A VARIATIONAL VIEW AT THE TIME-DEPENDENT MINIMAL SURFACE EQUATION

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Abstract. We present a global variational approach to the L^2-gradient flow of the area functional of cartesian surfaces through the study of the so called *weighted energy-dissipation* (WED) functional. In particular, we prove a relaxation result which allows us to show that minimizers of the WED converge in a quantitatively prescribed way to gradient-flow trajectories of the relaxed area functional. The result is then extended to general parabolic quasilinear equations arising as gradient flows of convex functionals with linear growth.

1. Introduction

This note concerns the L^2-gradient flow of the *area functional*,

$$A(u) := \int_\Omega \sqrt{1 + |\nabla u|^2}\,\mathrm{d}x.$$

Here, $u : \Omega \to \mathbb{R}$ is defined in some bounded open set of $\mathbb{R}^d$ with Lipschitz boundary $\partial\Omega$ and the domain of A is assumed to be $D(A) := \{u \in W^{1,1}(\Omega) \mid u = \varphi \text{ on } \partial\Omega\}$, where the equality on the boundary is to be intended in the usual sense of traces and $\varphi \in W^{1,1}(\Omega)$ is a prescribed boundary value. The gradient flow of A gives rise to the *time-dependent minimal surface equation*,

$$\partial_t u - \mathrm{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = 0 \quad \text{a.e. in } \Omega \times (0, T), \tag{1.1}$$

along with the initial and boundary condition $u = \varphi$ on the parabolic boundary of $\Omega_T := \Omega \times (0, T)$.

Problem (1.1) admits, in general, no classical solution unless $\partial\Omega$ is (basically) of non-negative mean curvature (but see also [12, 16, 17] for more general conditions). Indeed, as the functional A is convex and proper but fails to be lower semicontinuous on $L^2(\Omega)$, its gradient flow generally does not admit strong solutions.

A first possible way out from this obstruction relies on relaxation. Namely, one could consider the gradient flow in $L^2(\Omega)$ of the *relaxed area functional*,

$$\underline{A}(u) := \int_\Omega \sqrt{1 + |\nabla u|^2}\,\mathrm{d}x + |D^s u|(\Omega) + \int_{\partial\Omega} |\varphi - u|\,\mathrm{d}\mathcal{H}^{d-1},$$

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along with $D(\underline{A}) := BV(\Omega)$, where $D^s u := Du - \nabla u \mathcal{L}^d$ denotes the singular part of the Radon measure Du (the boundary integral in $\underline{A}$ is to be intended in the sense of traces throughout and without extra notation). Now, the functional $\underline{A}$ is lower semicontinuous on $L^2(\Omega)$ and convex. Hence, the classical nonlinear semigroup theory [8] applies and the gradient flow of $\underline{A}$ is well-posed.

A second option is that of formulating the gradient flow of A with the help of the so-called *weighted energy-dissipation* (WED) formalism [20]. This consists in translating the whole evolution problem in a global-in-time minimization plus a limit passage. In particular, one looks for minimizing trajectories $t \mapsto u_\varepsilon(x,t)$ with $u = \varphi$ on the parabolic boundary of Ω_T of the global-in-time WED functional

$$W^\varepsilon(u) := \int_0^T \mathrm{e}^{-t/\varepsilon} \left(\frac{1}{2} \|u_t\|^2_{L^2(\Omega)} + \frac{1}{\varepsilon} A(u) \right) \mathrm{d}t$$

and ascertain the so-called *causal limit* limit $u_\varepsilon \to u$. Details on this approach are recalled in Section 2. Note that W^ε is convex and proper but again generally not lower semicontinuous in $L^1(\Omega_T)$. Hence, one is forced to minimize its $L^1(\Omega_T)$ relaxation $\underline{W}^\varepsilon$ before taking the ε-limit. Indeed, as $\underline{W}^\varepsilon$ is lower semicontinuous and coercive in $H^1(0,T;L^2(\Omega))$, the functional $\underline{W}^\varepsilon$ coincides with the relaxation of W^ε in $H^1(0,T;L^2(\Omega))$ as well.

The main aim of this paper is to prove that these two a priori different approaches indeed coincide. This is translated into our main result as follows.

Theorem 1.1 (Convergence)**.** *Let u_ε minimize $\underline{W}^\varepsilon$ along with $u_\varepsilon(\cdot,0) = \varphi$ at time $t = 0$. Then $u_\varepsilon \to \underline{u}$ uniformly in $L^2(\Omega)$ where $\underline{u}$ solves*

$$\underline{u}'(t) + \partial \underline{A}(\underline{u}) \ni 0 \quad \text{in } L^2(\Omega), \text{ a.e. in } (0,T), \qquad \underline{u}(0) = \varphi. \tag{1.2}$$

Theorem 1.1 consists in a novel limiting variational formulation of the evolution of cartesian surfaces with vertical velocity equal to the (double of the) mean curvature. As such, we expect it to be interesting in view of possible approximating procedures. Although we do not believe that the minimization of W_ε will turn out to be numerically convenient with respect to the direct solution of (1.2) (by the implicit Euler method, say), we have to point out that indeed the convergence result of Theorem 1.1 is valid in much greater generality, and in particular can be adapted to sequences of *approximate minimizers*. The precise statement of this fact along with extra motivations and comments are given in Section 2. Here we anticipate that the crucial tool in the direction of the proof of Theorem 1.1 is the following relaxation result.

Theorem 1.2 (Relaxation)**.**

$$\underline{W}^\varepsilon(u) = \int_0^T \mathrm{e}^{-t/\varepsilon} \left(\frac{1}{2} \|u_t\|^2_{L^2(\Omega)} + \frac{1}{\varepsilon} \underline{A}(u) \right) dt.$$

This last result is proved separately in Section 3 and could be formulated as *the relaxation of the WED functional is the WED functional of the relaxation.* One could be tempted to extend this paradigm to general situations. However, this is generally not the case as shown by a counterexample of Mielke & Ortiz [19].

