UNIVERSITI PUTRA MALAYSIA

# NECESSARY AND SUFFICIENT CONDITION FOR EXTENTION OF CONVOLUTION SEMIGROUP

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MASTER OF SCIENCE
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**Thesis submitted to the School of Graduate Studies, Universti Putra Malaysia, in Fulfilment of the Requirement for the Degree of Master of Science**

**July 2007**

# DEDICATION

This thesis is dedicated to my parents; Allahyarham Mohd Jaffar bin Yeop Ibrahim, (1931 - 1999) and Sharifah Fatimah binti Syed Baharum.

**Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for degree of Master of Science**

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**Chairman : Professor Adem Kilicman, PhD**

**Faculty : Science**

Let $f(t,p)$ and $g(t,q)$ be real - valued continuous functions and $p$ and $q$ be $Q^+$ constants, where $Q^+$ denotes a set of positive rational numbers. Convolution of $f(t,p)$ and $g(t,q)$, denoted by $f(t,p)*g(t,q)$ is defined by

$$f(t,p)*g(t,q)=\int_0^t f(v,p)g(t-v,q)\,dv,$$

where $*$ denotes convolution operation, provided that the integral exists. From the definition of convolution, we introduce a new relation as follows

$$f(t,p)*g(t,q)=f(t,p+q) \text{ or } g(t,p+q),$$

where $+$ denotes an ordinary addition. The new relation is called extension of convolution semigroup. Objective of the study is to discover the necessary and sufficient condition for the new relation. The study is based on Laplace transformable functions. Convolution Theorem in Laplace transform is used to verify the new relation. It is impossible to achieve the new relation directly since most of the transforms are rational

polynomial functions. Furthermore, any transform in terms of exponential function is different from one another. However, we overcome the problem by

(a) Identity property under convolution such that $f(t)*\delta(t)=f(t)$, where $f(t)$ is a real - valued continuous function, which has Laplace transform and $\delta(t)$ is the delta function and it is the identity function under convolution. The Laplace transform of delta function $\delta(t)$ is 1.

(b) Under certain condition, the delta function $\delta(t)$ is a convolution semigroup such that $\delta(t,p)*\delta(t,q)=\delta(t,p+q)$.

(c) Delta function $\delta(t)$ can be replaced by other function under certain condition.

With (a), (b) and (c), we discover the following results:

**Proposition 1** Let $f(t,p)=pf_{\varepsilon}(t)$ and $g(t,q)=qg(t)$ for $t\geq 0$ such that $f_{\varepsilon}(t)\neq g(t)$

and $\lim_{\varepsilon\to 0}\int_{I_{\varepsilon}\subset R} f_{\varepsilon}(t)\,dt=\int_{R} g(t)\,dt=1$.

Then

$$f(t,p)*g(t,q)=f(t,p+q) \text{ if and only if } L[f_{\varepsilon}(t)]\neq 0 \text{ and } L[g(t)]=1,$$

or

$$f(t,p)*g(t,q)=g(t,p+q) \text{ if and only if } L[f_{\varepsilon}(t)]=1 \text{ and } L[g(t)]\neq 0,$$

where $p$ and $q$ are $Q^{+}$ constants with $\frac{1}{p}+\frac{1}{q}=1$ and $I_{\varepsilon}$ is an interval of the point with $\varepsilon$ neighborhood.

**Proposition 2** Let $f_{\varepsilon}(t)$ and $g(t)$ be given real - valued functions with $f_{\varepsilon}(t)=g(t)=0$ for $t<0$. Let $f(t,p)=f_{\varepsilon}(t-p)$ and $g(t,q)=g(t-q)$. $f_{\varepsilon}(t)\neq g(t)$ and $\lim_{\varepsilon\to 0}\int_{I_{\varepsilon}\subset R} f_{\varepsilon}(t)\,dt=\int_{R} g(t)\,dt=1$.

Then

$$f(t,p)*g(t,q)=f(t,p+q) \text{ if and only if } L[f_{\varepsilon}(t)]\neq 0 \text{ and } L[g(t)]=1,$$

or

$$f(t,p)*g(t,q)=g(t,p+q) \text{ if and only if } L[f_{\varepsilon}(t)]=1 \text{ and } L[g(t)]\neq 0,$$

where $p$ and $q$ are $Q^{+}$ constants and $I_{\varepsilon}$ is an interval of the point with $\varepsilon$ neighborhood.

