_o) using a loop to get formula 1

Vo(t) = Vi(t) – RI(t) -L(di(t)/dt) [formula 1]

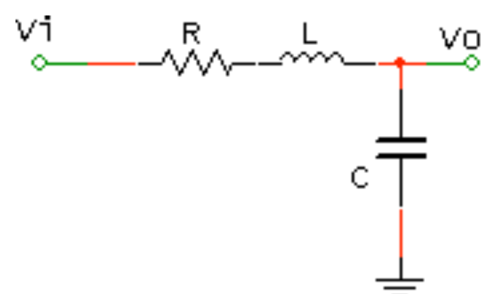

Figure 1: Circuit 1

We can also solve for the current through the circuit using formula 2 below, which was derived in lab 4.

$$I(t) = C\,(dVo(t)/dt) \text{ [formula 2]}$$

If we plug formula 2 into formula 1 we get:

$$LC\,(d^2tVo(t)/dt^2) + RC\,(dtVo(t)/dt) + Vo(t) = Vi(t) \text{ [formula 3]}$$

With formula 3, we can now solve for the homogeneous and particular solutions. To solve for the homogeneous solution, we first set Vi(t) equal to zero and Vo equal to Voh, getting formula 4. Then we substitute formula 5 into formula 4 to get formula 6, which simplifies to formula 7 as shown below.

$$LC\,(d^2tVoh(t)/dt^2) + RC\,(dtVoh(t)/dt) + Voh(t) = 0 \text{ [formula 4]}$$

$$Voh(t) = Ae^{st} \text{ [formula 5]}$$

$$LCS^2Ae^{st} + RCSAe^{st} + Ae^{st} = 0 \text{ [formula 6]}$$

$$S^2 + (R/L)S + (1/LC) = 0 \text{ [formula 7]}$$

We can rewrite formula 7 substituting in the damping ratio, ζ, which equals (R/2L) * $(LC)^{(1/2)}$ and the resonant frequency, ω_o, which equals $(1/LC)^{(1/2)}$ to get formula 8 below.

$$S\text{^}2 + 2\zeta\,\omega o\ s + \omega o^2 = 0 \text{ [formula 8]}$$

We can solve this equation using the quadratic formula of to get formula 9.

$$S = -\zeta\,\omega o \pm \omega o(\zeta^2-1)^{(1/2)} \text{ [formula 9]}$$

We can see from formula 7 that the sign of the root, or more specifically ζ, determines the oscillation form of the circuit's output voltage. If ζ is larger than 1, then there will be two real values of S. This is called the overdamped solution. If ζ is less than 1, then the two values of S will both have imaginary and real components. This is called the underdamped solution. If ζ is equal to 1, then S will only have one unique value. This is called the critically damped solution. However, because there is not a distinct difference in the voltage vs. time graph when S is approximately equal to 1, we shall overlook this case. Table 1 below summarizes the output voltage response if the input voltage is equal to one for both the under and overdamped cases.

	Overdamped	Underdamped
Homogeneous Response	$s_{1,2} = -\zeta\omega_0 \pm \omega_0\sqrt{\zeta^2 - 1}$ $V_{o,h}(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$	$s_{1,2} = -\zeta\omega_0 \pm j\omega_0\sqrt{1-\zeta^2}$ $= -\alpha \pm j\omega_d$ (book's notation) $V_{o,h}(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$ $= e^{-\zeta\omega_0 t}\left(B_1\cos\left(\omega_0\sqrt{1-\zeta^2}t\right) + B_2\sin\left(\omega_0\sqrt{1-\zeta^2}t\right)\right)$ $= Ce^{-\zeta\omega_0 t}\cos\left(\omega_0\sqrt{1-\zeta^2}t + \phi\right)$
Particular Response (for 1 volt input)	$V_{o,p}(t) = 1$	$V_{o,p}(t) = 1$
Complete Response (Find coefficients from initial conditions)	$V_{0,c}(t) = V_{0,p}(t) + V_{0,h}(t)$ $= 1 + A_1 e^{s_1 t} + A_2 e^{s_2 t}$	$V_{0,c}(t) = 1 + Ce^{-\zeta\omega_0 t}\cos\left(\omega_0\sqrt{1-\zeta^2}t + \phi\right)$

Table 1: Summary of Solutions for Overdamped and Underdamped RLC circuits

From Table 1, we note that, to find the complete output voltage response, we must add the homogeneous and particular solutions and apply initial conditions (usually Vo and dVo/dt at time equals 0+) of the circuit to find the unknown constants. We also note that the underdamped and overdamped case have different forms to their solution, i.e., ζ determines the form of the solution. To further examine ζ, we graph the complete responses above with respect to time in Figure 2 keeping ω_o constant. This shows how ζ effects oscillation.

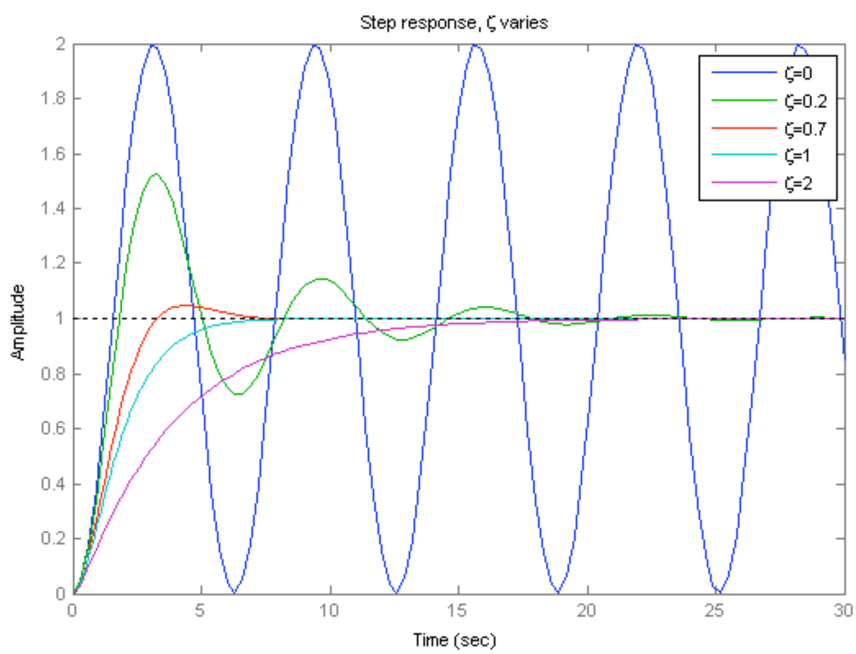

Figure 2: Demonstration of ζ Effect on Oscillation

From Figure 2, we see that as ζ increases the damping effect also increases. The damping effect occurs when the amplitude decreases over time. If there is a large damping effect then the amplitude decreases faster over time, and if there is no damping effect then the amplitude remains constant over time. Now we graph the complete responses again in Figure 3 with respect to the time, keeping ζ constant, to see how ω_o effects the oscillation. We note that, in Figure 3, the damping effect remains constant, but the speed increases as ω_o increases.

