LP in Standard and Slack Forms

$$
\begin{aligned}
max \quad & \sum_{j=1}^{n} c_j x_j \\
s.t. \quad & \sum_{j=1}^{n} a_{ij} x_j \quad \leq \quad b_i \quad for \quad i = 1, 2, \ldots, m \\
& \qquad\qquad x_j \quad \geq \quad 0 \quad for \quad j = 1, 2, \ldots, n
\end{aligned}
$$

$$
\begin{aligned}
z \quad & = \quad 0 + \sum_{j=1}^{n} c_j x_j \\
x_{n+i} \quad & = \quad b_i - \sum_{j=1}^{n} a_{ij} x_j \quad for \quad i = 1, 2, \ldots, m
\end{aligned}
$$

Auxiliary Linear Program

- L: LP in standard form:

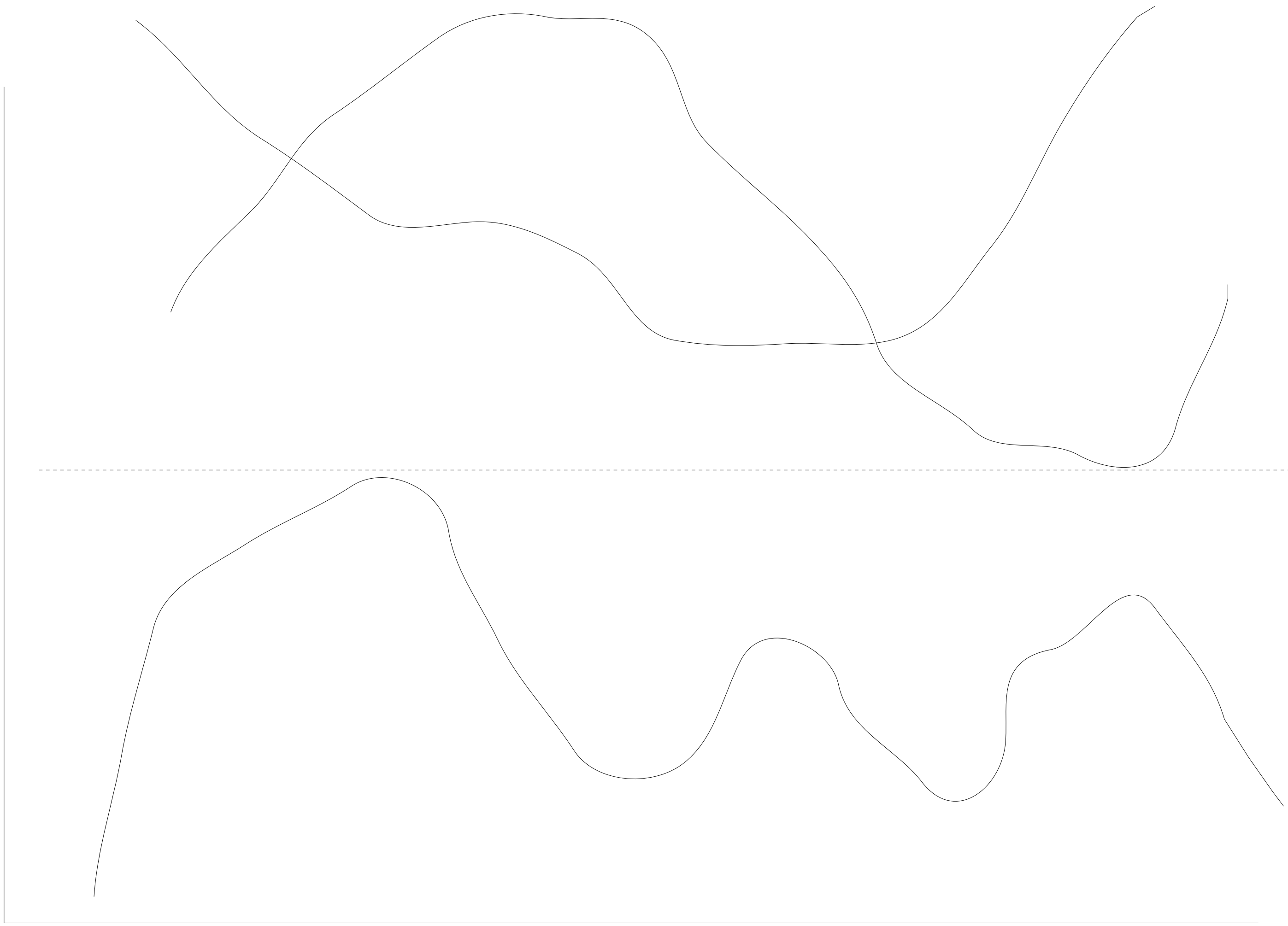
$$
\begin{aligned}
max \quad & \sum_{j=1}^{n} c_j x_j \\
s.t. \quad & \sum_{j=1}^{n} a_{ij} x_j \quad \leq \quad b_i \quad for \quad i = 1, 2, \ldots, m \\
& \qquad\qquad x_j \quad \geq \quad 0 \quad for \quad j = 1, 2, \ldots, n
\end{aligned}
$$