Proposition 1 is called scale form of the functions *f* and *g*, while Proposition 2 is called shift form of the functions *f* and *g*. The extension of convolution semigroup is formed by a non - impulsive and an impulsive function such that the non - impulsive function is an approximation of the impulsive function under certain condition, where all functions in this study are both real - valued continuous and of exponential order.

The study has shown that it is not necessary depend on the same function in order to get the new relation. This study is only true for the conditions described by Propositions 1 and 2.

**Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Master Sains**

**SYARAT PERLU DAN CUKUP UNTUK LANJUTAN KEPADA SEMIKUMPULAN KONVOLUSI**

Oleh

**MAI ZURWATUL AHLAM BINTI MOHD JAFFAR**

**Julai 2007**

**Pengerusi : Profesor Adem Kilicman, PhD**

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Katakan $f(t,p)$ dan $g(t,q)$ ialah fungsi nyata yang selanjar dan $p$ dan $q$ ialah pemalar dari $Q^+$, dimana $Q^+$ ialah set nombor rasional positif. Konvolusi $f(t,p)$ dan $g(t,q)$, iaitu $f(t,p)*g(t,q)$, ditakrifkan sebagai

$$f(t,p)*g(t,q)=\int_0^t f(v,p)g(t-v,q)\,dv,$$

dimana $*$ mewakili konvolusi dan dengan syarat kamiran wujud. Daripada definisi konvolusi, diperkenalkan satu hubungan baru seperti berikut:

$$f(t,p)*g(t,q)=f(t,p+q) \text{ atau } g(t,p+q),$$

dimana $+$ mewakili operasi penambahan. Hubungan baru tersebut dikenali sebagai lanjutan kepada semikumpulan konvolusi. Objektif kajian ini ialah mengkaji syarat perlu dan cukup untuk hubungan baru tersebut. Kajian ini berdasarkan fungsi penjelmaan Laplace. Teorem Konvolusi dalam jelmaan Laplace digunakan untuk mengesahkan hubungan baru tersebut. Adalah mustahil untuk mencapai hubungan baru tersebut secara langsung kerana kebanyakan jelmaan berbentuk pecahan fungsi polinomial. Malahan,

sebarang jelmaan dalam bentuk fungsi eksponen adalah berbeza antara satu sama lain. Walau bagaimanapun, masalah ini ditangani dengan

(a) Sifat identiti bagi konvolusi sedemikian hingga $f(t) * \delta(t) = f(t)$, dimana $f(t)$ ialah sebarang fungsi selanjar iaitu jelmaan Laplace dan $\delta(t)$ adalah fungsi delta dengan fungsi identiti untuk konvolusi. Jelmaan Laplacenya adalah 1.

(b) Bagi keadaan tertentu, fungsi delta $\delta(t)$ merupakan semikumpulan konvolusi sedemikian hingga $\delta(t, p) * \delta(t, q) = \delta(t, p+q)$.

(c) Fungsi delta $\delta(t)$ boleh digantikan dengan fungsi lain bagi keadaan tertentu.

Dengan (a), (b) dan (c), keputusan berikut diperolehi:

**Cadangan 1** Katakan $f(t, p) = pf_\varepsilon(t)$ dan $g(t, q) = qg(t)$ untuk $t \geq 0$ sedemikian hingga $f_\varepsilon(t) \neq g(t)$ dan $\underset{\varepsilon \to 0}{\text{had}} \int_{I_\varepsilon \subset R} f_\varepsilon(t)dt = \int_R g(t)dt = 1$.

Maka

$$f(t, p) * g(t, q) = g(t, p+q) \text{ jika dan hanya jika } L[f_\varepsilon(t)] \neq 0 \text{ dan } L[g(t)] = 1,$$

atau

$$f(t, p) * g(t, q) = g(t, p+q) \text{ jika dan hanya jika } L[f_\varepsilon(t)] = 1 \text{ dan } L[g(t)] \neq 0,$$

dimana $p$ dan $q$ adalah pemalar $Q^{+}$ dengan $\frac{1}{p}+\frac{1}{q}=1$ dan $I_{\varepsilon}$ adalah selang suatu titik dengan kejiranan $\varepsilon$.

**Cadangan 2** Katakan $f_{\varepsilon}(t)$ dan $g(t)$ ialah fungsi nyata yang selanjar diberi, dengan

$f_{\varepsilon}(t)=g(t)=0$ untuk $t<0$. Katakan $f(t,p)=f_{\varepsilon}(t-p)$ dan $g(t,q)=g(t-q)$. $f_{\varepsilon}(t)\neq g(t)$ dan $\underset{\varepsilon\to 0}{\text{had}} \int_{I_{\varepsilon}\subset R} f_{\varepsilon}(t)dt=\int_{R} g(t)dt=1$.

