Closed-form expression for the collision probability in the IEEE Ethernet Passive Optical Network registration scheme

Swapnil Bhatia and Radim Bartoš

Department of Computer Science, University of New Hampshire, Durham, New Hampshire 03824
sbhatia@cs.unh.edu; rbartos@cs.unh.edu

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We derive a closed-form expression for the message collision probability in the IEEE 802.3ah Ethernet Passive Optical Network (EPON) registration scheme. The expression obtained, although based on an approximation, shows a good match with simulation results. We use the results of our analysis to compute the size of the most efficient contention window and the most efficient number of nodes serviced by a given window size. © 2005 Optical Society of America

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1. Motivation

Protocols for emerging access network technologies such as Data-Over-Cable Service Interface Specification (DOCSIS) [1], Ethernet Passive Optical Network (EPON) [2], and some wireless technologies include a preliminary phase where the subscriber device must register with a head end or base station residing at a central office. Since this is the first communication between the head end and the subscriber device, no information about key parameters, such as latency or timing, is available to either party. Subscriber devices may be located at random distances unknown to the head end. As a result, most protocols rely on some collision avoidance scheme to reduce contention in the use of the communication channel. The recently adopted IEEE 802.3ah EPON standard prescribes the random delay scheme for this purpose. In this scheme, the head end broadcasts the size of a contention interval. The nodes, upon receiving this message, wait for a uniformly random interval and then transmit their registration message. In this paper, we derive a closed-form expression for the probability of message collision in this scheme and validate our result through simulation.

To our knowledge, this is the first attempt at computing the probability of collision for the IEEE EPON registration scheme. Although previous work in this area [3, 4] serves as an excellent general reference, its focus has primarily been on the stability and throughput of multiaccess schemes. Moreover, most of the assumptions (Poisson arrivals, backlogged nodes, etc.) are either not relevant to the IEEE EPON registration scheme or are out of the scope of our current work. For example, before the average success probability in a single registration cycle—the focus of our present work—is known, the multistep performance of the scheme cannot be analyzed. Our own past work [5] focuses on the high load performance characterization of the IEEE EPON registration scheme through simulations. A more recent analysis [6] is restricted only to the simpler case of identically distanced nodes. We propose a more generic model applicable to identically distanced as well as randomly distributed nodes. Our model includes the random round-trip delay together with the random contention window size. The model is parameterized by message size, contention window size, and round-trip time and is therefore directly applicable to a practical analysis of the IEEE EPON registration protocol.

2. Modeling Registration of an IEEE EPON Device

Figure 1 illustrates the IEEE EPON registration scheme. In this paper, we will focus on only the first step in the registration scheme: the transmission of the registration request. Subsequent steps cannot occur until the first step is completed successfully. As per the EPON protocol, the head end broadcasts a discovery message to signal the beginning of a special interval reserved for new-node registration. A new node, upon receiving it, replies with a registration request message transmitted after a random wait. If two such registration messages, say of length k each, arrive at the head end overlapping in time, then there is a collision. Thus, to model a collision we must calculate the arrival time of a message at the head end. We observe that this arrival time is a sum of the one-way propagation time of the broadcast message from the head end to the node, a random wait at the node, and another one-way propagation time of the registration request message from the node to the head end.

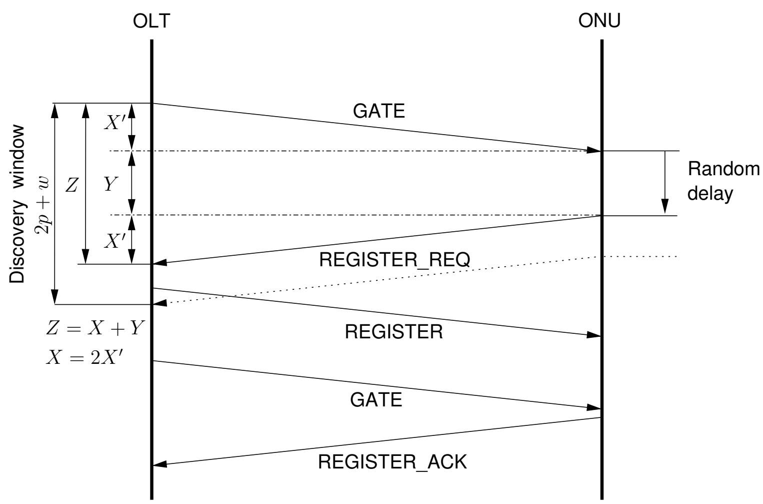

Fig. 1. IEEE EPON registration scheme.