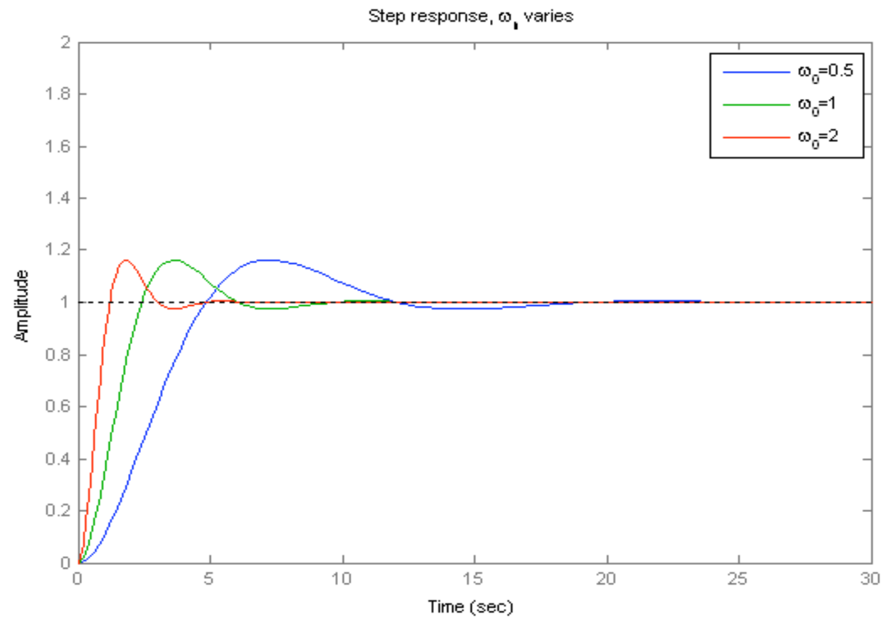

Figure 3: Demonstration of ω_o Effect on Oscillation

For the op-amp circuit in Figure 4 we can find Vi(t) in relation to Vo(t) using Kierchoff's current law or KCL about node o and node a, resulting in formulas 10 and 11.

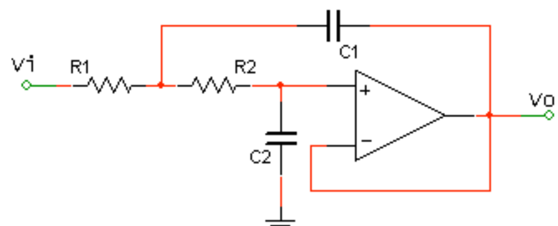

Figure 4: Circuit 2

*∑I = -C2 * (dVo/dt) - (Vo-Va)/R2 = 0 [formula 10]* (at node o)

∑I = C1 ((dVo/dt) - (dVa/dt)) - (Va-Vi)/R1 + (Vo - Va)/R2 = 0 [formula 11] (at node a)

Here we have two equations and two unknowns. Solving for $V_i(t)$, we get formula 12.

$$R_1R_2C_1C_2\frac{d^2v_o(t)}{dt^2}+(R_1+R_2)C_2\frac{dv_o(t)}{dt}+v_o(t)=v_i(t) \quad \text{[formula 12]}$$

The following page contains a more extensive derivation for this op-amp circuit.

Space for added formulas

Procedure:

1. We connected a Wavetek signal generator to the oscilloscope and had the signal generator create a square voltage oscillation that went from 0 to 1 Volts.
2. We created "Circuit 1" as seen in the introduction with R equaling 100 Ohms, C equaling 0.01μF and L equaling 112 mH. We had Vi come from the signal generator and Vout going to the oscilloscope. We also set the frequency to be 100 Hz.
3. We set the triggering to the rising edge and centered it at the origin of our axis on the oscilloscope.
    a. We measure and recorded the input and output of our circuit. We also downloaded the output voltage and a screenshot of our voltage from the oscilloscope.
4. By finding the period and decay of Vout's oscillation, we recorded estimated α and ω_d.
5. We predicted what would happened if R increased.
6. We observed what happened when we increased R and then decreased it back to its original value.
7. We also observed what happened when we increased the charged of C and then discharged it back to its original.
8. We set R to 300 Ohms, 1000 Ohms, 3000 Ohms, and 10,000 Ohms.
    a. We measure and recorded the input and output of our circuit. We also downloaded the output voltage and a screenshot of our voltage from the oscilloscope.
    b. By finding the period and decay of Vout's oscillation when R equaled 1000 Ohms, we recorded estimated α and ω_d and then returned R to its original value.
9. We found what happened when the input frequency was set to 4.6 kHz and found the frequency where the maximum output amplitude would occur.
10. We created "circuit 1" in Multisim where R equals 100 Ohms and 1000 Ohms, C equals 0.01μF and L equals 112 mH.
11. We created the circuit below using a breadbox where R1 equals 1.2 kOhm, R2 equals 9.1 kOhms, C1 equals 0.1 μF, and C2 equals 0.001 μF. We also set the op-amp to a voltage high enough where so it would not become saturated.
12. We repeated steps 3 and 4.
13. We also recreated this circuit in Multisim.

Results:

	100Ω RLC Circuit	300Ω RLC Circuit	1000Ω RLC Circuit	Op Amp Circuit
a theoretical	446.4	----	4464	4714
a experimental	868	1760	4810	4590
a % error	-94.44	----	-7.75	2.63
w_d theoretical	29876.7	----	29544.7	29891
w_d experimental	29500	29300	29100	29700
w_d % error	1.26	----	1.51	0.64
ζ theoretical	0.01494	----	0.1494	0.1558
ζ experimental	0.0294	0.0599	0.1628	0.1545
ζ %error	-96.79	----	-8.97	0.83
ω_o theoretical	29880	----	29880	30261
ω_o experimental	29500	29300	29400	29800
ω_o %error	1.27	----	1.61	1.52
τ theoretical	0.00224	----	0.000224	0.000212
τ experimental	0.0011521	0.0005682	0.0002079	0.0002179
τ %error	48.57	----	7.19	-2.76
T theoretical	0.00021	----	0.000213	0.00021
T experimental	0.0002130	0.0002144	0.0002159	0.0002116
T %error	-1.42	----	-1.37	-0.74