Maka

$$f(t,p)*g(t,q)=f(t,p+q) \text{ jika dan hanya jika } L[f_{\varepsilon}(t)]\neq 0 \text{ dan } L[g(t)]=1,$$

atau

$$f(t,p)*g(t,q)=g(t,p+q) \text{ jika dan hanya jika } L[f_{\varepsilon}(t)]=1 \text{ dan } L[g(t)]\neq 0,$$

dimana $p$ dan $q$ adalah pemalar $Q^{+}$ dan $I_{\varepsilon}$ adalah selang suatu titik dengan kejiranan $\varepsilon$.

Cadangan 1 dikenali sebagai bentuk skala untuk fungsi $f$ dan $g$, manakala Cadangan 2 pula dikenali sebagai bentuk alih untuk fungsi $f$ dan $g$. Lanjutan kepada semikumpulan konvolusi ini dibentuk oleh fungsi bukan jenis denyut dan fungsi jenis denyut sedemikian hingga fungsi bukan jenis denyut tersebut merupakan penghampiran kepada fungsi jenis denyut dibawah keadaan tertentu, dimana semua fungsi dalam kajian ini adalah nyata, selanjar dan tertib eksponen.

Kajian ini menunjukkan bahawa untuk mendapatkan hubungan baru adalah tidak bergantung kepada fungsi yang sama. Kajian ini hanya benar untuk syarat yang digariskan oleh Cadangan 1 dan 2.

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Love to my parents, Allahyarham Mohd Jaffar bin Yeop Ibrahim (1931 - 1999) and Sharifah Fatimah binti Syed Baharum.

A special appreciation to my best friend, Asya, for her existence through thick and thin of my life.

Finally, I wish to thank to my postgraduate - mates.

## APPROVAL

I certify that an Examination Committee has met on **9th July 2007** to conduct the final examination of name of **Mai Zurwatul Ahlam binti Mohd Jaffar** on her **Degree of Master of Science** thesis entitled **"Necessary and Sufficient Condition for Extention of Convolution Semigroup"** in accordance with Universiti Pertanian Malaysia (Higher Degree) Act 1980 and Universiti Pertanian Malaysia (Higher Degree) Regulations 1981. The Committee recommends that the candidate be awarded the relevant degree. Members of the Examination Committee are as follows:

**Habshah binti Midi, PhD**
Associate Professor
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**Zainidin K. Eshkuvatov, PhD**
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Date:

This thesis was submitted to the Senate of Universiti Putra Malaysia and has been accepted as fulfilment of the requirement for the degree of Master of Science. The members of the Supervisory Committee were as follows:

**Adem Kilicman**, **PhD**
Professor
Faculty of Science
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**Mohd Rizam Abu Bakar, PhD**
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**AINI IDERIS, PhD**
Professor and Dean
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Date: 22 January 2008

# DECLARATION

I hereby declare that the thesis is based on my original work except for quotations and citations which have been duly acknowledged. I also declare that is has not been previously or concurrently submitted for any other degree at UPM or other institutions.

**MAI ZURWATUL AHLAM BINTI MOHD JAFFAR**

Date:

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# CHAPTER I

# INTRODUCTION

## 1.1 Convolution

According to Zayed (1996), Hirschman and Widder (1955) were pioneers the theory of convolution presented in a book entitled by The Convolution Transform. Particularly, in that book they showed that for a certain class of kernels, the convolution covers a variety of different integral transform. While, according to Bronstein (2007), integral transforms have a large field of application in the solution of practical problems in physics and engineering. They are suitable to solve ordinary and partial differential equations, integral equations and difference equations.

Krabbe (1958) was studied on convolution from a point of spectral theory. Convolution in a convex domain and singular support was studied by Hormander (1968). Essein (1963) was analyzed in detailed on a convolution inequality. Treatment of convolution in distribution theory was studied by Csiszár (1966). Lin (1966) was studied on convolution semigroup associated with measured - valued branching processes. Numerical solution for a convolution integral equation was analyzed by Day (1969). Terms of convolution semigroup were frequently addressed in a book written by Berg (1976a). Berg (1976b) was studied on the support of the measures in a symmetric convolution semigroup. Hill (1980) was studied on spectral analysis of finite convolution operators with matrix kernels. Babenko (1987) was studied on approximation of convolution classes.