Due to technological constraints on the power and reach of a transmitted signal, the IEEE EPON standard [2] fixes the maximum distance from the head end at which a node may be located. We assume the maximum reach of our network to be such as to result in a maximum one-way propagation time of p and the maximum random wait time to be w (also fixed by the head end). Thus, the arrival time can vary between a minimum of 0 and a maximum of $2p+w$ as shown in Fig. 1. However, the IEEE EPON standard [2] places no constraints on the contention window size $w \geq 0$. For convenience, we also define $M = \max(2p, w)$ and $m = \min(2p, w)$. Thus, $M \geq m$.

Let X, Y, and Z be random variables. Let X and Y be independent and have a uniform distribution with $X \in \text{Uniform}[0, M]$ and $Y \in \text{Uniform}[0, m]$, where $M \geq m \geq 0$. Depending on the magnitudes of p and w, X and Y will each model the two-way propagation time or the random wait. To simplify the exposition, we first consider $m > 0$ and add $m = 0$ as a separate case later. Let f_X and f_Y denote the probability mass functions of X and Y, respectively. Thus, $f_X(x) = M^{-1}$ and $f_Y(y) = m^{-1}$. Let $Z = X + Y$. Thus, Z models the arrival time of a message from a node at the head end as shown in Fig. 1. Since X and Y are independent, their joint density is $f_{XY}(x, y) = (Mm)^{-1}$. Let $F_Z(z) = P(Z \leq z)$ denote the cumulative distribution function (CDF) of Z. We compute the CDF of Z by integrating $X + Y = Z$ with respect to z [7]. Our main derivation involves the computation of many such integrals and we omit the details of these calculations due to space constraints. The limits used for each integral are specified in the accompanying figures and should aid the reader in computing the integrals, if desired. Thus, we have

$$F_Z(z) = \begin{cases} 0 & \text{if } z < 0 \\ z^2/2Mm & \text{if } z \le m \\ (2z-m)/2M & \text{if } m < z \le M \\ \left(2zM + 2zm - m^2 - z^2 - M^2\right)/2mM & \text{if } M < z \le M+m \\ 1 & \text{if } z > M+m \end{cases} . \tag{1}$$

To find the probability mass function $f_Z(z)$, we differentiate with respect to z to get

$$f_Z(z) = \begin{cases} 0 & \text{if } z < 0 \\ f_1(z) = z/mM & \text{if } z \le m \\ f_2(z) = 1/M & \text{if } m < z \le m \\ f_3(z) = (m+M-z)/mM & \text{if } M < z \le M+m \\ 0 & \text{if } z > M+m \end{cases} . \tag{2}$$

3. Formulating the Collision Event

Let n denote the total number of devices attempting to send their respective registration request messages. Let Z_i denote the arrival time of the message from the device $1 \le i \le n$ at the head end. (We will use Z_i to refer to the device i as well as the random variable, depending on the context.) Since all the devices behave identically, any result for a single device will be true for any device. Fix Z_1 as the device under observation. Then, a successful transmission by Z_1 can be expressed as the event

$$\begin{aligned} \bigcap_{i=2}^{n} [\mid Z_1 - Z_i \mid > k] &= \bigcap_{i=2}^{n} [(Z_1 - Z_i > k) \cup (Z_1 - Z_i < -k)] \\ &= \bigcap_{i=2}^{n} [(Z_i < Z_1 - k) \cup (Z_i > Z_i + k)]. \end{aligned} \tag{3}$$

Suppose $Z_1 = t$, where $0 \le t \le M + m$; i.e., the transmission from device 1 arrives at the head end at some time t. Under this condition, a successful transmission event for Z_1 can be expressed as

$$\bigcap_{i=2}^{n} [\mid Z_1 - Z_i \mid > k \mid Z_1 = t\ ] = \bigcap_{i=2}^{n} [(Z_i < t-k) \cup (Z_i > t+k)]. \tag{4}$$