- L_{aux}: Auxiliary LP:

$$
\begin{aligned}
max \quad & \qquad -x_0 \\
s.t. \quad & \sum_{j=1}^{n} a_{ij} x_j - x_0 \quad \leq \quad b_i \quad for \quad i = 1, 2, \ldots, m \\
& \qquad\qquad x_j \quad \geq \quad 0 \quad for \quad j = 0, 1, 2, \ldots, n
\end{aligned}
$$

- L_{aux} is bounded and feasible.

Duality



Upper Bounds on Maximization LP

$$\begin{array}{lccccccccc} max & 4x_1 & + & x_2 & + & 5x_3 & + & 3x_4 & & \\ s.t. & x_1 & - & x_2 & - & x_3 & + & 3x_4 & \leq & 1 \\ & 5x_1 & + & x_2 & + & 3x_3 & + & 8x_4 & \leq & 55 \\ & -x_1 & + & 2x_2 & + & 3x_3 & - & 5x_4 & \leq & 3 \\ & x_1, & & x_2, & & x_3, & & x_4 & \geq & 0 \end{array}$$

- Multiply second constraint by 5/3:

$$\frac{25}{3}x_1 + \frac{5}{3}x_2 + 5x_3 + \frac{40}{3}x_4 \leq \frac{275}{3}$$

$$4x_1 + x_2 + 5x_3 + 3x_4 \leq \frac{25}{3}x_1 + \frac{5}{3}x_2 + 5x_3 + \frac{40}{3}x_4 \leq \frac{275}{3}$$

Upper Bounds on Maximization LP

$$
\begin{array}{lrcrcrcrcr}
max & 4x_1 & + & x_2 & + & 5x_3 & + & 3x_4 & & \\
s.t. & x_1 & - & x_2 & - & x_3 & + & 3x_4 & \leq & 1 \\
 & 5x_1 & + & x_2 & + & 3x_3 & + & 8x_4 & \leq & 55 \\
 & -x_1 & + & 2x_2 & + & 3x_3 & - & 5x_4 & \leq & 3 \\
 & x_1, & & x_2, & & x_3, & & x_4 & \geq & 0
\end{array}
$$

- Add the second constraint to the third constraint:

$$4x_1 + 3x_2 + 6x_3 + 3x_4 \leq 58$$

$$4x_1 + x_2 + 5x_3 + 3x_4 \leq 4x_1 + 3x_2 + 6x_3 + 3x_4 \leq 58$$

Upper Bounds on Maximization LP

$$
\begin{array}{lrcrcrcrcl}
max & 4x_1 & + & x_2 & + & 5x_3 & + & 3x_4 & & \\
s.t. & x_1 & - & x_2 & - & x_3 & + & 3x_4 & \leq & 1 \\
 & 5x_1 & + & x_2 & + & 3x_3 & + & 8x_4 & \leq & 55 \\
 & -x_1 & + & 2x_2 & + & 3x_3 & - & 5x_4 & \leq & 3 \\
 & x_1, & & x_2, & & x_3, & & x_4 & \geq & 0
\end{array}
$$

