Problem 1: (20 points)

Consider a LTI SISO system with state-space representation

$$\begin{aligned} \dot{x} &= \begin{pmatrix} 0 & 1 \\ 200 & -10 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \\ y &= \begin{pmatrix} 1 & -10 \end{pmatrix} x. \end{aligned}$$

1. (7 points) Determine the eigenvalues of the system matrix using the method described in lecture.
2. (2 points) Using your result from part 1, Is the system BIBS stable?
3. (7 points) Determine the system transfer function $Y(s)/U(s)$ using the matrices in the state-space representation.
4. (2 points) what are the poles of the system transfer function?
5. (2 points) Using your result from part 4, is the system BIBO stable?

Problem 2: (20 points)

Specifications for control systems frequently involve constraints on the time response characteristics of the closed-loop system. The standard characteristics for quantifying the unit-step response are

1. **rise time, t_r:** Rise time, t_r, is the time for the response, on its initial rise, to go from 0.1 to 0.9 times the steady-state value.

2. **settling time, t_s:** Settling time, t_s, is the time required for the response to first reach and thereafter remain within 1% of the steady-state value.

3. **percent overshoot, M_p:** The term overshoot is applicable when the response temporarily exceeds the steady-state value of the response. The parameter M_p is the peak value of the first overshoot, and is typically given as a percentage

$$M_p(\%) = 100 \left| \frac{M_p - y_{ss}}{y_{ss}} \right|,$$

where y_{ss} is the steady-state value of the response $y(t)$.

4. **peak time, t_p:** The time-to-peak is the time instant at which the transient response reaches the first overshoot, and so $y(t_p) = M_p$.

1. (15 points) Write a MATLAB function that returns the parameters t_r, t_s, $M_p(\%)$, t_p, and y_{ss} given a vector y containing the unit-step response of a system. To receive credit you must include a copy of the m-file along with your problem set solutions. The m-file must contain your name and section as comments, as well as comments describing how the m-file works.

2. (5 points) Using your m-file, find the parameters t_r, t_s, $M_p(\%)$, t_p, and y_{ss} for the signal $y(t)$ contained in the file *response.mat*.

Problem 3: (20 points)

Dynamic second-order linear time-invariant physical systems are often encountered in system analysis and control engineering. The mathematical model of these physical systems is the second-order linear differential equation

$$\frac{d^2y}{dt^2} + 2\zeta\omega_n\frac{dy}{dt} + \omega_n^2 y(t) = K\omega_n^2 u(t),$$

where ζ is the dimensionless damping ratio and $\omega_n > 0$ is the natural frequency. By calculating the zero-state response to a unit-step input for three different ranges of ζ, you will review how ζ and ω_n affect the time response characteristics of the system.

1. (2 points) Find the input-output transfer function $H(s) = Y(s)/U(s)$. What is the physical significance of the parameter K when both poles of $H(s)$ have strictly negative real parts ?

2. (4 points) When $\zeta > 1$, the poles are real and distinct

$$\begin{aligned} s_1 &= -\zeta\omega_n + w_n\sqrt{\zeta^2 - 1} \\ s_2 &= -\zeta\omega_n - w_n\sqrt{\zeta^2 - 1}. \end{aligned}$$

Show that the unit-step response of the system when $\zeta > 1$ is

$$y(t) = K\left[1 + \frac{s_2}{s_1 - s_2}e^{s_1 t} - \frac{s_1}{s_1 - s_2}e^{s_2 t}\right]1(t).$$

This response is termed *overdamped*. **Hint:** Use the fact that $s_1 s_2 = \omega_n^2$ to express $H(s)$ as

$$H(s) = \frac{K s_1 s_2}{(s - s_1)(s - s_2)}$$

when representing $Y(s)$ with a partial fraction expansion.