Since all the devices follow the same registration protocol and transmit registration messages independently of each other, the Z_i are all independent and identically distributed with densities described by Eq. (1). Therefore, the probability of a successful transmission by Z_1 can be expressed as

$$\begin{aligned} P\left(\bigcap_{i=2}^{n} [\mid Z_1 - Z_i \mid > k \mid Z_1 = t]\right) &= \prod_{i=2}^{n} P[(Z_i < t-k) \cup (Z_i > t+k)] \\ &= P[(Z_2 < t-k) \cup (Z_2 > t+k)]^{n-1}, \end{aligned} \tag{5}$$

where we use Z_2 to represent any other single device. Since

$$P[(t-k) < Z_2 < (t+k)] = F_{Z_2}(t+k) - F_{Z_2}(t-k),$$

the remaining probability in the tails can be expressed as

$$P[(Z_2 < t-k) \cup (Z_2 > t+k)]^{(n-1)} = \{1 - [F_{Z_2}(t+k) - F_{Z_2}(t-k)]\}^{(n-1)}. \tag{6}$$

Hence, the probability of a successful transmission by Z_1 given that $Z_1 = t$ is

$$P\left(\bigcap_{i=2}^{n}[| Z_1 - Z_i |> k \mid Z_1 = t]\right) = \{1 - [F_{Z_2}(t+k) - F_{Z_2}(t-k)]\}^{(n-1)}. \tag{7}$$

Finally, applying the law of total probability,

$$\begin{aligned} P\left(\bigcap_{i=2}^{n}[| Z_1 - Z_i |> k]\right) &= \int_{-\infty}^{\infty} P\left(\bigcap_{i=2}^{n}[| Z_1 - Z_i |> k \mid Z_1 = t]\right) P(Z_1 = t)\,\mathrm{d}t \\ &= \int_{-\infty}^{\infty} [1 - F_{Z_2}(t+k) + F_{Z_2}(t-k)]^{(n-1)} f_{Z_2}(t)\,\mathrm{d}t. \end{aligned} \tag{8}$$

$F_{Z_2}(z)$ and $f_{Z_2}(z)$ are available from Eqs. (1) and (2). Figure 2 shows the probability mass function $f_{Z_i}(z)$ of any device Z_i along with an illustration of the condition $Z_1 = t$. From Fig. 2, we can see that due to the piecewise structure of $f_{Z_i}(z)$, the limits for integrating with respect to $Z = t$ will depend heavily on the relative magnitudes of k, m, and M. Moreover, the limits will also depend on the position of t relative to k, m, and M. For example, consider the simplest case where $m = 0$, $M > 0$, and $0 < k \leq M - k$. For this one permutation of parameters, we must again split the calculation of Eq. (8) for various relative positions of t with respect to the remaining intervals shown in Fig. 2. Thus, we would have to consider separately $t < 0$, $0 \leq t < k$, $k \leq t < M - k$, $M - k \leq t < M$, and $M \leq t < M + k$. Note that these subintervals are specific to the single simple case above. For the other more complex cases, the description of subintervals to be considered will be different and their number will be larger.

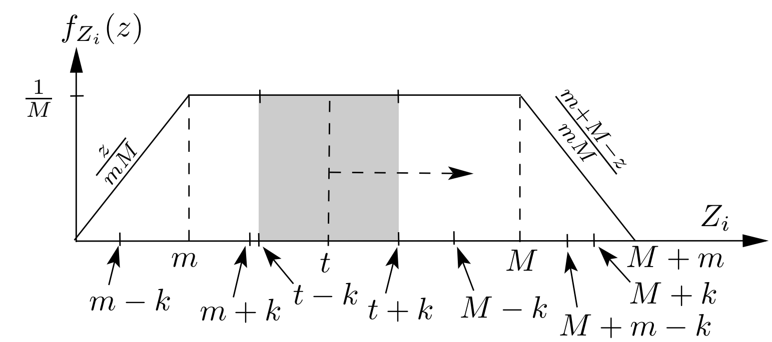

Fig. 2. Probability mass function $f_{Z_i}(z)$ with the condition $Z_1 = t$.

Clearly, the present approach will lead to a large number of subintervals over which the calculation of Eq. (8) will have to be performed—a tedious process. Notice that, out of all the cases, those introduced due to the relative magnitudes of our parameters k, m, and M are unavoidable. However, the subcases due to the conditional $Z_1 = t$ have been introduced only because the events in Eq. (3) are not independent. If an assumption were made as to the independence of those events, then a number of subcases would be avoided. Specifically, all the subintervals introduced due to the conditional $Z_1 = t$ could be avoided at the expense of introducing some error into the calculation. In Section 4, we explore this approach and obtain an approximation that shows a good match with values obtained from simulations.