- Construct a linear combination of the constraints using nonnegative multipliers y_1, y_2, and y_3:

$$(y_1+5y_2-y_3)x_1+(-y_1+y_2+2y_3)x_2+(-y_1+3y_2+3y_3)x_3+(3y_1+8y_2-5y_3)x_4\leq y_1+55y_2+3y_3$$

- Left-hand side will be an upper bound for the LP if the coefficients at each x_j are at least as big as the corresponding coefficients in the objective function

$$y_1+5y_2-y_3\geq 4 \qquad -y_1+y_2+2y_3\geq 1 \qquad 3y_1+8y_2-5y_3\geq 3 \qquad -y_1+3y_2+3y_3\geq 5$$

- Any set of nonnegative multipliers y_i satisfying these inequalities also satisfies $4x_1+x_2+5x_3+3x_4\leq y_1+55y_2+3y_3$
- Good upper bound: minimize right-hand side s.t. constraints.

Good Upper Bound

$$
\begin{array}{lrcrcrcl}
min & y_1 & + & 55y_2 & + & 3y_3 & & \\
s.t. & y_1 & + & 5y_2 & - & y_3 & \geq & 4 \\
 & -y_1 & + & y_2 & + & 2y_3 & \geq & 1 \\
 & -y_1 & + & 3y_2 & + & 3y_3 & \geq & 5 \\
 & 3y_1 & + & 8y_2 & - & 5y_3 & \geq & 3 \\
 & y_1, & & y_2, & & y_3 & \geq & 0
\end{array}
$$

Duality

- The identification of a dual problem is almost always coupled with the discovery of a polynomial-time algorithm.
- Duality provides a proof that a solution is optimal.

LP in Standard Form and Its Dual

$$
\begin{array}{lccccccccc}
max & 3x_1 & + & x_2 & + & 2x_3 & & \\
s.t. & x_1 & + & x_2 & + & 3x_3 & \leq & 30 \\
 & 2x_1 & + & 2x_2 & + & 5x_3 & \leq & 24 \\
 & 4x_1 & + & x_2 & + & 2x_3 & \leq & 36 \\
 & x_1 & , & x_2 & , & x_3 & \geq & 0
\end{array}
$$

$$
\begin{array}{lccccccccc}
min & 30y_1 & + & 24y_2 & + & 36y_3 & & \\
s.t. & y_1 & + & 2y_2 & + & 4y_3 & \geq & 3 \\
 & y_1 & + & 2y_2 & + & y_3 & \geq & 1 \\
 & 3y_1 & + & 5y_2 & + & 2y_3 & \geq & 2 \\
 & y_1 & , & y_2 & , & y_3 & \geq & 0
\end{array}
$$

LP in Standard Form and Its Dual

$$\begin{aligned} max \quad & \sum_{j=1}^{n} c_j x_j \\ s.t. \quad & \sum_{j=1}^{n} a_{ij} x_j \le b_i \quad for \quad i = 1, 2, \ldots, m \\ & x_j \ge 0 \quad for \quad j = 1, 2, \ldots, n \end{aligned}$$

$$\begin{aligned} min \quad & \sum_{i=1}^{m} b_i y_i \\ s.t. \quad & \sum_{i=1}^{m} a_{ij} y_i \ge c_j \quad for \quad j = 1, 2, \ldots, n \\ & y_i \ge 0 \quad for \quad i = 1, 2, \ldots, m \end{aligned}$$