Table 2: Experimental and Theoretical Results

*Percent Error: ((|theo-exp|/theo)*100%))*

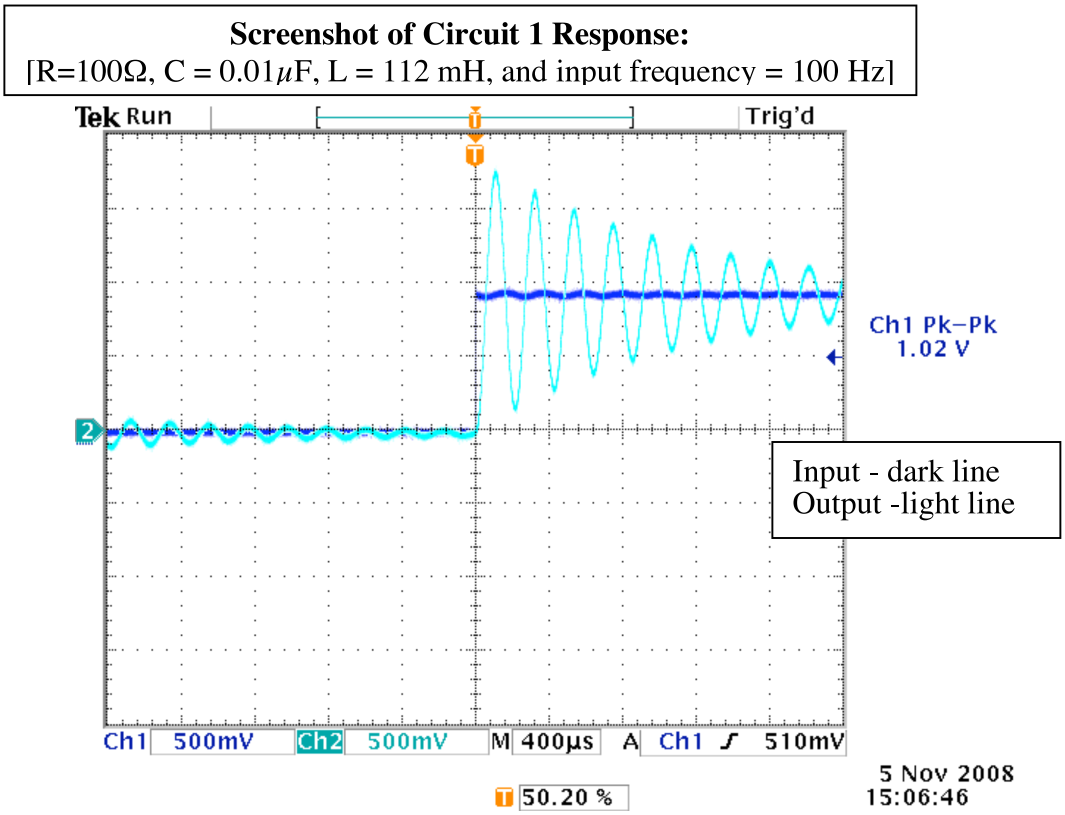

Figure 5: Screenshot of Circuit 1 with the 100Ω Resistor

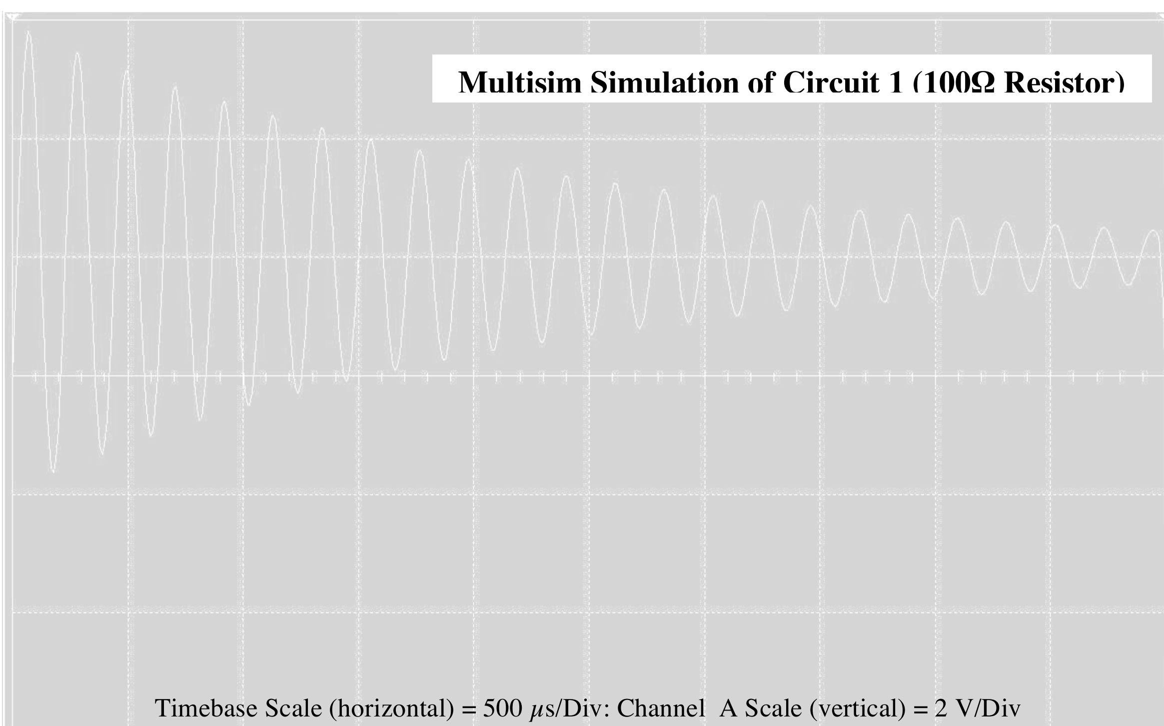

Figure 6: Multisim Simulation of Circuit 1 with the 100Ω Resistor

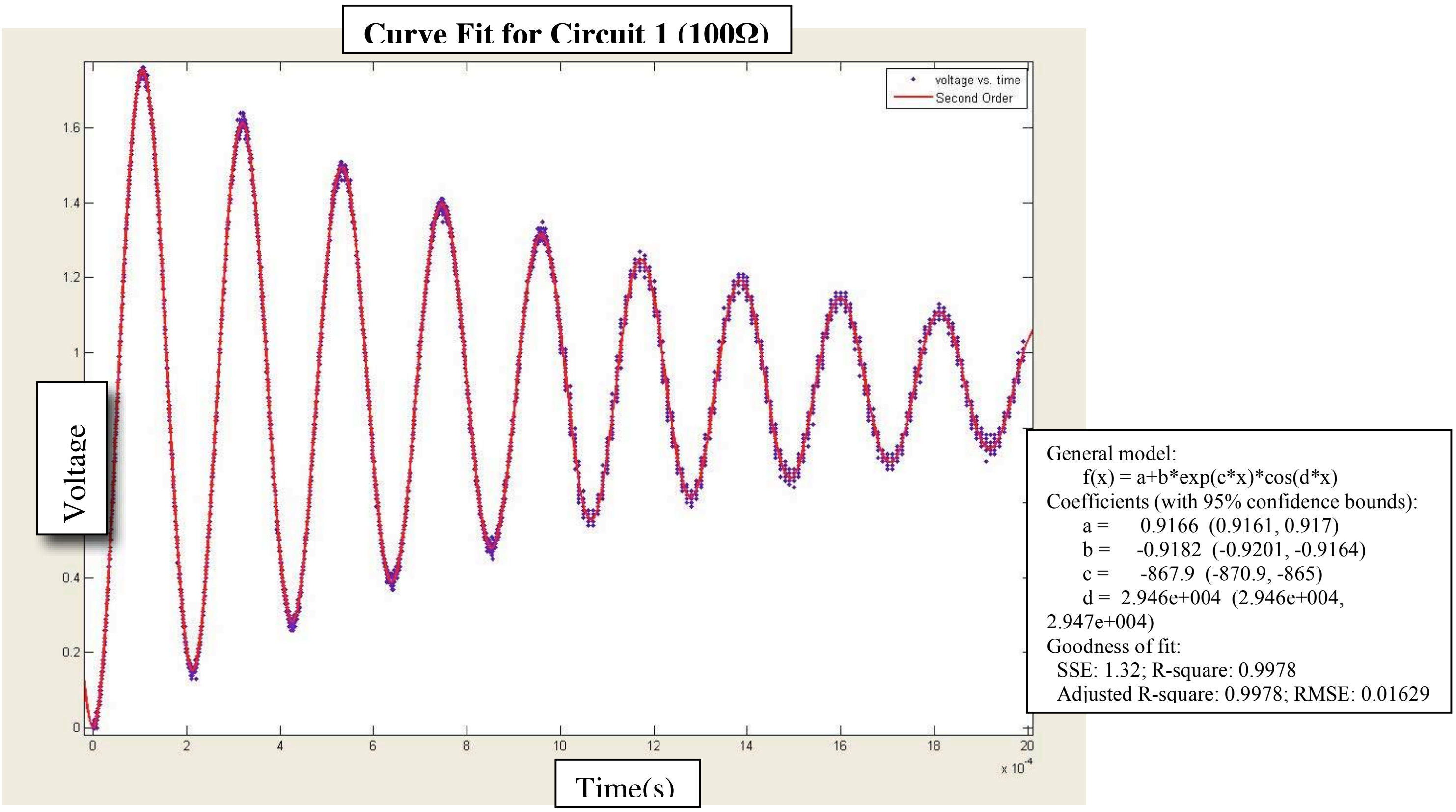