3. (4 points) For a *critically damped* response, $\zeta = 1$ and the poles are real and identical

$$s_1 = s_2 = -\zeta\omega_n = -\omega_n.$$

Show that the second-order step-response for $\zeta = 1$ is

$$y(t) = K\left[1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t}\right]1(t).$$

4. (4 points) When $0 \le \zeta < 1$ the poles are complex conjugates

$$\begin{aligned} s_1 &= -\sigma + \jmath\omega_d \\ s_2 &= -\sigma - \jmath\omega_d, \end{aligned}$$

where the damped frequency ω_d is

$$\omega_d = \omega_n\sqrt{1 - \zeta^2}$$

And

$$\sigma = \zeta\omega_n.$$

Solve for the unit-step response when $0 \le \zeta < 1$ and show that

$$y(t) = K\left[1 - e^{-\sigma t}\left(\cos\omega_d t + \frac{\sigma}{\omega_d}\sin\omega_d t\right)\right]1(t).$$

Observe that the real (σ) and imaginary (ω_d) part of the complex pole pair determines the rate of exponential decay and the frequency of oscillation, respectively. Due to the oscillatory behavior of $y(t)$, the system response is said to be *underdamped*.

5. (4 points) In part 4 the complex poles of $H(s)$ can also be written in polar form. Show that the poles are located at a radius ω_n in the s-plane and at an angle $\theta = \sin^{-1}\zeta$ from the imaginary axis. Sketch the location of the complex pole pair in the s-plane. Clearly label σ, ω_d, ω_n and θ in your plot.

6. (2 points) When $\zeta < 0$, does the unit-step response approach a steady-state value? Explain your answer in one or two sentences.

Problem 4: (20 points)

This problem considers the effects of a single finite zero on the transient response of the second-order system

$$\frac{Y(s)}{U(s)} = \frac{(s/\alpha\zeta\omega_n)+1}{(s/\omega_n)^2+2\zeta(s/\omega_n)+1}.$$

The zero is located at

$$s = -\alpha\zeta\omega_n.$$

If α is large, than the zero is far removed from the poles and will have little effect on the response. In fact, as $\alpha\to\infty$, we have

$$\lim_{\alpha\to\infty}\frac{Y(s)}{U(s)} = \frac{1}{(s/\omega_n)^2+2\zeta(s/\omega_n)+1}.$$

1. (8 points) Let $\zeta = 0.5$ and $\omega_n = 1$ rad/sec. Using MATLAB, plot the unit-step response for $\alpha = 1, 2, 4,$ and 100 in a single figure. Label the individual responses using the **legend** command . In order to obtain the Greek symbol α in the legend, use the MATLAB symbol \alpha. Add your name and section using **gtext**, an include a copy of the m-file specifying the commands used to generate the plots.

2. (7 points) For each system in part 1, sketch the pole-zero map, and using MATLAB, determine the percent peak overshoot, the time-to-peak, rise-time, and settling time. Based on these time response characteristics, summarize the effect of moving the zero towards the imaginary axis in two or three short sentences.

3. (5 points) In part 1 the zero is always located in the left-half plane. Now consider the affect of an *unstable zero*, that is, a zero located in the right-half plane. With $\zeta = 0.5$ and $\omega_n = 1$ rad/sec, simulate the step-response for $\alpha = -1$ using MATLAB. In comparison to the results obtained in part 1, what is the salient feature of the step-response obtained with a system that has an unstable zero?

Problem 5: (20 points)

Consider the following second order system with an added pole

$$H(s) = \frac{1}{(s/p+1)(s^2+s+1)}.$$

1. (10 points) Let $p = \alpha/2$ and plot the unit-step response using MATLAB for $\alpha = 0.1, 1,$ and 10 in a single figure. Label the responses using the **legend** command and add your name and section number using **gtext**. Include a copy of an m-file showing the MATLAB commands used to generate the plots.

2. (10 points) For each system in part 1, sketch the pole-zero map, and using MATLAB, determine the percent peak overshoot, the time-to-peak, rise-time, and settling time. Based on these time response characteristics, summarize the effect of moving the pole towards the imaginary axis in two or three short sentences.