In the last section of the paper, we generalize Theorem 1.2 to the case of convex functionals F with linear growth at infinity. Indeed, such WED approach can be used for an entire class of parabolic quasilinear equations of the form:

$$u_t = \operatorname{div}\big(a(x, \nabla u)\big) \quad \text{a.e. in } \Omega \times (0, T), \tag{1.3}$$

along with the same initial and boundary condition $u = \varphi$, where $a(x, \xi) = \nabla_\xi f(x, \xi)$ with f convex in ξ and having at most linear growth at infinity. Apart from the time-dependent minimal surface equation, another prototypical example in this class is the *total variation flow*,

$$u_t = \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right), \quad \text{a.e. in } \Omega \times (0, T).$$

Formally, (1.3) arises as the L^2-gradient flow of the functional

$$F(u) := \int_\Omega f(x, \nabla u(x))\, \mathrm{d}x,$$

and the trajectories of the relaxed energy $\underline{F}$ can be recovered by the casual limit of the approximate minimizers of the corresponding WED functional once we prove an analogous relaxation result for

$$I^\varepsilon(u) = \int_0^T e^{-T/\varepsilon} \left(\frac{\|u_t\|^2_{L^2(\Omega)}}{2} + \frac{F(u)}{\varepsilon}\right) \mathrm{d}t.$$

We point out that the WED approach has already been exploited in the framework of evolution by mean curvature. Indeed, Ilmanen uses the WED functional in [14] for proving existence and partial regularity of the so-called Brakke mean curvature flow of varifolds. In comparison with Ilmanen's paper, our results are weaker on one side as we concentrate on finite-time evolution of cartesian surfaces only. On the other hand, our functional frame is more classical as we deal with graphs of BV functions and the related convergence results are of a quantitative nature. Moreover, we recall that the interest on the WED approach (or, as it will turn out to be equivalent, to elliptic-in-time regularizations of evolution problems) starts at least from the classical monograph by Lions & Magenes [18] (linear), the already cited [14] and Hirano [13] (gradient flows). As mentioned, the general discussion of the WED functional approach to gradient flows is in [21] whereas two applications are in [9]. As for the doubly nonlinear dissipative evolution case, one shall mention the rate-independent theory of Mielke & Ortiz [19] (see also [20]), as well as the general convergence results of [2, 1]. Finally, the semilinear hyperbolic case has been tackled via the WED approach in [22].

2. WED formalism and main results

Let us start this section by recalling the WED theory for gradient flows from [21]. Assume we are given a real Hilbert space H with scalar product $(\cdot, \cdot)$ and corresponding norm $\|\cdot\|_H$. Moreover, let the functional $F : H \to (-\infty, \infty]$ be proper and convex. Starting from some given initial datum $u_0 \in D(F) := \{u \in H \mid F(u) < \infty\}$, we shall be concerned with the classical *gradient flow*,

$$u' + \partial F(u) \ni 0 \quad \text{in } H, \ \text{a.e. in } \ (0, T), \ \ u(0) = u_0. \tag{2.1}$$

Here, the equation is intended to be solved in H, the prime stands for time-differentiation, and the symbol ∂ denotes the subdifferential in the sense of Convex

Analysis, namely

$$v \in \partial F(u) \quad \Longleftrightarrow \quad u \in D(F) \ \text{ and } \ (v, w-u) \le F(w) - F(u) \quad \forall w \in H.$$

Problem (2.1) stands as the paradigm of nonlinear dissipative evolution and may arise as the variational formulation of a variety of parabolic problems. As such, the well-posedness of (2.1) has been extensively considered. In case F is lower semicontinuous, the operator ∂F is maximal monotone. Hence, the classical nonlinear semigroup theory applies and strong solutions to (2.1) exists uniquely and continuously depending on the initial datum [15, 10, 7, 8]. Note that $u_0 \in \overline{D(\partial F)}$ would be enough for proving the well-posedness of (2.1) in the lower semicontinuous setting.

The case of non lower semicontinuous functionals F is more delicate and has recently emerged as a new interesting benchmark, especially in connection with micro-structure evolution. At the stationary level, states which minimize the energy F may not exists and one resorts in relaxing F. The situation is less clear at the evolution level and a natural idea seems that of introducing a suitable global-in-time variational functional on entire trajectories whose minimizers solve (2.1) (and then possibly relax it). This perspective has been followed by Mielke & Ortiz [19] who reformulated a large class of evolution problems as (limits of) minimizers of a class of global-in-time functionals featuring the sum of the (scaled) energy and the dissipation, integrated in time via an exponentially decaying weight. The resulting so-called *weighted energy-dissipation* (WED) functionals $I^\varepsilon : H^1(0,T;H) \to (-\infty, \infty]$ read, in the case of the gradient flow (2.1), as

$$I^\varepsilon(v) := \int_0^T \mathrm{e}^{-t/\varepsilon} \left(\frac{1}{2} \|v'\|_H^2 + \frac{1}{\varepsilon} F(v) \right) dt. \tag{2.2}$$

Note that, whenever restricted to the closed convex set

$$K := \{u \in H^1(0,T;H) \mid u(0) = u_0\}, \tag{2.3}$$

the WED functionals I^ε are uniformly convex by virtue of the term $\|v'\|_H^2$, namely, without assuming any strict convexity of F.

In case F is lower semicontinuous and bounded below, the WED functional admits a unique minimizer $u_\varepsilon \in K$ which, in particular, solves the Euler-Lagrange system:

$$-\varepsilon u_\varepsilon'' + u_\varepsilon' + \partial F(u_\varepsilon) \ni 0 \quad \text{a.e. in } \ (0,T), \tag{2.4}$$
$$u_\varepsilon(0) = u_0, \tag{2.5}$$
$$u_\varepsilon'(T) = 0. \tag{2.6}$$

Namely, the minimizer of I^ε in K solves an *elliptic-in-time regularization* of the gradient flow (2.1). As the problem above is of second order in time, an extra boundary condition (2.6) at the final point T arises.

Note that, at all levels $\varepsilon > 0$, causality is lost. This motivates the name of *causal limit* for the limit $\varepsilon \to 0$ into (2.4)-(2.6). The main result of [21] consists in proving that by performing the causal limit one indeed recovers the solution of (2.1). More precisely, by letting u_ε be the unique minimizer of I^ε in K one has the quantitative error bound

$$\max_{t\in[0,T]} \|u(t)-u_\varepsilon(t)\|_H \le C\varepsilon^{1/2},$$

where u is the unique solution of the gradient flow (2.1) and the error constant C depends just on $F(u_0)$ and T. The above convergence rate is sharp and is indeed valid in much greater generality (non autonomous case, more general initial data) in the interpolation space $(C([0,T];H), H^1(0,T;H))_{\eta,1}$ [6] for $0\le\eta<1$ with order $\varepsilon^{(1-\eta)/2}$.