Figure 7: Curve Fit for the Output of Circuit 1 with the 100Ω Resistor

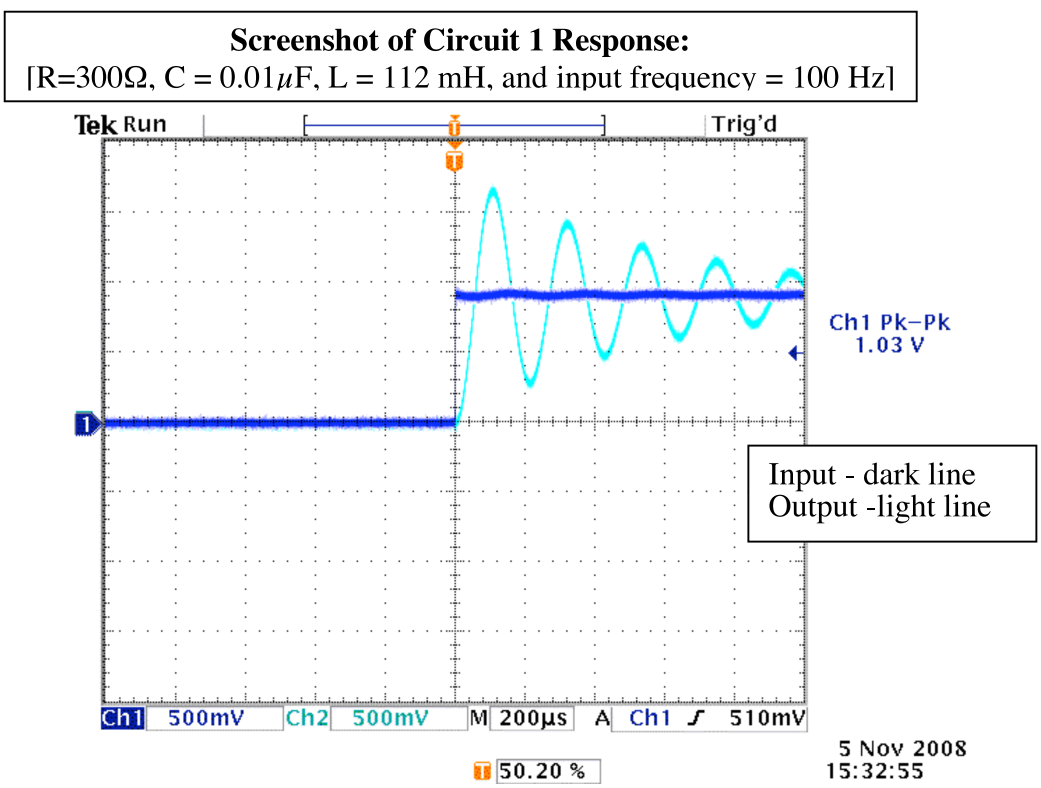

Figure 8: Screenshot of Circuit 1 with the 300Ω Resistor

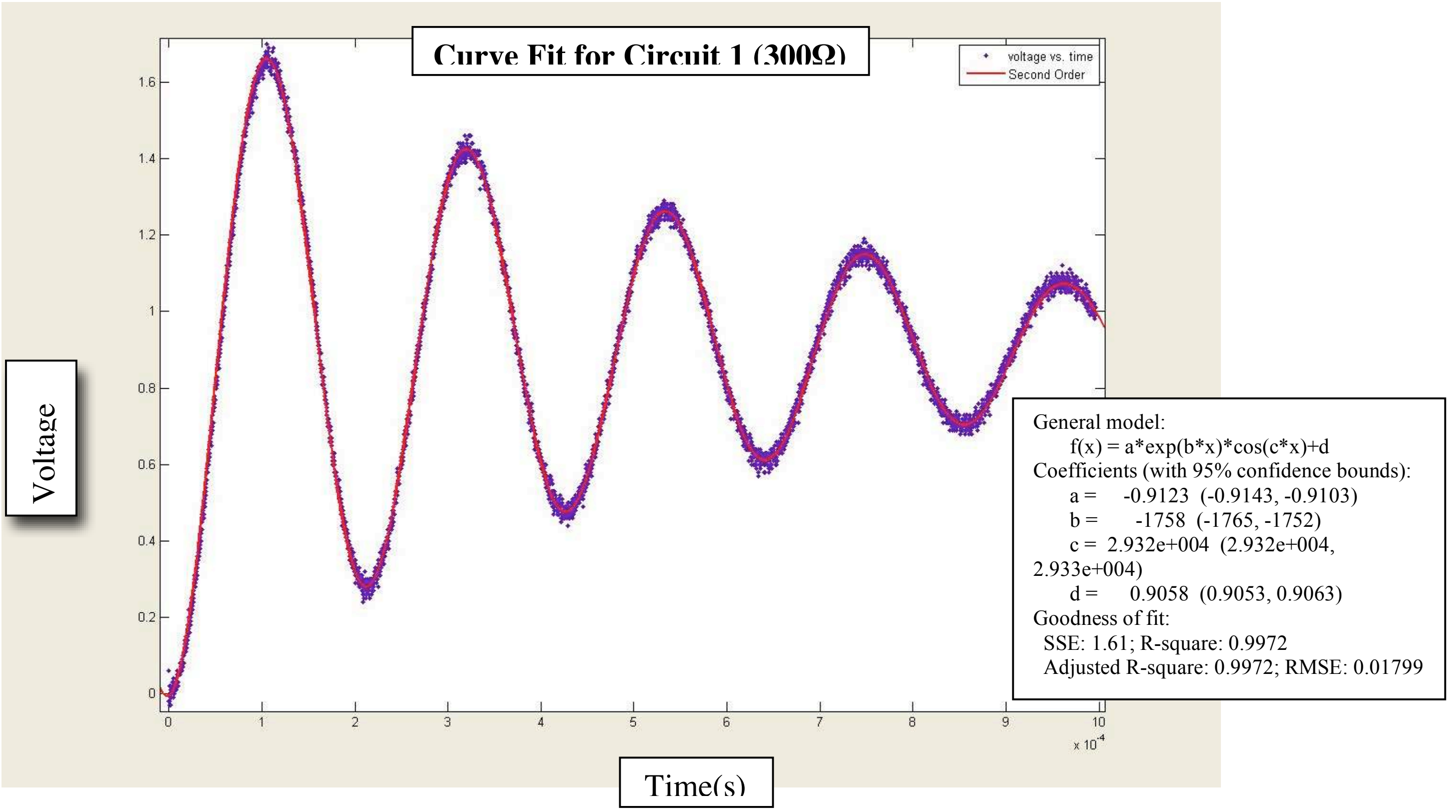

Figure 9: Curve Fit for the Output of Circuit 1 with the 300Ω Resistor

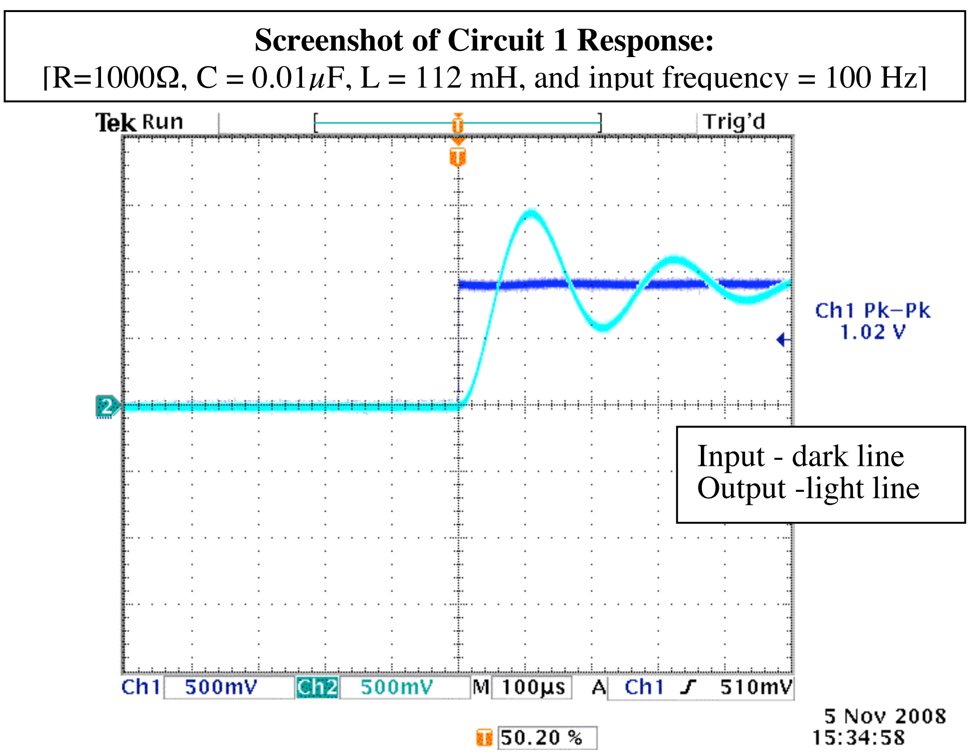

Figure 10: Screenshot of Circuit 1 with the 1000Ω Resistor