4. Derivation of an Approximating Expression

Consider again, two independent identically distributed random variables Z_1 and Z_2 with probability mass functions $f_{Z_1}(z_1)$ and $f_{Z_2}(z_2)$ as derived in Eq. (2). Consider their joint density. Due to the piecewise structure of $f_Z(z)$, the joint density of Z_1 and Z_2 will comprise nine regions defined by the three cases each for Z_1 and Z_2 as illustrated in Fig. 3. Figure 3 (right) shows the joint density of Z_1 and Z_2 for examples $M = 10\,\mu s$ and $m = 5\,\mu s$. As is clear from Eq. (2), the joint density will plateau for $m \leq Z_1$ and $Z_2 \leq M$ when $m < M$. The shaded area in Fig. 3 (left) shows the region where $\mid Z_1 - Z_2 \mid \leq k$. Let P be the event $\mid Z_1 - Z_2 \mid \leq k$. To find the probability of event P, as mentioned earlier, we must consider several cases arising from the magnitudes of the parameters k, m, and M relative to each other. In each case, we also need to consider whether $m, M \leq k$ or $m, M > k$. Finally, $M + m < k$ or $m = 0$ are other special cases. Together, all the situations result in a function of the form expressed in Fig. 4. Due to the absence of the conditional, the number of cases to be considered is significantly reduced. Note that, were the conditional present, each branch of the tree in Fig. 4 would result in many more branches. Due to our assumption about the independence of events in Eq. (3), we are able to prune the tree much earlier.

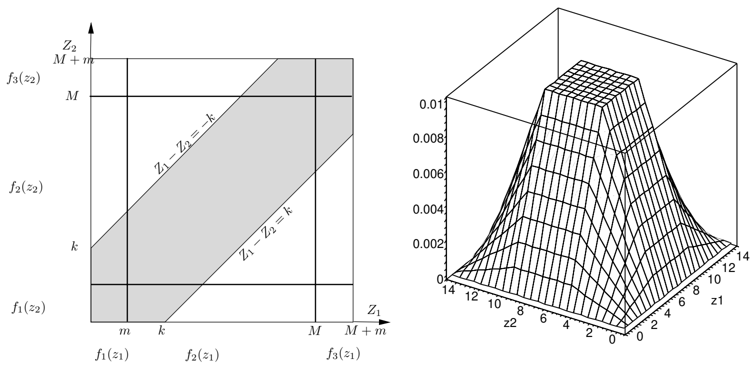

Fig. 3. Computing the probability of $\mid Z_1 - Z_2 \mid \leq k$ (left). Joint density of Z_1 and Z_2 (in μs) with $M = 10$ and $m = 5$ (right).

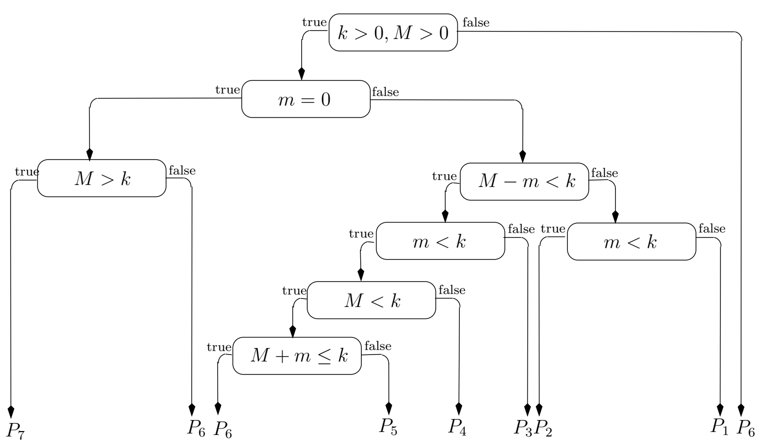

Fig. 4. $P(\mid Z_1 - Z_2 \mid \leq k)$.