The situation is clearly much more complicated when F is not lower semicontinuous as I^ε may fail to admit a minimizer in K. A possible solution is that of considering quantitatively qualified *approximate* minimizers. Namely, we say that $v_\varepsilon \in K$ are ε^3-approximate minimizers if

$$I^\varepsilon(v_\varepsilon) \le \inf_K I^\varepsilon + \varepsilon^3.$$

Let us denote by $\underline{I}^\varepsilon$ the relaxation of I^ε in $H^1(0,T;H)$. If $\underline{I}^\varepsilon$ happens to be itself the WED functional of a convex and lower semicontinuous functional $\widehat{F}$, we have that [21, Cor. 5.5] any sequence of ε^3-approximate minimizers v_ε converges uniformly to $\widehat{u}$, namely the gradient-flow evolution driven by $\widehat{F}$. In particular, if $\underline{u}_\varepsilon$ minimizes $\underline{I}^\varepsilon$ on K, we have that

$$\max_{t\in[0,T]} \|\widehat{u}(t)-\underline{u}_\varepsilon(t)\|_H \le C\varepsilon^{1/2}.$$

One may now wonder whether $\widehat{u}$ is the gradient-flow trajectory driven by the relaxed energy $\underline{F}$ or not. In other words, one could ask if $\underline{F}\equiv\widehat{F}$, that is to say if, by taking the relaxation of the WED functional, we are actually dealing with the WED functional on the relaxed energy. We shall remark right away that this is not generally the case as the counterexample in [19, Thm. 5.1] shows. On the other hand, some first nontrivial examples of $\underline{F}\equiv\widehat{F}$ are detailed by CONTI & ORTIZ [9]. In particular, they consider both a model of *martensite branching* in a one-dimensional bar and the two-dimensional description of the phenomenon of *island growth and coarsening* during the epitaxial growth of thin films. In both cases, they provide an explicit characterization of the relaxation $\underline{I}^\varepsilon$ which itself turns out to be the WED functional on the relaxed energy.

The aim of this note is to present yet another example of the circumstance $\underline{F}\equiv\widehat{F}$ in the context of mean curvature evolution. In particular, by referring to the notation of Section 1, Theorem 1.2 entails that the relaxation $\underline{W}^\varepsilon$ of the WED functional W^ε is itself a WED functional on the relaxed area functional $\underline{A}$. A proof of the crucial Theorem 1.2 is detailed in Section 3 below. Given the relaxation result, Theorem 1.1 follows from the above recalled theory of [21]. In particular, the statement of Theorem 1.1 can be sharpened as follows.

Theorem 2.1 (Convergence, sharper statement)**.** *Let $H=L^2(\Omega)$ and v_ε be ε^3-approximate minimizers of W^ε on K as in* (2.3)*. Then, there exists $C>0$ depending on $A(u_0)$ and T but not on ε such that*

$$\max_{t\in[0,T]} \|\underline{u}(t)-v_\varepsilon(t)\|_H \le C\varepsilon^{1/2},$$

where $\underline{u}$ is the unique solution of

$$\underline{u}'(t) + \partial\underline{A}(\underline{u}) \ni 0 \quad \text{a.e. in } (0,T), \quad u(0)=u_0.$$

The same uniform convergence result holds for the minimizers $\underline{u}_\varepsilon$ of the relaxed WED functional $\underline{W}^\varepsilon$ on K.

Note that a direct characterization of the subdifferential $\partial\underline{A}$ has been provided by Demengel & Temam in [11].

3. Relaxation: the area functional

We shall now turn to the proof of the relaxation Theorem 1.2 which, as commented above, is the core ingredient for the validity of the convergence Theorem 2.1.

We recall the following notation. In the sequel $\Omega \subset \mathbb{R}^d$ is a bounded open domain with Lipschitz boundary and, for $T \in (0,\infty)$, $\mathbb{R}^d_T := \mathbb{R}^d \times (0,T)$, $\Omega_T := \Omega \times (0,T)$, $\partial\Omega_T := \partial\Omega \times (0,T)$ and $\partial_p\Omega_T$ is the usual parabolic boundary of Ω_T,

$$\partial_p\Omega_T := \partial\Omega_T \cup (\overline{\Omega} \times \{0\}).$$

For the initial value and the boundary data, we fix $\varphi \in W^{1,1}(\Omega) \cap L^2(\Omega)$ and let $u_0 = \varphi$.

Given the global-in-time WED functional W^ε, we consider its relaxation $\underline{W}^\varepsilon$ in $L^1(\Omega_T)$ and the WED functional $\widehat{W}^\varepsilon$ of the relaxed area functional $\underline{A}$, respectively given by

$$\underline{W}^\varepsilon(u) := \inf\left\{\liminf_{j\to\infty} W^\varepsilon(u_j) \mid u_j \to u \;\text{ in }\; L^1(\Omega_T)\right\},$$
$$\widehat{W}^\varepsilon(u) := \int_0^T \mathrm{e}^{-t/\varepsilon}\left(\int_\Omega \frac{|u_t|^2}{2}\,\mathrm{d}x + \frac{\underline{A}(u)}{\varepsilon}\right) dt,$$

where the domains of definition are respectively given by

$$D(W^\varepsilon) := \{u \in L^1(0,T;W^{1,1}(\Omega)) \mid u_t \in L^2(\Omega_T), \quad u = \varphi \;\text{ on }\; \partial_p\Omega_T\},$$
$$D(\underline{W}^\varepsilon) = D(\widehat{W}^\varepsilon) := \{u \in L^1(0,T;BV(\Omega)) \mid u_t \in L^2(\Omega_T), \quad u(\cdot,0) = u_0\}.$$

Once again, let us note that $\underline{W}^\varepsilon$ indeed coincides with the $H^1(0,T;L^2(\Omega))$-relaxation of W^ε.