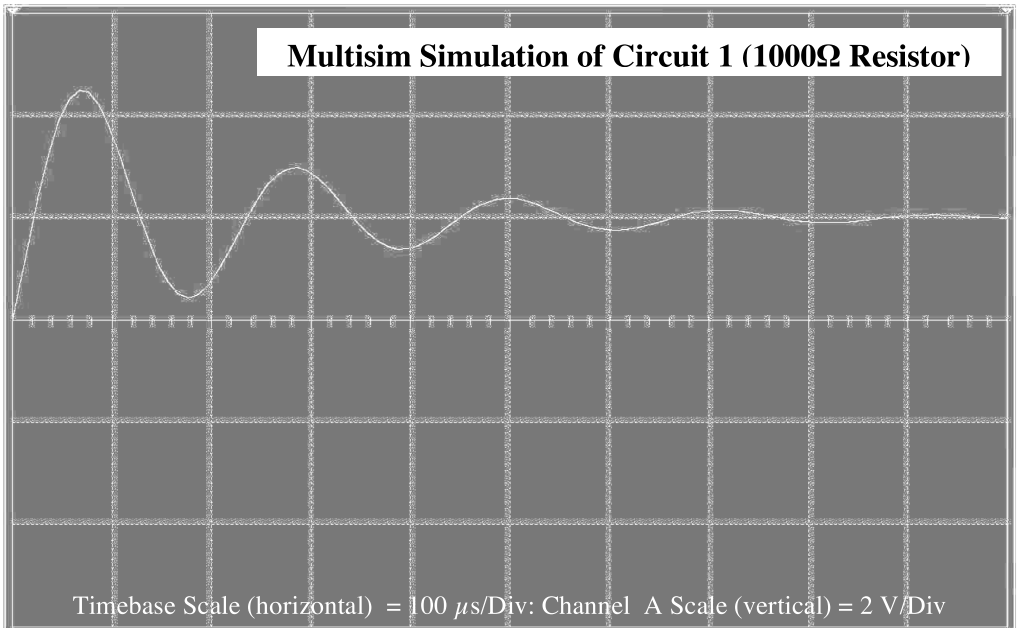

Figure 11: Multisim Simulation of Circuit 1 with the 1000Ω Resistor

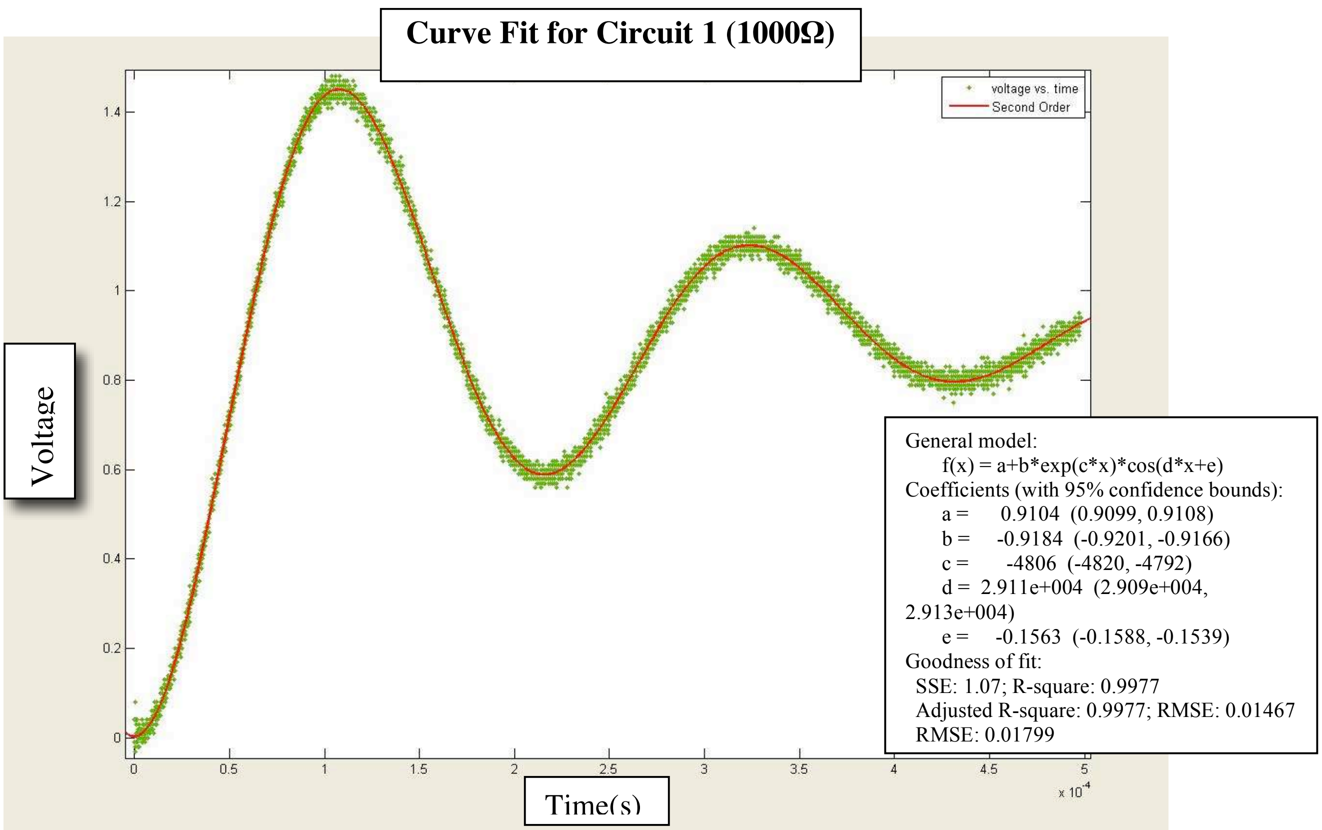

Figure 12: Curve Fit for the Output of Circuit 1 with the 1000Ω Resistor

Screenshot of Circuit 1 Response:
[R=3000Ω, C = 0.01μF, L = 112 mH, and input frequency = 100 Hz]

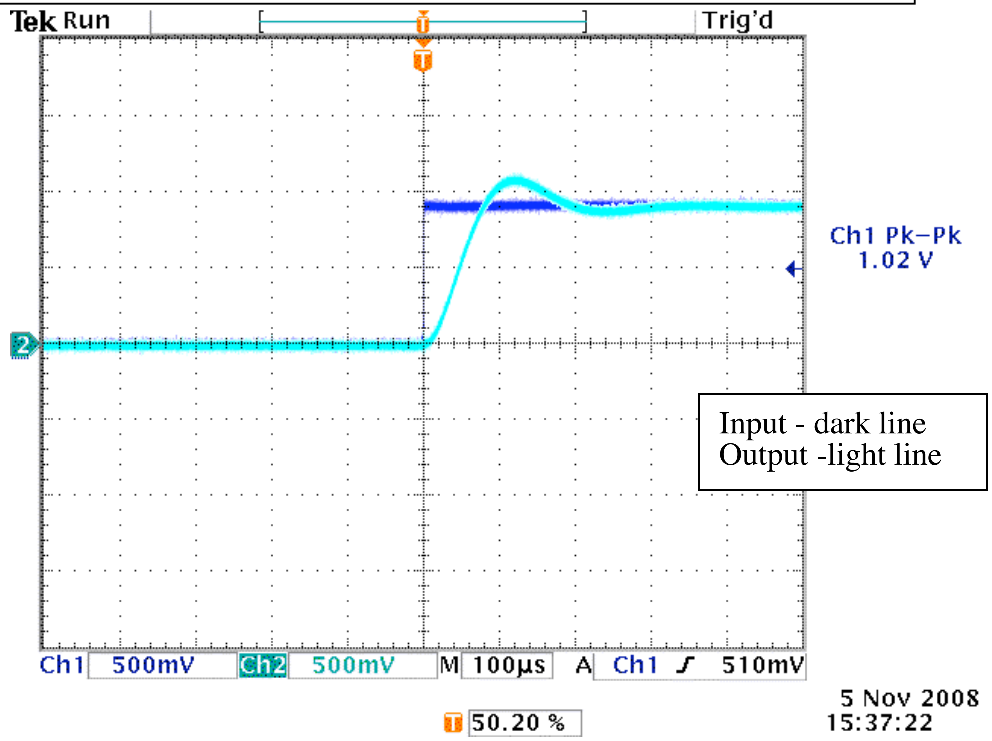