Budzban and Mukherjea (1992) studied on convolution products of non - identical distributions on a topological semigroup. Necessary and sufficient conditions for the convergence of convolution products of non - identical distributions on finite abelian semigroups were obtained by Budzban (1994). Budzban and Rusza (1997) were studied on convergence of convolution products of probability measures on discrete semigroups. Mukherjea and Hognas (2003) were analyzed on maximal homomorphic group image and convergence of convolution sequences on a semigroup. Mukherjea and Budzban (2004) were studied on sub - semigroups of completely simple semigroups and weak convergence of convolution products of probability measures.

There are various developments of convolution in various disciplines. Most recently, biological cybernetics, Wyler (2007) has studied on neural network firing rate - models on integral form. Classification and matricial interpretation of infinitely divisible distributions for rectangular free convolution was obtained by Benaych - Georges (2007).

**Definition 1.1**: Convolution of two real - valued continuous functions $f$ and $g$ over a finite range $[0,t]$ is defined by

$$f(t)*g(t)=\int_0^t f(\tau)g(t-\tau)\,d\tau, \tag{1.1}$$

where the symbol $f*g$ denotes convolution of $f$ and $g$. Constant functions, arbitrary periodic functions, polynomials and exponential functions are suitable for finite type.

**Definition 1.2**: Convolution of two real - valued continuous functions $f$ and $g$ over an infinite range is defined by

$$f(t)*g(t)=\int_{-\infty}^{\infty} f(\tau)g(t-\tau)\,d\tau . \tag{1.2}$$

Usually, convolution satisfies the following properties:

a) Commutative $\quad f(t)*g(t)=g(t)*f(t)$.

b) Associative $\quad (f(t)*g(t))*h(t)=f(t)*(g(t)*h(t))$.

c) Distributive over addition $\quad f(t)*(g(t)+h(t))=f(t)*g(t)+f(t)*h(t)$

.

## 1.2 Integral Transform and Convolution

Let $T$ represents a general integral transform for arbitrarily function $f(t)$. An integral transformation or simply integral transform is a correspondence between two functions $f(t)$ and $T[f(t)]$ in the form of

$$T[f(t)]=\int_{a}^{b} K(s,t)f(t)\,dt , \tag{1.3}$$

where $K(s,t)$ is called the kernel of the transform, $f(t)$ is a given function and $T[f(t)]$ is the transform function. If limits $a$ and $b$ are finite, then $T[f(t)]$ is said to be the finite transform of $f(t)$. There are more than ten types of integral transform. More recently, some new integral transforms have been introduced for use in pattern recognition and characterizing signals such as the Wavelet transform, the Gabor transform and the Walsh transform. The inverse transform of a transform for the given function $f$ is denoted by

$$T^{-1}[T[f(t)]] = f(t).$$

The determination of the inverse transform means the solution of the integral equation, where the function $T[f(t)]$ is given and $f(t)$ is to be determined. The explicit determination of inverse transform for different integral transform for different kernels belongs to the fundamental problems of the theory of integral transform. In the present study, we narrowed to Laplace transform since it is a center of the present study.

We will discuss briefly on Fourier and Mellin transform because they are closely related to Laplace transform.

For a function $f(t)$ defined on $[0,\infty)$, its Laplace transform is denoted as $L[f(t)]$ obtained by the following integral

$$L[f(t)] = F(s) = \int_0^\infty \exp(-st) f(t)\, dt, \tag{1.4}$$

where $s$ is a complex variable such that the integral converges. For an absolutely integrable $f(t)$ defined on $R$, its Fourier transform $F$ is

$$F[f(t)] = F(\omega) = \int_{-\infty}^\infty \exp(-i\omega t) f(t)\, dt, \tag{1.5}$$

where $s = i\omega$, $\omega$ is a real number such that the integral converges. For a function $f(t)$ defined on $(0,\infty)$, its Mellin transform $M$ is

$$M[f(t)] = \int_0^\infty t^{s-1} f(t)\, dt, \tag{1.6}$$

where $s$ is a complex variable such that the integral converges. From the above formulas (1.4), (1.5) and (1.6), $t$ and $s$ are often taken to be real and $f(t)$ is assumed to be a real - valued continuous function. However the most general integral transform $T$ to be considered will be one in which all variables are complex. We only treated for real case since it is relevant to the study.

Next, we discuss the presence of convolution theorem in different kinds of integral transform that centered on Laplace transform.