Remark 3.1. *For what concerns the domain $D(W_\varepsilon)$, we point out that time-dependent boundary data $\varphi \in L^1(0,T;W^{1,1}(\Omega))$ may be considered as well.*

With this notation, Theorem 1.2 reads

$$\widehat{W}^\varepsilon \equiv \underline{W}^\varepsilon.$$

Since $\widehat{W}^\varepsilon$ is clearly lower semicontinuous in $L^1(\Omega_T)$, it follows that $\widehat{W}^\varepsilon \le \underline{W}^\varepsilon$. Therefore, in order to establish the theorem, we need only to prove the converse inequality, which, in turn, is equivalent to show the following claim: for every $u \in D(\underline{W}^\varepsilon)$, there exists a sequence $u_j \in D(W^\varepsilon)$ such that $u_j \to u$ strongly in $L^1(\Omega_T)$ and

$$\liminf_{j\to\infty} W^\varepsilon(u_j) \le \widehat{W}^\varepsilon(u). \tag{3.1}$$

Remark 3.2. *For later purposes, we stress that, by taking into account the inequality $\widehat{W}^\varepsilon \le \underline{W}^\varepsilon$, any sequence satisfying* (3.1) *realizes in fact the equality.*

We shall proceed in three steps. At first, we define z_j to be a suitable regularization by convolution of u. Then, in Subsection 3.2 we perform a shift in time of z_j and a linear interpolation in order to construct functions v_j achieving the initial datum $v_j(x,0) = u_0(x)$. Finally, in Subsection 3.3 we modify v_j into a sequence u_j such that $u_j = \varphi$ on $\partial_p\Omega_T$. The main point is to check that we can make these constructions in such a way that inequality (3.1) holds.

3.1. **Bulk construction.** Let $u \in D(\underline{W}^\varepsilon)$ be given and, without introducing new notation, assume u to be extended with respect to space to $u \in L^1(0,T;BV(\mathbb{R}^d))$ along with

$$|Du(\cdot,t)|(\partial\Omega) = 0 \quad \text{for every} \quad t \in (0,T). \tag{3.2}$$

Note that this can be done since Ω is a Lipschitz domain [3, Prop. 3.21, p. 131].

Next we fix a decreasing sequence $\tau_j \to 0^+$, a radially symmetric non-negative mollifier $\rho \in C_c^\infty(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} \rho = 1$, and let $\rho_j(x) := \tau_j^{-d}\rho(x/\tau_j)$. We start by defining $z_j := \rho_j * u$, namely

$$z_j(x,t) = \int_{\mathbb{R}^d} \rho_j(x-y)u(y,t)\,\mathrm{d}y.$$

Clearly, we have that $z_j \in L^1(0,T;W^{1,1}(\Omega))$ and $z_j \to u$ in $L^1(\Omega_T)$. Moreover, since ρ_j is independent of time, one directly checks that $z_{j,t} \to u_t$ strongly in $L^2(\Omega_T)$. On the other hand, by the standard estimate on the convolution (see, for instance, [3, Thm. 2.2, p. 42]), we have that, for every $t \in (0,T)$,

$$\int_\Omega \sqrt{1+|\nabla z_j(x,t)|^2}\,\mathrm{d}x \le \int_{N_\varepsilon(\Omega)} \sqrt{1+|\nabla u(x,t)|^2}\,\mathrm{d}x + |D^s u|(N_\varepsilon(\Omega)),$$

where $N_\varepsilon(\Omega) := \{x \in \mathbb{R}^d \mid \operatorname{dist}(x,\Omega) < \varepsilon\}$ denotes the ε neighborhood of Ω. Hence, integrating in $\mathrm{d}t_\varepsilon := \mathrm{e}^{-t/\varepsilon}\mathrm{d}t$, we deduce from (3.2) that

$$\limsup_{j\to\infty} \int_{\Omega_T} \sqrt{1+|\nabla z_j|^2}\,\mathrm{d}x\,\mathrm{d}t_\varepsilon \le \int_{\Omega_T} \sqrt{1+|\nabla u|^2}\,\mathrm{d}x\,\mathrm{d}t_\varepsilon + \int_0^T |D^s u|(\Omega)\,\mathrm{d}t_\varepsilon. \tag{3.3}$$

Remark 3.3. *Note that, by semicontinuity of $\underline{A}$, the inequality in* (3.3) *is fact an equality.*

3.2. **Initial value.** Let us now modify the sequence z_j in such a way that the initial value u_0 is achieved. To this aim, let

$$s_j^2 := \int_\Omega |u_0(x) - z_j(x,0)|^2\mathrm{d}x.$$

Note that, since $z_j(\cdot,0) = \rho_j * u_0$ and $u_0 \in L^2(\Omega)$, we have that $s_j \to 0$ as $j \to \infty$. We shall define the functions v_j by shifting z_j in time by s_j and then recovering the initial value u_0 in a linear fashion. In particular, if $s_j = 0$ we set $v_j = z_j$. Otherwise, we set

$$v_j(x,t) := \begin{cases} z_j(x,t-s_j) & \text{for} \ \ s_j \le t, \\ \dfrac{s_j - t}{s_j}\,u_0(x) + \dfrac{t}{s_j} z_j(x,0) & \text{for} \ \ 0 \le t < s_j. \end{cases}$$

Clearly, $v_j(\cdot,0)=u_0$. We claim that

$$v_j \to u \quad \text{strongly in } \ L^1(\Omega_T), \tag{3.4}$$

$$v_{j,t} \to u_t \quad \text{strongly in } \ L^2(\Omega_T), \tag{3.5}$$

$$\begin{aligned}\limsup_{j\to\infty} \int_{\Omega_T} \sqrt{1+|\nabla v_j|^2}\,\mathrm{d}x\,\mathrm{d}t_\varepsilon \\ \le \int_{\Omega_T} \sqrt{1+|\nabla u|^2}\,\mathrm{d}x\,\mathrm{d}t_\varepsilon + \int_0^T |D^s u|(\Omega)\,\mathrm{d}t_\varepsilon.\end{aligned} \tag{3.6}$$