Figures 5, 6, 7, 8, 9, 10, 11 illustrate each of the P_i in Fig. 4. Using these figures, we calculate the probability contained in the shaded region for each P_i. Integrating piecewise within the limits assigned to the shaded region in each figure, we obtain the following expressions for each of the P_i in Fig. 4:

$$P_1 = \frac{k\left(k^3 - 4m^3 - 4mk^2 + 12Mm^2\right)}{6m^2M^2}, \tag{9}$$

$$P_2 = \frac{\left(12Mk - m^2 - 6k^2\right)}{6M^2}, \tag{10}$$

$$\begin{aligned} P_3 = \Bigg(& \frac{12Mkm^2 + 12mkM^2 - 12mk^2M + 6m^2M^2 + m^4 + 3k^4 - 4km^3}{12m^2M^2} \\ & + \frac{6k^2m^2 - 4mk^3 - 4kM^3 + 6k^2M^2 - 4Mk^3 + M^4}{12m^2M^2} - \frac{4mM^3 + 4Mm^3}{12m^2M^2}\Bigg), \end{aligned} \tag{11}$$

$$\begin{aligned} P_4 = \Bigg(& \frac{6m^2M^2 - 4mM^3 + 12mM^2k - 12mMk^2 + 4mk^3 - m^4}{12m^2M^2} \\ & + \frac{12Mkm^2 - 4M^3k + 6M^2k^2 - 4Mk^3 + 4km^3 - 4Mm^3}{12m^2M^2} + \frac{k^4 + M^4 - 6k^2m^2}{12m^2M^2}\Bigg), \end{aligned} \tag{12}$$

$$\begin{aligned} P_5 = \Bigg(& \frac{12km^2M + 12kmM^2 - 12k^2mM + 6m^2M^2 - k^4 - 4m^3M - 4mM^3}{12m^2M^2} \\ & + \frac{4mk^3 + 4Mk^3 - M^4 - m^4 + 4km^3 + 4kM^3 - 6k^2m^2 - 6k^2M^2}{12m^2M^2}\Bigg), \end{aligned} \tag{13}$$

$$P_6 = 1, \tag{14}$$

If $m = 0$, then $Y \in \text{Uniform}\,[0,0]$ and hence $Z = X$. Therefore, $f_Z = f_X$, and

$$P_7 = \frac{k\,(2M - k)}{M^2}. \tag{15}$$

Figure 12 shows the probability of collision for two nodes participating in the IEEE EPON registration scheme with a total message length of 316 bytes (64 byte actual message with additional overhead [6]) as specified in the IEEE EPON standard (equivalent to $k = 2.528\,\mu\text{s}$). Although the value for the parameter p is also specified in the standard as $100\,\mu\text{s}$ (equivalent to 20 km), the range of values for p in the figure allows us to use the same model to compute the probabilities for clustered nodes or nodes situated at an identical distance. The range of values for the wait period w is unspecified by the standard and is open to various implementation schemes.

We now extend our two-node model to n nodes. Let $P_s\,(k)$ and $P_c\,(k)$ denote the probability of successful and unsuccessful transmission (i.e., collision), respectively, for a node in the presence of $k-1$ other nodes. The probability of successful transmission in the two-node case is thus $P_s\,(2) = 1 - P_c\,(2)$. We already derived $P_c\,(2)$ earlier in this section, since $P_c\,(2) = P\,(\mid Z_1 - Z_2 \mid \leq k)$, which is available from Fig. 4 and Eqs. (9)–(15). A successful transmission by a node in the presence of $n-1$ other nodes implies that its transmission did not collide with any of the other $n-1$ nodes. If we assume independence of each pairwise collision event of Eq. (8), we can write

$$P_s\,(n) = P_s(2)^{n-1}. \tag{16}$$

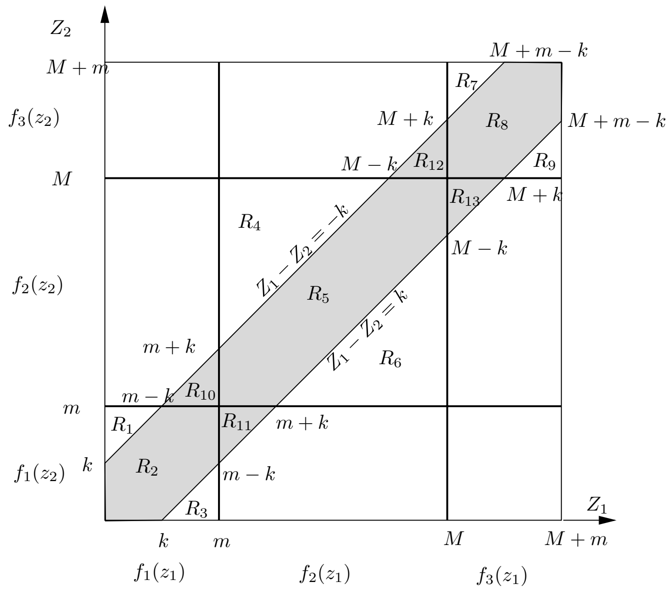

Fig. 5. $P_1 : m \geq k, M > m$ and $M - m \geq k$.