Figure 13: Screenshot of Circuit 1 with the 3000Ω Resistor

Screenshot of Circuit 1 Response:
[R=10000Ω, C = 0.01μF, L = 112 mH, and input frequency = 100 Hz]

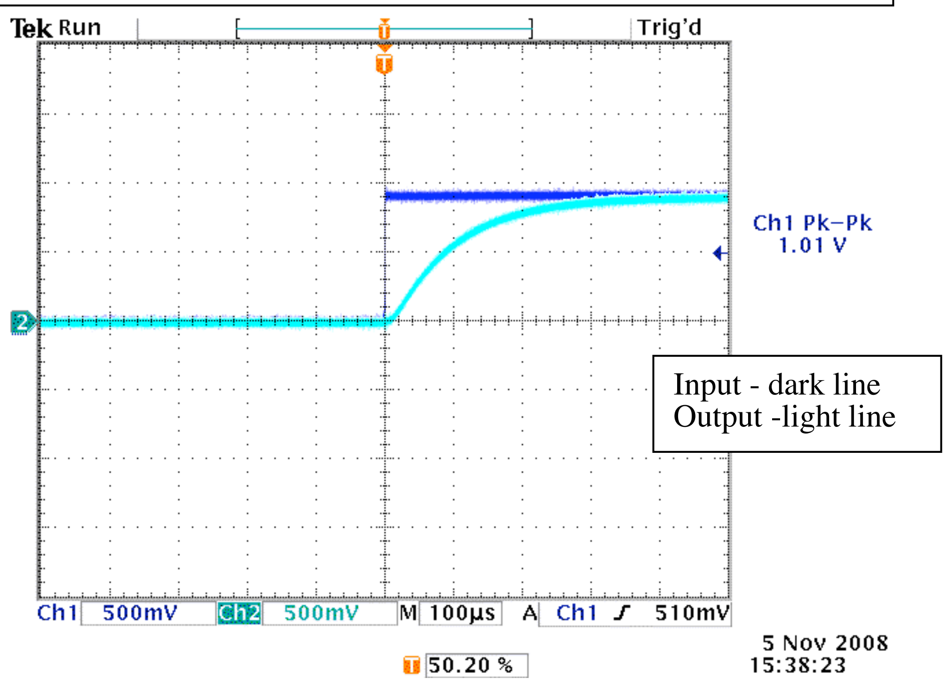

Figure 14: Screenshot of Circuit 1 with the 10000Ω Resistor

Screenshot of Circuit 1 Response:
[R=100Ω, C = 0.01μF, L = 112 mH, and input frequency = 4.6 kHz]