Indeed, since $u \in C([0,T];L^1(\Omega))$, the convergence (3.4) follows easily from the following estimate

$$\begin{aligned}\|u-v_j\|_{L^1(\Omega_T)} &= \int_0^{s_j} \|u(\cdot,t)-v_j(\cdot,t)\|_{L^1(\Omega)}\mathrm{d}t + \int_{s_j}^T \|u(\cdot,t)-z_j(\cdot,t-s_j)\|_{L^1(\Omega)}\mathrm{d}t \\ &\le C\,s_j\,\|u_0\|_{L^1(\Omega)} + \int_{s_j}^T \|u(\cdot,t)-u(\cdot,t-s_j)\|_{L^1(\Omega)}\mathrm{d}t + \|u-z_j\|_{L^1(\Omega_T)} \to 0.\end{aligned}$$

In the same way, (3.5) follows from the continuity of translations in L^2, namely,

$$\begin{aligned}&\int_{s_j}^T \|u_t(\cdot,t)-v_{j,t}(\cdot,t)\|^2_{L^2(\Omega)}\mathrm{d}t \\ &\le \int_{s_j}^T \|u_t(\cdot,t)-u_t(\cdot,t-s_j)\|^2_{L^2(\Omega)}\mathrm{d}t + \|u_t-z_{j,t}\|^2_{L^2(\Omega_T)} \to 0,\end{aligned}$$

and, from the choice of s_j,

$$\begin{aligned}\int_0^{s_j} \|v_{j,t}-u_t\|^2_{L^2(\Omega)} &\le 2\int_0^{s_j} \|u_t(\cdot,t)\|^2_{L^2(\Omega)}\,dt + 2\int_0^{s_j} \|v_{j,t}(\cdot,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t \\ &\le o(1) + 2\int_0^{s_j}\int_\Omega \left|\frac{u_0(x)-z_j(x,0)}{s_j}\right|^2 \mathrm{d}x\,\mathrm{d}t = o(1)+2s_j \to 0.\end{aligned}$$

Finally, inequality (3.6) is an easy consequence of (3.3) and the convexity of the relaxed area functional which in particular implies that

$$\int_0^{s_j} A(v_j(\cdot,t))\,\mathrm{d}t \le s_j\,\left(\frac{A(u_0)}{2} + \frac{A(\rho_j * u_0)}{2}\right) \to 0.$$

3.3. **Boundary value matching.** Now we modify the sequence v_j in a neighborhood of $\partial\Omega_T$ in order to match the boundary data φ. To this aim, let $\delta(x) := \operatorname{dist}(x,\partial\Omega)$. Note that $\delta \in W^{1,\infty}(\Omega)$ with

$$|\nabla\delta(x)| \le 1 \quad \text{for almost every} \quad x \in \Omega.$$

For all $\mu < 1$ small enough, we consider a decreasing, convex cut-off function $\eta_\mu \in C^1([0,\infty))$ such that

$$\eta_\mu(0)=1, \quad \eta'_\mu(0) = -1/\mu \quad \text{and} \quad \eta_\mu(t)=0 \text{ for } t \ge \mu+\mu^2.$$

For $\mu_j \to 0^+$ given by

$$\mu_j^2 := \|u-v_j\|_{L^1(\Omega_T)}, \tag{3.7}$$

we define

$$\sigma_j(x,t) := \big(\varphi(x)-v_j(x,t)\big)\,\eta_{\mu_j}\big(\delta(x)\big).$$

Clearly $\sigma_j \equiv \varphi - v_j$ on $\partial\Omega_T$ and, from $v_j(x,0) = u_0(x) = \varphi(x)$, it follows that $\sigma_j(\cdot,0) \equiv 0$. We claim the following:

$$\sigma_j \to 0 \quad \text{strongly in} \quad L^1(\Omega_T), \tag{3.8}$$

$$\sigma_{j,t} \to 0 \quad \text{strongly in} \quad L^2(\Omega_T), \tag{3.9}$$

$$\limsup_{j\to\infty} \int_{\Omega_T} |\nabla\sigma_j(x,t)| \,\mathrm{d}x \,\mathrm{d}t_\varepsilon \le \int_{\partial\Omega_T} |\varphi(x) - u(x,t)| \,\mathrm{d}\mathcal{H}^{d-1}(x) \,\mathrm{d}t_\varepsilon. \tag{3.10}$$

The proof of convergences (3.8) and (3.9) is straightforward. Indeed, by the construction of σ_j we have that

$$\begin{aligned}\int_{\Omega_T} |\sigma_j| \,\mathrm{d}x \,\mathrm{d}t = \int_{\Omega_{2\mu_j}} |\sigma_j| \,\mathrm{d}x \,\mathrm{d}t &\le \int_{\Omega_{2\mu_j}} |\varphi - v_j| \,\mathrm{d}x \,\mathrm{d}t \\ &\le \int_{\Omega_{2\mu_j}} |\varphi - u| \,\mathrm{d}x \,\mathrm{d}t + \|u - v_j\|_{L^1(\Omega_T)},\end{aligned}$$

where, for all $\mu > 0$, we have set $\Omega_\mu := \{x \in \Omega \mid \delta(x) < \mu\}$. Note that, from the Lipschitz regularity of $\partial\Omega$ it follows that there exit constants $C, \mu_0 > 0$ such that

$$|\Omega_\mu| \le C\,\mu \quad \text{for every} \quad \mu \le \mu_0.$$

In particular this implies that $|\Omega_\mu| \to 0$ as $\mu \to 0$. Hence, convergence (3.8) follows. Analogously, since $v_{j,t} \to u_t$ in $L^2(\Omega_T)$, from

$$\sigma_{j,t}(x,t) = -v_{j,t}(x,t)\,\eta_{\mu_j}\big(\delta(x)\big),$$

we deduce the convergence (3.9).