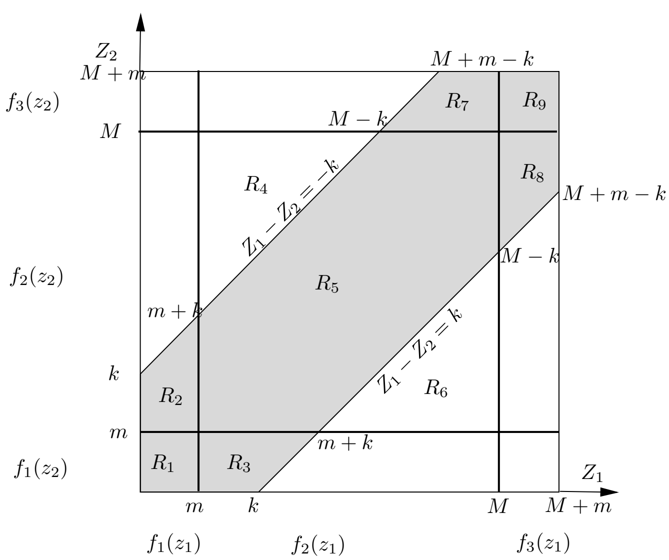

Fig. 6. $P_2 : m < k, M > m$ and $M - m \geq k$.

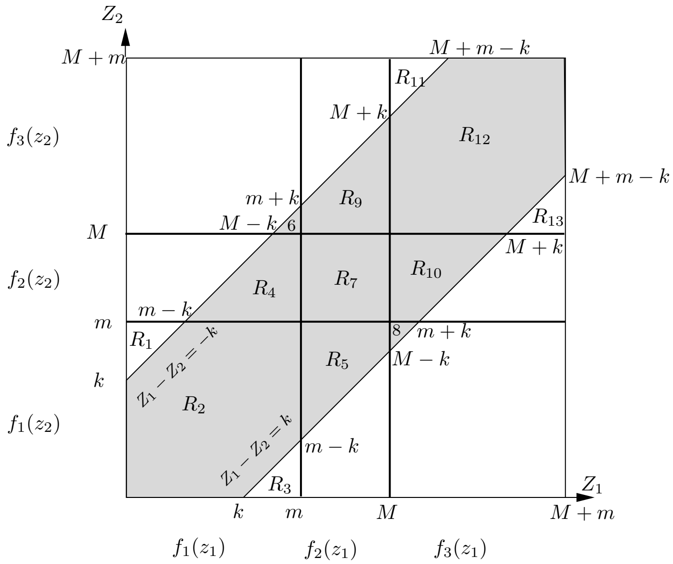

Fig. 7. $P_3 : m \geq k, M \geq m$ and $M - m < k$.

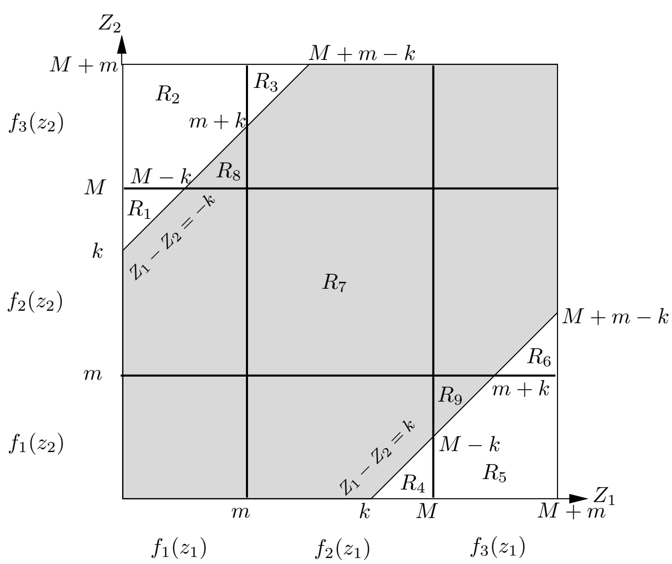

Fig. 8. $P_4 : m < k$, $M > m$, $M \geq k$, and $M - m < k$.