**1.3 Laplace Transform**

Deakin (1981) was studied and followed the development of Laplace transform from its earliest beginnings 1737 to 1880, where it was addressed that Laplace (1779) was a pioneer the theory of Laplace transform. Historically the development begins when the search for solutions to differential equations in the form of definite integrals of certain types. The theory of Laplace transform has a long history, dating back to Euler (1737). The development of the Laplace transform as an attempt to rigorize the Calculus of Operator. Petzval (1858) was studied on the technique of Laplace's greatest exponent, where he had brought the theory to its highest development, but even this did not incorporate contour integrals in the fresh sense. He did have a version of the inversion for-mula; it was not a tractable one, although a theorem equivalent to our modern inversion formula was available in the contemporaneous work of Riemann (1860). One of the earliest previous histories of the Laplace transform have been by Spitzer (1878). Shortly, the years

1737 - 1880 saw the gradual development of a body of theory centered on what we would now term the Laplace transform. However, the theory was incomplete and most noticeably in its failure to incorporate the full power of complex analysis.

Deakin (1982) was also studied and followed the development of Laplace transform from 1880 to 1937. Poincare (1898) inaugurated the new era of Laplace transform together with independent work by Schlesinger (1992), particularly in the hands of Pincherle (1915). In between 1880 - 1937, there were rapid developments followed, ultimating in Doetsch (1937), in which the transform took its modern shape.

On the ascendancy of Laplace transform was also studied by Deakin (1992), where it was addressed that the transform is now very widely used in mathematics itself and in its applications, particularly in electrical engineering. The Laplace transform is employed in the solution of differential equations, difference equations and functional equations; it allows ready evaluation of certain integrals and claims connection with number theory, which all in addition to the interest that the transform itself holds within functional analysis. In addition, it was addressed that the modern Laplace transform is relatively recent.

**1.4 Existence and Uniqueness**

Existence and uniqueness of Laplace transform are very important in the study of Laplace transform.

**Definition 1.3:** A function $f(t)$ is a piecewise continuous on a finite interval $[a,b]$ if $f$ is continuous on $[a,b]$ , except possibly at finitely many points $c_1, c_2, \cdots, c_n$.

**Definition 1.4:** A function $f(t)$ is said to be of exponential order if there exists constants $M$ and $\alpha$, such that $|f(t)| \le M \exp(\alpha t)$ for all $t \in [0,\infty)$, and $\alpha \ge 0$.

**Theorem 1.1**: Existence Theorem for Laplace transform: If

a) $f(t)$ is piecewise continuous on $[0,\infty)$.

b) $f(t)$ is an exponential order on $[0,\infty)$.

then the Laplace transform of $f(t)$, that is, $L[f(t)] = F(s)$ exists for $s > \alpha$.

**Proof**

Let $f(t)$ is piecewise continuous on $[0,\infty)$, then $\exp(-st)f(t)$ is integrable on $[0,\infty)$.

Assume

$$|L[f(t)]| = \left| \int_0^\infty \exp(-st)\, f(t)\, dt \right|$$

$$\le \int_0^\infty \exp(-st) |f(t)|\, dt$$

$$\le \int_0^\infty \exp(-st)\, M \exp(\alpha t)\, dt$$

$$= \frac{M}{\alpha - s} \left( \exp(-(s-\alpha)t) \Big|_0^\infty \right)$$

$$= \frac{M}{s-\alpha}.$$

Then, $s > \alpha$ for $| \, L[f(t)] \, | < \infty$.

There are real - valued continuous functions $f(t)$, which are not exponential order, but Laplace transform of $f(t)$ exist or otherwise. Therefore conditions for Laplace tranform to exist are sufficient. For example, $f(t) = \frac{1}{\sqrt{t}}$ does not satisfy the exponential order condition but $L\left[\frac{1}{\sqrt{t}}\right]$ exists.

$$L\left[\frac{1}{\sqrt{t}}\right] = \int_0^\infty \exp(-st)\, t^{-\frac{1}{2}} dt.$$

Taking $st = x$

$$\int_0^\infty \exp(-st)\, t^{-\frac{1}{2}}\, dt = \int_0^\infty \exp(-x) \left(\frac{x}{s}\right)^{-\frac{1}{2}} \frac{dx}{s}$$

$$= \left(\frac{1}{s}\right)^{\frac{1}{2}} \int_0^\infty \exp(-x)(x)^{-\frac{1}{2}}\, dx$$

$$= \frac{1}{\sqrt{s}} \Gamma\left(\frac{1}{2}\right)$$

$$= \sqrt{\frac{\pi}{s}}.$$

All functions in this study are both real - valued continuous and of exponential order.