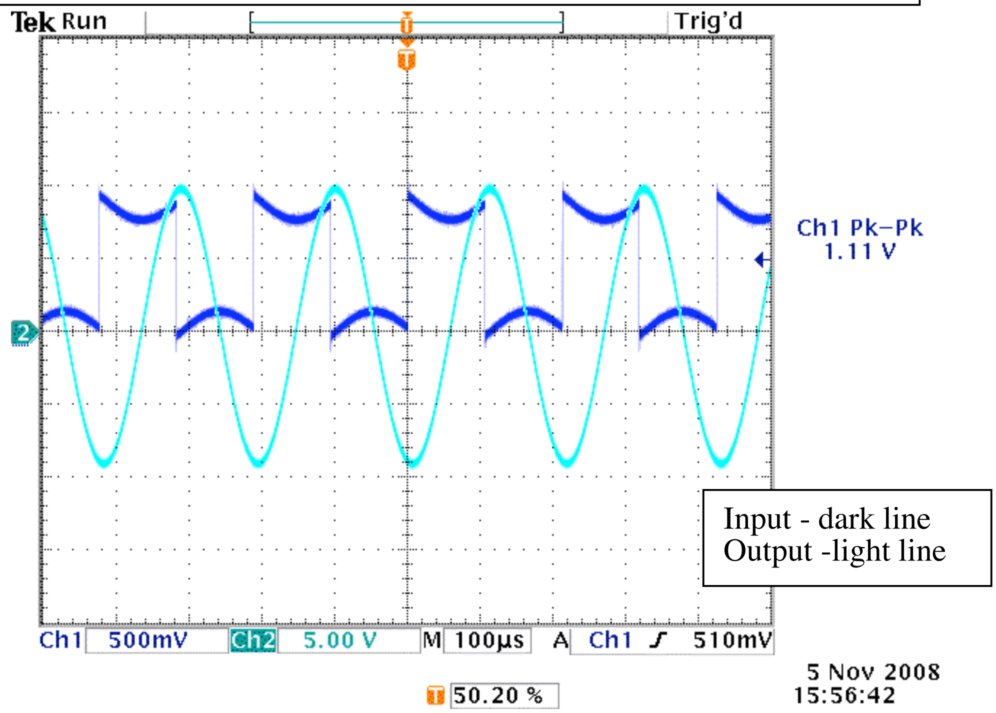

Figure 15: Screenshot of Circuit 1 with the 100Ω Resistor, 4.6 kHz Input Frequency

Screenshot of Op Amp Circuit Response:
[R1 = 1.2 kΩ, R2 = 9.1 kΩ, C1 = 0.1 μF, C2 = 0.001 μF]