As for the inequality (3.10), we start by noticing that, since $\eta_{\mu_j} \circ \delta \in W^{1,\infty}(\Omega)$, then $\sigma_j \in L^1(0,T;W^{1,1}(\Omega))$ and, by the chain rule,

$$\begin{aligned}\int_{\Omega_T} |\nabla\sigma_j| \,\mathrm{d}x \,\mathrm{d}t_\varepsilon &\le \int_0^T \int_{\Omega_{\mu_j+\mu_j^2}} |\nabla\varphi - \nabla v_j|\,|\eta_{\mu_j}| \,\mathrm{d}x \,\mathrm{d}t_\varepsilon \\ &+ \int_0^T \int_{\Omega_{\mu_j+\mu_j^2}} |\varphi - v_j|\,|\eta'_{\mu_j} \circ \delta|\,|\nabla\delta| \,\mathrm{d}x \,\mathrm{d}t_\varepsilon =: I_{1,j} + I_{2,j}.\end{aligned}$$

Recalling the construction of v_j, which is obtained from the regularization z_j and a linear interpolation with u_0, the first integrand $I_{1,j}$ can be easily estimated as follows:

$$\begin{aligned}I_{1,j} \le \int_0^T \int_{\Omega_{2\mu_j}} \big(|\nabla\varphi| + |\nabla v_j|\big) \,\mathrm{d}x \,\mathrm{d}t_\varepsilon &\le \int_0^T \int_{\Omega_{2\mu_j}} |\nabla\varphi| \,\mathrm{d}x \,\mathrm{d}t_\varepsilon + \\ &+ \int_0^T |Du(\cdot,t)|(N_{\tau_j}(\Omega_{2\mu_j}))\,dt_\varepsilon.\end{aligned}$$

Therefore, since $\lim_j |Du(\cdot,t)|(N_{\tau_j}(\Omega_{2\mu_j})) = |Du(\cdot,t)|(\partial\Omega_T) = 0$ for every $t \in (0,T)$ and $|Du(\cdot,t)|(N_{\tau_j}(\Omega_{2\mu_j})) \le |Du(\cdot,t)|(\mathbb{R}^d) \in L^1(0,T)$, we conclude from the Dominated Convergence Theorem that $I_{1,j} \to 0$.

For what concerns $I_{2,j}$, we notice that by (3.7) we can proceed as follows:

$$
\begin{aligned}
I_{2,j} &\leq \int_0^T \frac{1}{\mu_j} \int_{\Omega_{\mu_j+\mu_j^2}} \big(|\varphi(x) - u(x,t)| + |u(x,t) - v_j(x,t)|\big)\,\mathrm{d}x\,\mathrm{d}t_\varepsilon \\
&\leq \int_0^T \frac{1}{\mu_j} \int_{\Omega_{\mu_j+\mu_j^2}} |\varphi(x) - u(x,t)|\,\mathrm{d}x\,\mathrm{d}t_\varepsilon + \mu_j^{-1}\|u - v_j\|_{L^1(\Omega_T)} \\
&= \int_0^T \frac{1}{\mu_j} \int_{\Omega_{\mu_j+\mu_j^2}} |\varphi(x) - u(x,t)|\,\mathrm{d}x\,\mathrm{d}t_\varepsilon + \mu_j.
\end{aligned}
$$

Then, the inequality (3.10) is proved once we show that

$$
\limsup_{j\to\infty} \int_0^T \frac{1}{\mu_j} \int_{\Omega_{\mu_j+\mu_j^2}} |\varphi(x) - u(x,t)|\,\mathrm{d}x\,\mathrm{d}t_\varepsilon \leq \int_{\partial\Omega_T} |\varphi - u|\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t_\varepsilon. \tag{3.11}
$$

Form the trace theory of BV functions we have, for every $t \in (0,T)$,

$$
\frac{1}{\mu_j} \int_{\Omega_{\mu_j+\mu_j^2}} |\varphi(x) - u(x,t)|\,\mathrm{d}x \to \int_{\partial\Omega} |\varphi(x) - u(x,t)|\,\mathrm{d}\mathcal{H}^{d-1}(x).
$$

Moreover, we can argue as follows

$$
\begin{aligned}
&\frac{1}{\mu_j} \int_{\Omega_{\mu_j+\mu_j^2}} |\varphi(x) - u(x,t)|\,\mathrm{d}x \\
&\leq C \int_{\partial\Omega} |\varphi(x) - u(x,t)|\,\mathrm{d}\mathcal{H}^{d-1}(x) + C\int_\Omega |\nabla\varphi(x)|\,\mathrm{d}x + C\,|Du(\cdot,t)|(\Omega),
\end{aligned}
$$

for some constant $C > 0$ depending on $\partial\Omega$, and dominated convergence entails inequality (3.11).

Now we set $u_j := v_j + \sigma_j$. Clearly, $u_j \in L^1(0,T;W^{1,1}(\Omega))$ and by convergences (3.8) and (3.9) we deduce the strong convergences $u_j \to u$ and $u_{j,t} \to u_t$ in $L^1(\Omega_T)$ and $L^2(\Omega_T)$, respectively. By construction $v_j = \varphi$ on $\partial\Omega_T$. Moreover, since $\sigma_j(\cdot,0) \equiv 0$, we also have that $u_j(\cdot,0) = u_0$.

In particular, we have checked that

$$
\int_{\Omega_T} |u_{j,t}|^2 \mathrm{d}x\,\mathrm{d}t_\varepsilon \to \int_{\Omega_T} |u_t|^2 \mathrm{d}x\,\mathrm{d}t_\varepsilon,
$$

and, using inequality (3.10) and $\sqrt{1+|a+b|^2} \leq \sqrt{1+|a|^2} + |b|$ for any $a, b \in \mathbb{R}^d$,

$$
\begin{aligned}
&\limsup_{j\to\infty} \int_{\Omega_T} \sqrt{1+|\nabla u_j|^2}\,\mathrm{d}x\,\mathrm{d}t_\varepsilon \\
&\leq \limsup_{j\to\infty} \left(\int_{\Omega_T} \sqrt{1+|\nabla v_j|^2}\,\mathrm{d}x\,\mathrm{d}t_\varepsilon + \int_{\Omega_T} |\nabla\sigma_j|\,\mathrm{d}x\,\mathrm{d}t_\varepsilon \right) \\
&\overset{(3.6),(3.10)}{\leq} \int_{\Omega_T} \sqrt{1+|\nabla u|^2}\,\mathrm{d}x\,\mathrm{d}t_\varepsilon + \int_0^T |D^s u|(\Omega)\,\mathrm{d}t_\varepsilon + \int_{\partial\Omega_T} |\varphi - u|\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t_\varepsilon.
\end{aligned}
$$

Eventually, we have proved that

$$
\limsup_{j\to\infty} W^\varepsilon(u_j) \leq \widehat{W}^\varepsilon(u),
$$

thus concluding the proof of Theorem 1.2.