Weak Duality

- ***x****: feasible solution to the primal LP.
- ***y****: feasible solution to the dual LP.
- Claim $\sum_{j=1}^{n} c_j x_j^* \leq \sum_{i=1}^{m} b_i y_i^*$
- Proof:

$$
\begin{aligned}
\sum_{j=1}^{n} c_j x_j^* &\leq \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} y_i^*\right) x_j^* && \text{from the dual} \\
&= \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j^*\right) y_i^* \\
&\leq \sum_{i=1}^{m} b_i y_i^* && \text{from the primal}
\end{aligned}
$$

Importance of Weak Duality

- ***x****: feasible solution to the primal LP.
- ***y****: feasible solution to the dual LP.
- If

$$\sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_i^*$$

then ***x**** and ***y**** are optimal solutions to the primal and to the dual LPs, respectively.

Final Feasible Basic Solution and Corresponding Dual Solution

$$
\begin{array}{llllllllll}
max & z & = & 28 & - & x_3/6 & - & x_5/6 & - & 2x_6/3 \\
s.t. & x_4 & = & 18 & - & x_3/2 & + & x_5/2 & + & 0x_6 \\
 & x_2 & = & 4 & - & 8x_3/3 & - & 2x_5/3 & + & x_6/3 \\
 & x_1 & = & 8 & + & x_3/6 & + & x_5/6 & - & x_6/3
\end{array}
$$

Basic variables: x_1=8, x_2=4, x_4=18

Objective value z = 28

$$
y_i = \begin{cases} -c'_{n+i} & \text{if } (n+i) \in N \\ 0 & \text{otherwise} \end{cases}
$$

y_1 = 0 (since x_4 is basic), y_2 = 1/6, y_3 = 2/3

$$
\begin{array}{llllllll}
min & 30y_1 & + & 24y_2 & + & 36y_3 & & \\
s.t. & y_1 & + & 2y_2 & + & 4y_3 & \geq & 3 \\
 & y_1 & + & 2y_2 & + & y_3 & \geq & 1 \\
 & 3y_1 & + & 5y_2 & + & 2y_3 & \geq & 2 \\
 & y_1 & , & y_2 & , & y_3 & \geq & 0
\end{array}
$$

Feasible Solution to the Dual

$$
\begin{array}{lccccccc}
min & 30y_1 & + & 24y_2 & + & 36y_3 & & \\
s.t. & y_1 & + & 2y_2 & + & 4y_3 & \geq & 3 \\
 & y_1 & + & 2y_2 & + & y_3 & \geq & 1 \\
 & 3y_1 & + & 5y_2 & + & 2y_3 & \geq & 2 \\
 & y_1 & , & y_2 & , & y_3 & \geq & 0
\end{array}
$$

$y_1 = 0$ (since x_4 is basic)
$y_2 = 1/6$
$y_3 = 2/3$

Objective value: 30 x 0 + 24 x 1/6 + 36 x 2/3 = 28

1 x 0 + 2 x 1/6 + 4 x 2/3 = 3
1 x 0 + 2 x 1/6 + 1 x 2/3 = 1
3 x 0 + 5 x 1/6 + 2 x 2/3 = 13/6

Duality Theorem

- Suppose that SIMPLEX terminates with a feasible basic solution ***x**** = (x_1*,x_2*,..., x_n*) with:
  - *N* and *B* denoting the nonbasic and basic variables for the final slack form
  - *c*' denoting the coefficients of the objective function in the final slack form.
- Let ***y**** = (y_1*, y_2*, ..., y_m*) be defined by

$$y_i^* = \begin{cases} -c'_{n+i} & \text{if } (n+i) \in N \\ 0 & \text{otherwise} \end{cases}$$

- Then ***x**** is an optimal solution to the primal LP, ***y**** is an optimal solution to the dual LP and

$$\sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_i^*$$

Proof of Duality Theorem

- We have to show that
  - y^* is feasible solution for the dual, and
  - $\sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_i^*$

