Experiment No. 2

Z –TRANSFORM

The *z-transform* $G(z)$ of a sequence $g[n]$ is defined as

$$G(z) = \mathcal{Z}\{g[n]\} \sum_{n=-\infty}^{\infty} g[n] z^{-n},$$

where z is a complex variable. The set of values of z for which the z-transform $G(z)$ converges is called its *region of convergence* (ROC). In general, the region of convergence of a z-transform of a sequence $g[n]$ is an annular region of the z-plane:

$$R_{g-} < |z| < R_{g+},$$

In the case of LTI discrete-time systems , all pertinent z-transforms are rational functions of z^{-1}, that is, they are ratios of two polynomials in z^{-1}:

$$G(z) = \frac{P(z)}{D(z)} = \frac{p_0 + p_1 z^{-1} + \ldots + p_{M-1} z^{-(M-1)} + p_M z^{-M}}{d_0 + d_1 z^{-1} + \ldots + d_{N-1} z^{-(N-1)} + d_N z^{-N}},$$

which can be alternately written in factored form as

$$G(z) = \frac{p_0}{d_0} \frac{\prod_{r=1}^{M}(1 - \xi_r z^{-1})}{\prod_{s=1}^{N}(1 - \lambda_s z^{-1})} = \frac{p_0}{d_0} z^{N-M} \frac{\prod_{r=1}^{M}(z - \xi_r)}{\prod_{s=1}^{N}(z - \lambda_s)}.$$

The *zeros* of $G(z)$ are given by $z = \xi_r$ while the *poles* are given by $z = \lambda_s$. There are additional $(N - M)$ zeros at $z = 0$ (the origin in the z-plane) if $N > M$ or additional $(M - N)$ poles at $z = 0$ if $N < M$.

For a sequence with a rational z-transform, the ROC of the z-transform cannot contain any poles and is bounded by the poles.

The *inverse z-transform* $g[n]$ of a z-transform $G(z)$ is given by

$$g[n] = \frac{1}{2\pi j} \oint_C G(z)\, z^{n-1}\, dz,$$

where C is a counterclockwise contour encircling the point z_0 in the ROC of $G(z)$.

A rational z-transform $G(z) = P(z)/D(z)$, where the degree of the polynomial $P(z)$ is M and the degree of the polynomial $D(z)$ is N, and with distinct poles at $z = \lambda_s$, $s = 1, 2, \ldots, N$, can be expressed in a partial-fraction expansion form given by

$$G(z) = \sum_{\ell=0}^{M-N} \eta_\ell z^{-\ell} + \sum_{s=0}^{N} \frac{\rho_s}{1 - \lambda_s z^{-1}},$$

Assuming $M \geq N$.

The constants ρ_s in the above expression, called the *residues* , are given by

$$\rho_s = (1 - \lambda_s z^{-1}) G(z)|_{z=\lambda_s}.$$

Evaluation of *z*-Transforms on Unit Circle

The function freqz can be used to evaluate the values of a rational *z*-transform on the unit circle. The evaluation of Z- Transform on a unit circle leads to the Frequency Response or Fourier Transform.

Sample Program 1

```
% Discrete-Time Fourier Transform Computation
%
% Read in the desired number of frequency samples
k = input('Number of frequency points = ');
% Read in the numerator and denominator coefficients
num = input('Numerator coefficients = ');
den = input('Denominator coefficients = ');
% Compute the frequency response/Evaluate Z transform on unit circle
w = 0:pi/(k-1):pi;
h = freqz(num, den, w);
% Plot the frequency response
subplot(2,2,1)
plot(w/pi,real(h));grid
title('Real part')
xlabel('\omega/\pi'); ylabel('Amplitude')
subplot(2,2,2)
plot(w/pi,imag(h));grid
title('Imaginary part')
xlabel('\omega/\pi'); ylabel('Amplitude')
subplot(2,2,3)
plot(w/pi,abs(h));grid
title('Magnitude Spectrum')
xlabel('\omega/\pi'); ylabel('Magnitude')
subplot(2,2,4)
plot(w/pi,angle(h));grid
title('Phase Spectrum')
xlabel('\omega/\pi'); ylabel('Phase, radians')
```