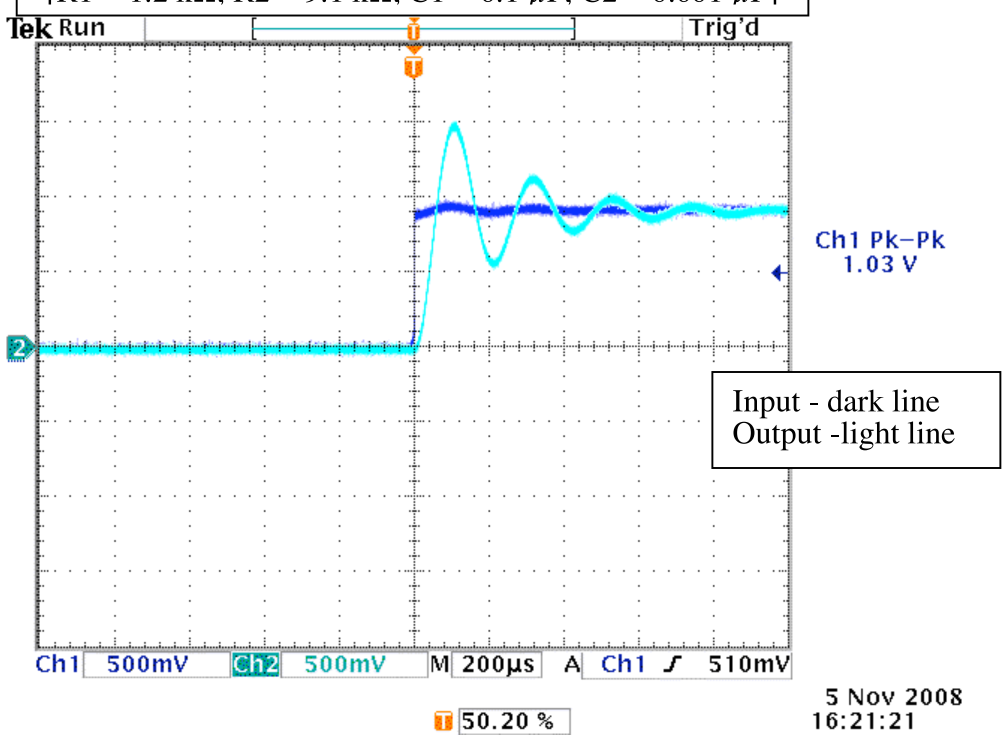

Figure 16: Screenshot of Op Amp Circuit

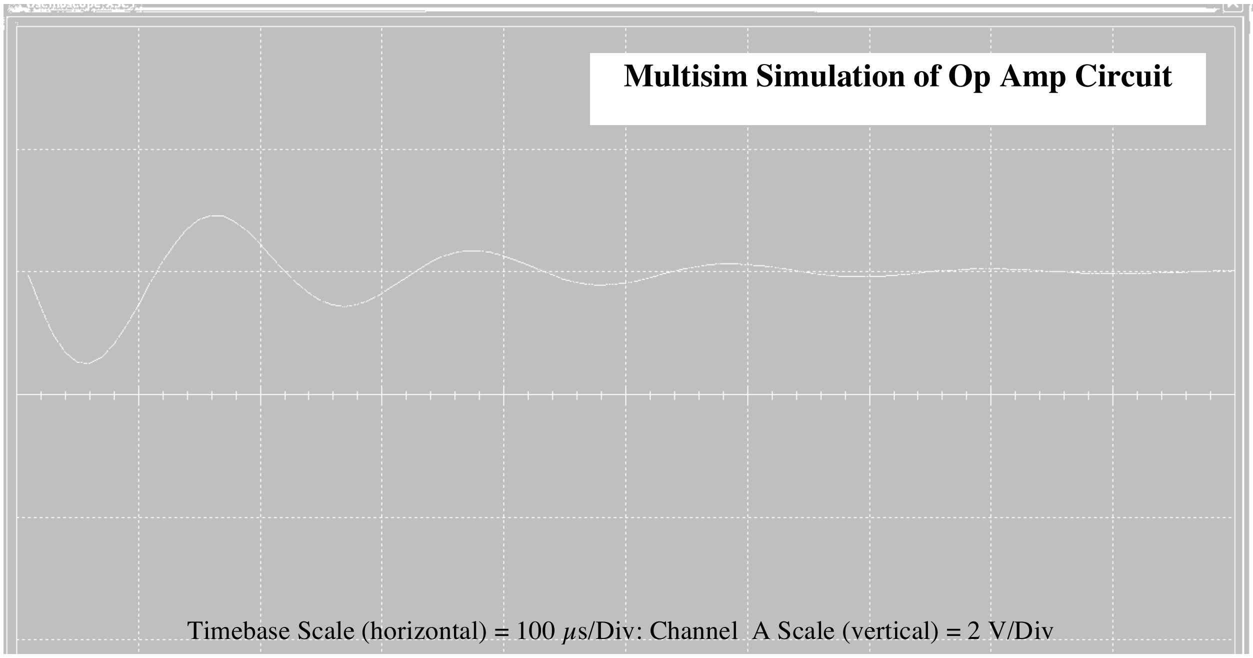

Figure 17: Multisim Simulation of Op Amp Circuit

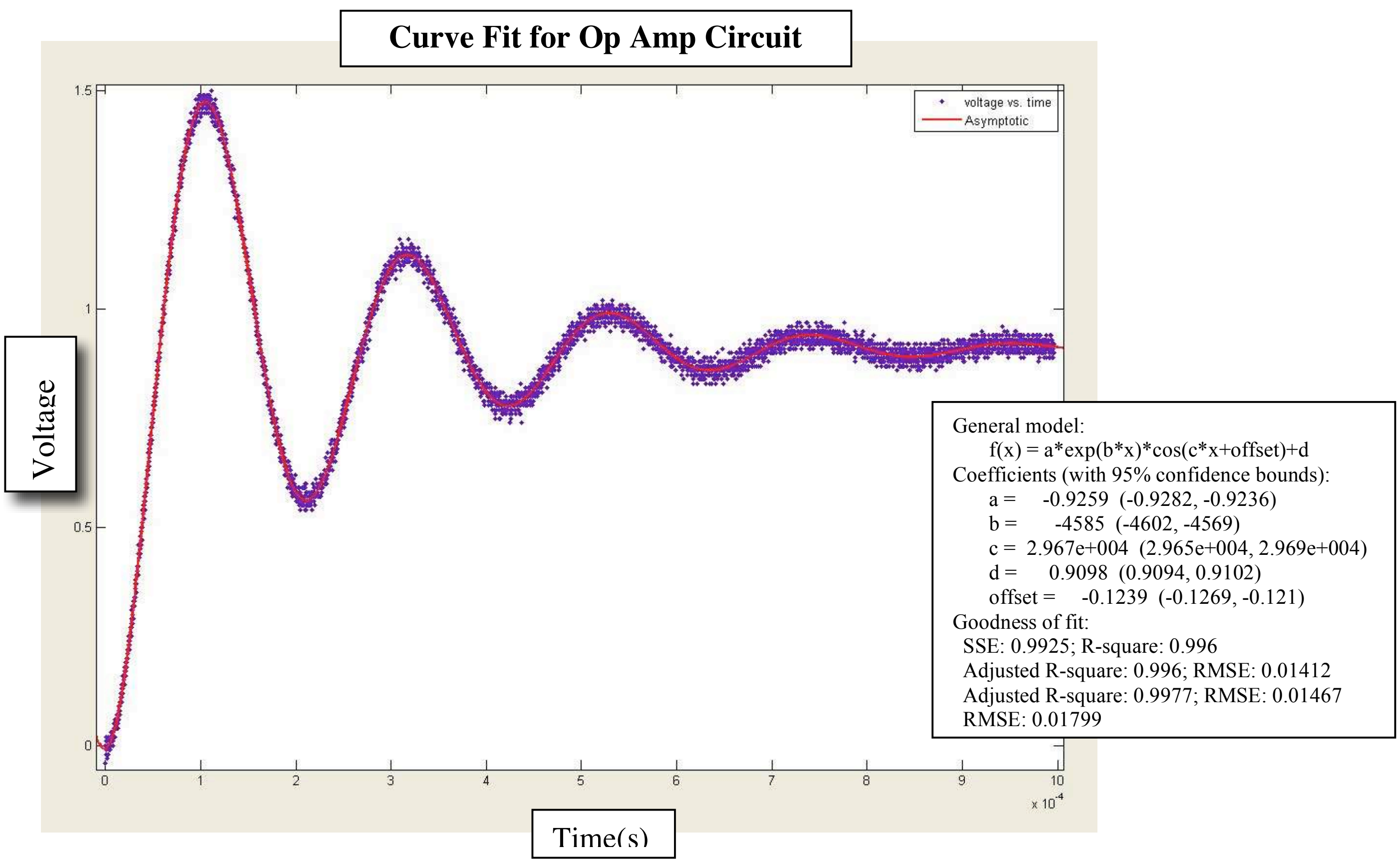

Figure 18: Curve Fit for the Output of the Op Amp Circuit

Discussion:

Taking a qualitative look at the RLC circuit, we note that if the resistance of the circuit is increased, the natural response decays at a faster rate. This is logical because of the following relation: $\alpha = R / 2L$. As R increases, α also increases, causing faster decay. Also, as capacitance is increased, the frequency of oscillation decreases. This is again logical, as the frequency of oscillation equals the following: $\omega = \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$. Here, when C is increased, ω clearly decreases.

Quantitatively, the 100Ω RLC circuit had the largest errors of all the evaluated circuits with a 94.44% error for α, a 96.79% error for ζ, and a 48.57% error for τ. Note that $\tau = \alpha^{-1}$ and, thus, will have the same sources of error. Recalling again that $\alpha = R / 2L$, one major source of error becomes clear: the resistance used to calculate the theoretical value of α was 100Ω. In reality, the function generator should be modeled as an ideal source in series with a 50Ω resistor. Taking this resistance into account, cuts the error by about half to 22.85%. The remainder of this error is likely due to non-ideal circuit elements, especially the inductor, which we know is the least ideal element. Non-ideal elements like the inductor and resistance built into the oscilloscope add resistance to the circuit, which increases α, as seen in the experimental result. Because the resistor is only 100Ω, relatively small resistances like resistance in the wire, for instance, will have a noticeable effect on the overall circuit resistance, where as these small resistances will not play nearly so large a role in the 1000Ω resistor RLC circuit.

Similar to α, $\zeta = \frac{R}{2L\sqrt{\omega^2{}_o - \alpha^2}}$ is largely affected by the error in resistance to the circuit. It should be noted that the major error comes from the numerator. The denominator will provide a small error, which causes ζ to have slightly more error than α as seen. This error is minimal because ζ is governed by ω_d, which is not affected by resistance and has a very small error. ω_d governs the denominator because $\omega_o = \sqrt{\omega^2 + \alpha^2}$. While α has some error associated with it, ω or ω_d is much larger than α, which causes ω_0 to have a small error. This was confirmed in our experimental results. ω_0 is also much larger than α, and L can be assumed to have little error, so the denominator retains minimal error.

For the 1000Ω RLC circuit, our error was very small as expected for the ω values (~1.5%) and a reasonable error for α (7.75%) and ζ (8.97%). This error is due to the same sources as the error for the 100Ω circuit; only here the overall effect of the added resistances is less significant because of the large resistor value. As expected, ζ has slightly more error than α just as in the 100Ω circuit. By comparing the errors from the 1000Ω circuit with that of the op amp circuit, it becomes apparent that the inductor has a significant resistance as the error for the op amp circuit is much smaller than that of the 1000Ω RLC circuit, which has similar theoretical values. Recall here that the op amp circuit was created to achieve a similar output to the 1000Ω RLC circuit, while eliminating the inductor, the least ideal element.

Conclusions and Future Work:

In conclusion, with the exception of an error due to resistance in the function generator, inductor, capacitor, oscilloscope, and wires, which is magnified for RLC circuits with small resistors, the RLC circuit created closely follows theoretical results. Some of this error, specifically resistance due to the inductor, was successfully eliminated by creating an op amp circuit with similar theoretical results as the 1000Ω RLC circuit. As a result, this op amp circuit followed most closely to the theory.

Acknoledgements:

Cheever, Erik. "E11 Lab 5 (2nd order time domain response) - Procedure." Swarthmore College. 23 Nov. 2008 <http://www.swarthmore.edu/NatSci/echeeve1/Class/e11/E11L5/Lab5(Procedure).html>.

Cheever, Erik. "E11 Lab 5 (2nd order time domain response) -Background." Swarthmore College. 23 Nov. 2008 <http://www.swarthmore.edu/NatSci/echeeve1/Class/e11/E11L5/Lab5(BackGround).html>.

Cheever, Erik. "E11 Lab 5 (2nd order time domain response) - Report." Swarthmore College. 23 Nov. 2008 <http://www.swarthmore.edu/NatSci/echeeve1/Class/e11/E11L5/Lab5(Report).html>.