Proof of Duality Theorem

- Objective function of the final slack form of the primal is:

$$v^* + \sum_{j \in N} c_j^* x_j = v^* + \sum_{j \in N} c_j^* x_j + \sum_{j \in B} 0\, x_j = v^* + \sum_{j=1}^{n+m} c_j^* x_j$$

- Optimal value of the primal objective function: $v^* = \sum_{j=1}^{n} c_j x_j^*$

$$\sum_{j=1}^{n} c_j x_j = v^* + \sum_{j=1}^{n} c_j^* x_j + \sum_{j=n+1}^{n+m} c_j^* x_j = v^* + \sum_{j=1}^{n} c_j^* x_j - \sum_{i=1}^{m} y_i^* (b_i - \sum_{j=1}^{n} a_{ij} x_j)$$

$$= (v^* - \sum_{i=1}^{m} b_i y_i^*) + \sum_{j=1}^{n} (c_j^* + \sum_{i=1}^{m} a_{ij} y_i^*) x_j$$

- This must hold for every choice of x_1, x_2, ..., x_n. Hence
  $v^* = \sum_{i=1}^{m} b_i y_i^*$ and $c_j = c_j^* + \sum_{i=1}^{m} a_{ij} y_i^*, \forall j = 1, 2, \ldots, n$

- Since $c_j^* \le 0, \forall k = 1, 2, \ldots, n+m$, we get

$$\sum_{i=1}^{m} a_{ij} y_i^* \ge c_j, \forall j = 1, 2, \ldots, n \quad \text{and} \quad y_i^* \ge 0, \forall i = 1, 2, \ldots, m$$

Primal Dual Combinations

		Dual		
		Optimal	**Infeasible**	**Unbounded**
	Optimal	*Possible*	*Impossible*	*Impossible*
Primal	**Infeasible**	*Impossible*	*Possible*	*Possible*
	Unbounded	*Impossible*	*Possible*	*Impossible*

Both Primal and Dual Infeasible

$$
\begin{array}{lrcrcr}
max & 2x_1 & - & x_2 & & \\
s.t. & x_1 & - & x_2 & \leq & 1 \\
 & -x_1 & + & x_2 & \leq & -2 \\
 & x_1, & & x_2 & \geq & 0
\end{array}
$$

$$
\begin{array}{lrcrcr}
min & y_1 & - & 2y_2 & & \\
s.t. & y_1 & - & y_2 & \geq & 2 \\
 & -y_1 & + & y_2 & \geq & -1 \\
 & y_1, & & y_2 & \geq & 0
\end{array}
$$

Practical Implications

- If $m >> n$ then the number of constraint in the dual will be much smaller than in the primal.
- Number of pivots in SIMPLEX is usually less than $1.5m$ and only rarely is higher than $3m$.
- Number of pivots increases very slowly with n.
- Solving dual will in such cases be more efficient.