Pole Zero Plot

The pole-zero plot of a rational *z*-transform $G(z)$ can be readily obtained using the function zplane. There are two versions of this function. If the *z*-transform is given in the form of a rational function , the command to use is zplane(num, den) where num and den are row vectors containing the coefficients of the numerator and denominator polynomials of $G(z)$ in ascending powers of z^{-1}. On the other hand, if the zeros and poles of $G(z)$ are given, the command to use is zplane(zeros, poles) where zeros and poles are column vectors. In the pole-zero plot generated by MATLAB, the location of a pole is indicated by the symbol × and the location of a zero is indicated by the symbol º.

Rational , Factored Forms of z-transform

The function tf2zp can be used to determine the zeros and poles of a rational *z*-transform $G(z)$. The program statement to use is [z, p, k] = tf2zp(num,den) where num and den are row vectors containing the coefficients of the numerator and denominator polynomials of $G(z)$ in ascending powers of z^{-1} and the output file contains the gain constant k and the computed zeros and poles given as column vectors z and p, respectively. The reverse process of converting a *z*-transform given in the form of zeros, poles, and the gain constant to a rational form can be implemented using the function zp2tf. The program statement to use is [num,den] = zp2tf(z,p,k).

The factored form of the z-transform can be obtained from the zero-pole description using the function sos = zp2sos(z,p,k). The function computes the coefficients of each second-order factor given as an $L \times 6$ matrix sos where

$$\text{sos} = \begin{bmatrix} b_{01} & b_{11} & b_{21} & a_{01} & a_{11} & a_{21} \\ b_{02} & b_{12} & b_{22} & a_{02} & a_{12} & a_{22} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{0L} & b_{1L} & b_{2L} & a_{0L} & a_{1L} & a_{2L} \end{bmatrix}$$

where the Lth row contains the coefficients of the numerator and the denominator of the Lth second-order factor of the z-transform $G(z)$:

$$G(z) = \prod_{\ell=1}^{L} \frac{b_{0\ell} + b_{1\ell} z^{-1} + b_{2\ell} z^{-2}}{a_{0\ell} + a_{1\ell} z^{-1} + a_{2\ell} z^{-2}}$$

Sample Program 2

```
%Analysis of Z-Transforms
%Definition of numerators and denominator coefficients
num=[5 6 -44 21 32];
den=[5 13 15 18 -12];
%Conversion from rational to Factored form
[z,p,k]=tf2zp(num,den);
disp('Zeros are at');disp(z);
disp('Poles are at');disp(p);
disp('Gain Constant');disp(k);
%Determination of radius of the poles
radius=abs(p);
disp('Radius of the poles');disp(radius);
%Pole Zero Map using numerator and denominator coefficients
zplane(num,den)
%Conversion from factored to secomd ordered factored
sos=zp2sos(z,p,k)
disp('Second Order Sections');disp(sos);
```

Lab Exercises:

Problem 1 Evaluate the following z-transform on the unit circle:

$$G(z) = \frac{2 + 5z^{-1} + 9z^{-2} + 5z^{-3} + 3z^{-4}}{5 + 45z^{-1} + 2z^{-2} + z^{-3} + z^{-4}}.$$

Problem 2 Write a MATLAB program to compute and display the poles and zeros, to compute and display the factored form, and to generate the pole-zero plot of a z-transform that is a ratio of two polynomials in z^{-1}. Using this program, Find and plot the poles and zeroes of $G(z)$ .Also Find the radius of the resulting poles.

Problem 3 From the pole-zero plot generated in Problem 2, determine the number of possible regions of convergence (ROC) of $G(z)$. Show explicitly all possible ROCs . Can you tell from the pole-zero plot whether or not the DTFT exists?