4. Convex functionals with linear growth

In this section we generalize the results of the previous sections to the case of general linear growth functionals. We focus, indeed, our attention to the following problem:

$$\begin{cases} u_t = \operatorname{div}\big(a(x, \nabla u)\big) & \text{in } \Omega_T, \\ u = \varphi & \text{on } \partial_p \Omega_T, \end{cases} \tag{4.1}$$

where Ω and φ are as in Section 3 and $a(x, \xi) = \nabla_\xi f(x, \xi)$ with f convex in ξ and having at most linear growth at infinity. Together with the area functional, other examples in this class are the already mentioned *total variation flow* $f(\xi) = |\xi|$, as well as the *Lagrangians*

$$f(x, \xi) = \sqrt{1 + \alpha_{ij}(x)\xi_i\xi_j}, \quad f(x, \xi) = \sqrt{1 + x^2 + |\xi|^2},$$

where the coefficients α_{ij} are continuous and elliptic and $\alpha_{ij} = \alpha_{ji}$. The reader is referred to [4] for additional examples and details.

Problem (4.1) formally arises as the L^2-gradient flow of the functional

$$F(u) := \int_\Omega f(x, \nabla u(x))\, \mathrm{d}x, \qquad D(F) := W^{1,1}(\Omega) \subset L^2(\Omega). \tag{4.2}$$

Since the area functional is a special case of (4.2) for $f(x, \xi) = \sqrt{1 + |\xi|^2}$, all the considerations for (1.1) hold as well for (4.1). In particular, the lack of semicontinuity in $L^2(\Omega_T)$ due to the linear growth condition prevents from applying directly the nonlinear semigroup theory.

Here we show that the WED functional can be employed in order to recover the gradient flow trajectories in $L^2(\Omega)$ of the relaxed energy $\underline{F}$. Exactly as for the area functional, this amounts to prove a relaxation result for

$$I^\varepsilon(u) = \int_0^T e^{-T/\varepsilon} \left(\frac{\|u_t\|^2_{L^2(\Omega)}}{2} + \frac{F(u)}{\varepsilon} \right) dt.$$

We shall start by recalling some notation for functionals defined on measures. Given a continuous function $g : \mathcal{O} \times \mathbb{R}^m \to [0, \infty)$, $\mathcal{O} \subseteq \mathbb{R}^n$ open, and a $\mathbb{R}^m$-valued Radon measure μ on $\mathcal{O}$, we define

$$\int_{\mathcal{O}} g(x, \mu) := \int_{\mathcal{O}} g\left(x, \frac{d\mu}{d\,|\mu|}\right) d|\mu|(x).$$

It is easy to verify that when g is positively 1-homogeneous with respect to ξ, namely if $g(x, \lambda\xi) = \lambda\, g(x, \xi)$ for every $(x, \xi) \in \mathcal{O} \times \mathbb{R}^m$ and $\lambda > 0$, then, by writing the classical decomposition $\mu = \mu^a \mathcal{L}^n + \vec{\mu}^s|\mu^s|$ where $\overrightarrow{\mu}^s := \mu^s/|\mu^s|$, it holds

$$\begin{aligned} \int_{\mathcal{O}} g(x, \mu) &= \int_{\mathcal{O}} g\left(x, \frac{\mu^a}{|\mu^a|}\right) |\mu^a| \mathrm{d}x + \int_{\mathcal{O}} g\left(x, \vec{\mu}^s\right) d|\mu^s| \\ &= \int_{\mathcal{O}} g\left(x, \mu^a\right) \mathrm{d}x + \int_{\mathcal{O}} g\left(x, \vec{\mu}^s\right) d|\mu^s|. \end{aligned} \tag{4.3}$$

Let now $f : \overline{\Omega} \times \mathbb{R}^d \to [0, \infty)$ be a continuous function. We shall assume that

(H_1) f is convex in ξ for every $x \in \Omega$;

(H_2) f has linear growth in ξ, i.e. there exists $M>0$ such that

$$\frac{1}{M}|\xi| - M \le f(x,\xi) \le M\,(1+|\xi|) \quad \forall\,(x,\xi)\in\Omega\times\mathbb{R}^d.$$

By convexity, the *recession function* $f^\infty : \bar\Omega\times\mathbb{R}^d \to [0,\infty)$

$$f^\infty(x,\xi) := \lim_{t\to\infty}\frac{1}{t} f(x,t\,\xi)$$

is well-defined and we let $\bar f : \bar\Omega\times\mathbb{R}^d\times[0,\infty)\to[0,\infty)$ given by

$$\bar f(x,\xi,t) := \begin{cases} f\left(x,\dfrac{\xi}{t}\right)\,t & \text{if } \ t>0,\\ f^\infty(x,\xi) & \text{if } \ t=0.\end{cases}$$

We shall additionally assume that

(H_3) $\bar f$ is continuous.

Under assumptions (H_1)-(H_3), the relaxation of F is $L^1(\Omega)$ is given by [5],

$$\begin{aligned}\underline{F}(u) = &\int_\Omega f(x,\nabla u(x))\,\mathrm{d}x + \int_\Omega f^\infty(x,\overrightarrow{D^s u}(x))\,d|D^s u|(x)\\ &+ \int_{\partial\Omega} f^\infty(x,n(x)\,[\varphi(x)-u(x)])\,d\mathcal{H}^{d-1}(x), \quad D(\underline{F}) = BV(\Omega).\end{aligned}$$

Here, $n(x)$ is the external unit normal to $\partial\Omega$ in x.

Moreover, we set

$$\underline{I}^\varepsilon(u) := \inf\left\{\liminf_{j\to\infty} I^\varepsilon(u_j)\ |\ u_j\to u \ \text{ in } \ L^1(\Omega_T)\right\},$$

along with the domains of definition

$$D(I^\varepsilon) := D(W^\varepsilon) \quad \text{and} \quad D(\underline{I}^\varepsilon) := D(\underline{W}^\varepsilon).$$

The main result of this section reads as follows.