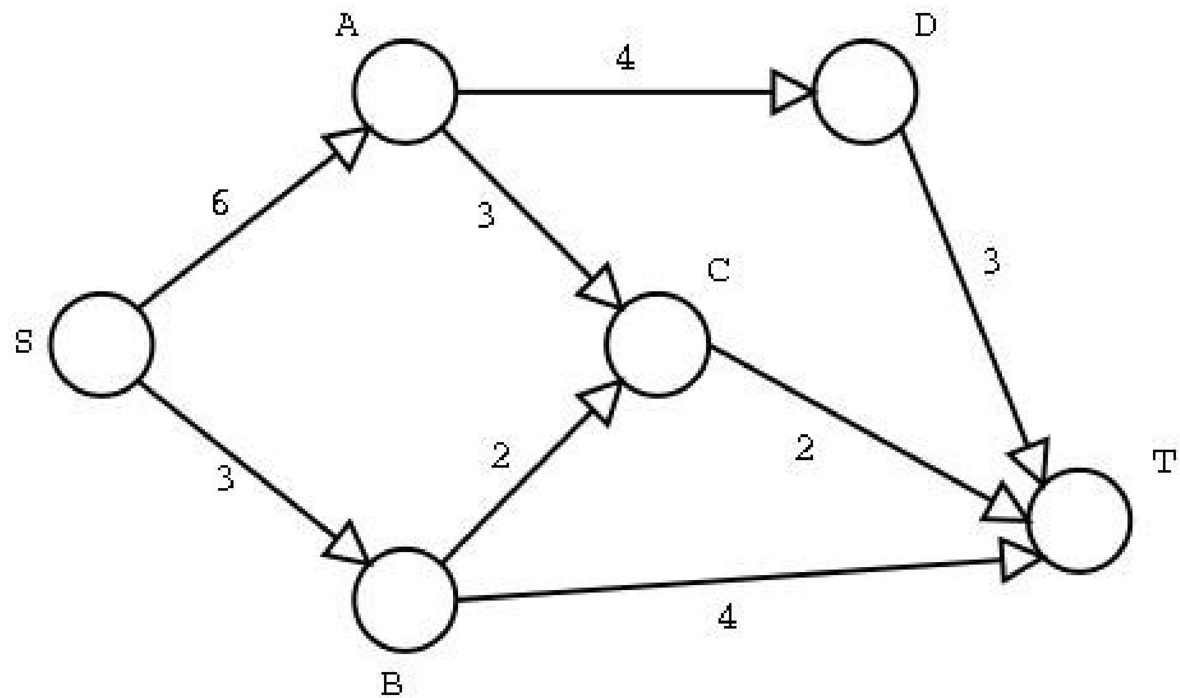
Certificate of Optimality

$$
\begin{aligned}
max \quad & \sum_{j=1}^{n} c_j x_j \\
s.t. \quad & \sum_{j=1}^{n} a_{ij} x_j \le b_i \quad && for \quad i = 1, 2, \dots, m \\
& x_j \ge 0 && for \quad j = 1, 2, \dots, n
\end{aligned}
$$

$$
\begin{aligned}
min \quad & \sum_{i=1}^{m} b_i y_i \\
s.t. \quad & \sum_{i=1}^{m} a_{ij} y_i \ge c_j \quad && for \quad j = 1, 2, \dots, n \\
& y_i \ge 0 && for \quad i = 1, 2, \dots, m
\end{aligned}
$$

Maximum Flow

- **Given**: A directed graph $G = (V,E)$ where each edge $(u,v) \in E$ has a real-valued, nonnegative capacity $c(u,v)$, a source vertex s and a destination vertex t.
- **Find**: A **maximum flow** $f: V \times V \rightarrow \boldsymbol{R}$ from s to t



Flow

- **Given**: A directed graph $G = (V,E)$ where each edge $(u,v) \in E$ has a real-valued, nonnegative **capacity** $c(u,v)$, a **source** vertex s and a **destination** vertex t.
- A **flow** from s to t in G is a real-valued function $f: V \times V \to \mathbf{R}$ satisfying:
  - Capacity constraints: $f(u,v) \leq c(u,v), \forall u, v \in V$.
  - Skew symmetry: $f(u,v) = - f(v,u), \forall u, v \in V$.
  - Flow conservation:
- Flow **value** $| f |$ is defined as

$$\sum_{v \in V} f(s, v)$$

Maximum Flow as LP Problem

- maximize $\sum_{v \in V} f(s, v)$

  subject to

  $f(u,v) \leq c(u,v)$ for all $u, v \in V$

  $f(u,v) = - f(v,u)$ for all $u, v \in V$

  $\sum_{v \in V} f(u, v) = 0$ for all $u \in V - \{ s, t \}$

Maximum Flow as LP Problem

- maximize $\sum_{v \in V} x_{sv}$

  subject to

  $x_{uv} \leq c_{uv}$ for all $u, v \in V$

  $x_{uv} = - x_{vu}$ for all $u, v \in V$

  $\sum_{v \in V} x_{uv} = 0$ for all $u \in V \setminus \{ s, t \}$