Problem 4 Write a MATLAB program to compute and display the rational z-transform from its zeros, poles and gain constant. Determine the rational form of a z-transform whose zeros are at $\xi_1 = 0.3$, $\xi_2 = 2.5$, $\xi_3 = -0.2+j\,0.4$, and $\xi_4 = -0.2-j\,0.4$; the poles are at $\lambda_1 = 0.5$, $\lambda_2 = -0.75$, $\lambda_3 0.6 + j\,0.7$, and $\lambda_4 = 0.6 - j\,0.7$; and the gain constant k is 3.9.

Inverse z-Transform

The inverse $g[n]$ of a rational z-transform $G(z)$ can be computed using MATLAB in basically two different ways. For finding the Inverse z-transform, it is necessary to know a priori the ROC of $G(z)$.

First Method

The function impz provides the samples of the time domain sequence, which is assumed to be causal. There are three versions of this function:

[g,t] = impz(num,den), [g,t]= impz(num,den, L), and [g,t] = impz(num,den, L, FT),

where num and den are row vectors containing the coefficients of the numerator and denominator polynomials of $G(z)$ in ascending powers of z^{-1}, L is the desired number of the samples of the inverse transform, g is the vector containing the samples of the inverse transform starting with the sample at $n = 0$, t is the length of g, and FT is the specified sampling frequency in Hz with default value of unity.

The function y=filter(num,den,x) can also be used to calculate the input response of a z-transform . Where num, den represent vectors containing numerator and denominator coefficients of z-transform.While x represents input to the filter / z-transform .The length of output y is equal to input x. If an impulse input sequence is passed to the z-transform , the output will be the inverse z-transform.

Sample Program 3

```
%Inverse Z-Transform using impz
%definition of numerator and denominator coefficients
num=[0.1+.4*i 5 .05];
den=[1 .9+0.3*i .12];
%Finding the inverse z transform of G(z)
[a,b]=impz(num,den);
%Evaluating on Unit Circle i.e. Fourier Transform
[c,d]=freqz(num,den);
% Plotting of x[n] and it's fourier transform
subplot(2,2,1)
stem(b,real(a))
title('Real Part of g[n]')
xlabel('Samples'); ylabel('Magnitude')
grid on
subplot(2,2,2)
stem(b,imag(a))
title('Imaginary Part of g[n]')
xlabel('Samples'); ylabel('Magnitude')
grid on
subplot(2,2,3)
plot(d/pi,abs(c))
title('Magnitude Spectrum of g[n]')
xlabel('\omega/\pi'); ylabel('Magnitude')
grid on
subplot(2,2,4)
plot(d/pi,angle(c))
title('Phase Spectrum of g[n]')
xlabel('\omega/\pi'); ylabel('Phase, radians')
grid on
```

Second Method

A closed-form expression for the inverse of a rational z-transform can be obtained by first performing a partial-fraction expansion using the function residuez and then determining the inverse of each term in the expansion by looking up a table of z-transforms. The function residuez can also be used to convert a z-transform given in the form of a partial-fraction expansion to a ratio of polynomials in z^{-1}.It has the format [r,p,k] = reiduez(num,den) where r,p,k are the residues, poles and direct terms of the partial-fraction expansion of z-transform described by num and den vectors which contain numerator and denominator coefficients.

$$\frac{num(z)}{den(z)} = \frac{r(1)}{1 - p(1)z^{-1}} + \ ... + \frac{r(n)}{1 - p(n)z^{-1}} + \ k(1) + k(2)z^{-1} + \cdots$$

Problem 5 Write a MATLAB program to compute the first L samples of the inverse of a rational z-transform where the value of L is provided by the user through the command input. Using this program compute and plot the first 50 samples of the inverse of $G(z)$. Use the command stem for plotting the sequence generated by the inverse transform.

Problem 6 Write a MATLAB program to determine the partial-fraction expansion of a rational z-transform. Using this program determine the partial-fraction expansion of $G(z)$ and then its inverse z-transform $g[n]$ in closed form. Assume $g[n]$ to be a causal sequence.