Theorem 4.1 (Relaxation, linear-growth functionals)**.**

$$\underline{I}^\varepsilon(u) = \int_0^T e^{-T/\varepsilon}\left(\frac{\|u_t\|^2_{L^2(\Omega)}}{2} + \frac{\underline{F}(u)}{\varepsilon}\right)dt.$$

We postpone the proof of this result to Subsection 4.1 below. However, we shall comment here that the above relaxation result together with the general theory of [20] entail the convergence of qualified approximate minimizers of I^ε to suitably weak variational solutions of (4.1). In particular, we have the following.

Theorem 4.2 (Convergence, linear-growth functionals)**.** *Let $H = L^2(\Omega)$ and v_ε be ε^3-approximate minimizers of I^ε on K as in (2.3). Then, there exists $C>0$ depending on $F(u_0)$ and T but not on ε such that*

$$\max_{t\in[0,T]}\|\underline{u}(t) - v_\varepsilon(t)\|_H \le C\varepsilon^{1/2},$$

where $\underline{u}$ is the unique solution of

$$\underline{u}'(t) + \partial\underline{F}(\underline{u}) \ni 0 \quad \text{in } L^2(\Omega), \quad \text{a.e. in } (0,T), \qquad \underline{u}(0) = u_0.$$

The same uniform convergence result holds for the minimizers $\underline{u}_\varepsilon$ of the relaxed WED functional $\underline{I}^\varepsilon$ on K.

Let us mention that, under suitable additional smoothness and structure assumptions on f, an explicit characterization of the subdifferential $\partial\underline{F}$ in $L^2(\Omega)$ is provided in [4, Prop. 6.8, p. 175].

4.1. **Relaxation: linear-growth functionals.** This subsection contains the proof of Theorem 4.1 which can be restated as

$$\underline{I}^\varepsilon(u) \equiv \widehat{I}^\varepsilon(u) := \int_0^T \mathrm{e}^{-t/\varepsilon}\left(\int_\Omega \frac{|u_t|^2}{2}\,\mathrm{d}x + \frac{\underline{F}(u)}{\varepsilon}\right)dt, \quad D(\widehat{I}^\varepsilon) := D(\underline{W}^\varepsilon).$$

As the semicontinuity of $\widehat{I}^\varepsilon$ implies that $\widehat{I}^\varepsilon(u) \le \underline{I}^\varepsilon(u)$ for every $u \in L^1(\Omega_T)$, it is sufficient to show the opposite inequality. Namely, for every $u \in D(\underline{I}^\varepsilon)$, we need to find a sequence $u_j \in D(I^\varepsilon)$ such that $u_j \to u$ in $L^1(\Omega_T)$ and

$$\liminf_{j\to\infty} I^\varepsilon(u_j) = \widehat{I}^\varepsilon(u). \tag{4.4}$$

We claim that (4.4) is an easy consequence of the construction in the proof of Theorem 1.2 and and Reshetnyak's continuity theorem [3, Thm. 2.39, p. 68], namely

Theorem 4.3 (Reshetnyak continuity)**.** *Let $\mathcal{O} \subseteq \mathbb{R}^n$ be open and ν_j, ν be $\mathbb{R}^m$-valued finite Radon measures in $\mathcal{O}$ such that*

$$\nu_j \to \nu \quad \textit{weakly-}* \textit{ in } \mathcal{O} \qquad \textit{and} \quad \|\nu_j\|(\mathcal{O}) \to \|\nu\|(\mathcal{O}).$$

Then, for every continuous and bounded function $f : \Omega \times \mathbb{S}^{d-1} \to \mathbb{R}$, it holds

$$\lim_{j\to\infty} \int_O f\left(x, \frac{d\nu_j}{d|\nu_j|}(x)\right) d|\nu_j|(x) = \int_O f\left(x, \frac{d\nu}{d|\nu|}(x)\right) d|\nu|(x).$$

Now, given $v \in BV(\Omega)$, we define the $\mathbb{R}^{d+1}$-valued measure in $\mathbb{R}^d$ given by

$$\mu = (\nabla v(t), 1)\,\mathcal{L}^d \llcorner \Omega + (\vec{D}^s v, 0)\,|D^s v| + ((\phi - v)\,n, 0)\,\mathcal{H}^{d-1} \llcorner \partial\Omega.$$

Analogously, given $u \in L^1(0,T;BV(\Omega))$, we consider the $\mathbb{R}^{d+1}$-valued measure $\nu := \mu(t) \otimes \mathrm{d}t_\varepsilon$ on $\mathbb{R}^d_T$, where $\mu(t)$ is as above for $v = u(\cdot,t)$, i.e.

$$\nu(A) := \int_0^T \mu(t)\big(A \cap (\mathbb{R}^d \times \{t\})\big)\,\mathrm{d}t_\varepsilon \quad \forall\, A \subseteq \mathbb{R}^d_T \quad \text{Borel.}$$

From (4.3) (note that $\bar{f}$ is 1-positively homogeneous) it is easy to verify that

$$\int_0^T \underline{F}(u(\cdot,t))\mathrm{d}t_\varepsilon = \int_{\Omega_T} \bar{f}(x,\nu). \tag{4.5}$$

Let now $u \in D(\underline{I}^\varepsilon)$, u_j be the sequence constructed in Section 3, and ν, ν_j the corresponding measures. Recall that:

(a) $u_j \equiv \varphi$ on $\partial_p\Omega$,
(b) $u_j \to u$ strongly in $L^1(\Omega_T)$,
(c) $u_{j,t} \to u_t$ strongly in $L^2(\Omega_T)$,
(d) $\int_0^T A(u_j)\mathrm{d}t_\varepsilon \to \int_0^T \underline{A}(u)\mathrm{d}t_\varepsilon$ (recall Remark 3.2).

From (b) it follows easily that $\nu_j \to \nu$ weakly-$*$. By (4.5) applied to the area functional and (d), one has that $\|\nu_j\|(\Omega_T) \to \|\nu\|(\Omega_T)$. Hence, from the representation (4.5) and Theorem 4.3, we infer that (note that $F(u_j) = \underline{F}(u_j)$)

$$\lim_{j\to\infty} \int_0^T F(u_j)\mathrm{d}t_\varepsilon = \int_0^T \underline{F}(u)\mathrm{d}t_\varepsilon,$$

which together with (c) gives (4.4).

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