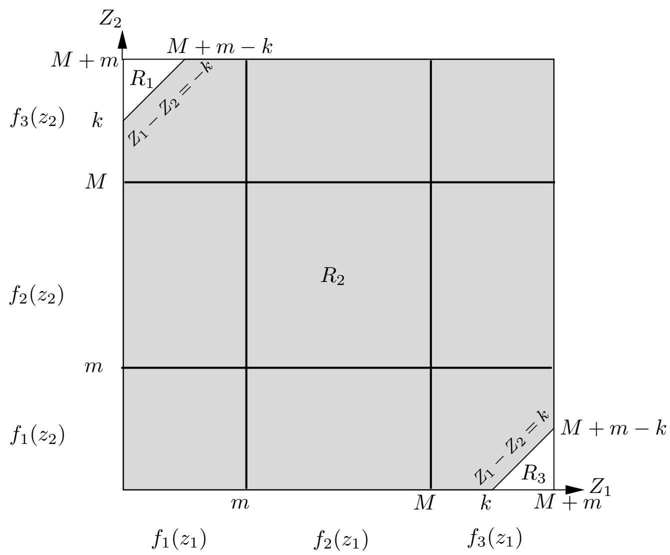

Fig. 9. $P_5 : m < k, M \geq m, M < k, M - m < k$, and $M + m > k$.

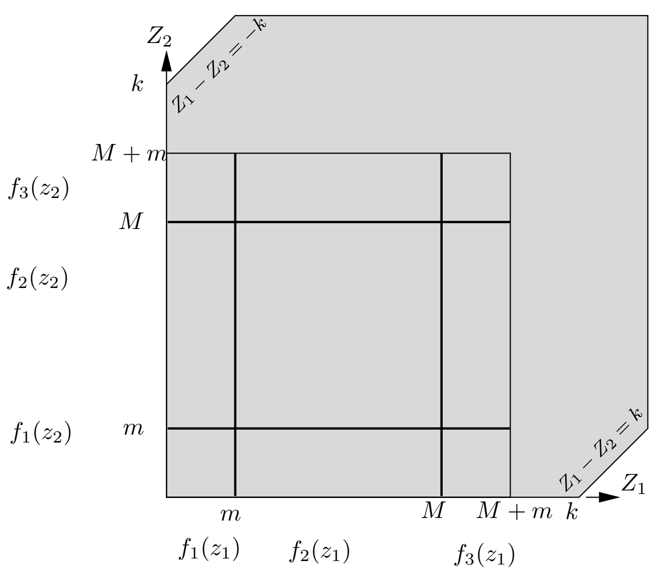

Fig. 10. $P_6 : m < k, M \geq m, M < k, M - m < k$, and $M + m \leq k$.

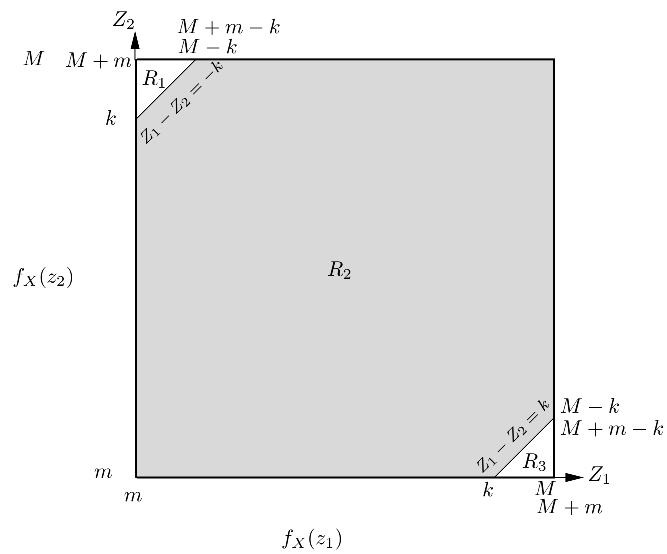

Fig. 11. $P_7 : m = 0$ and $M > k$.

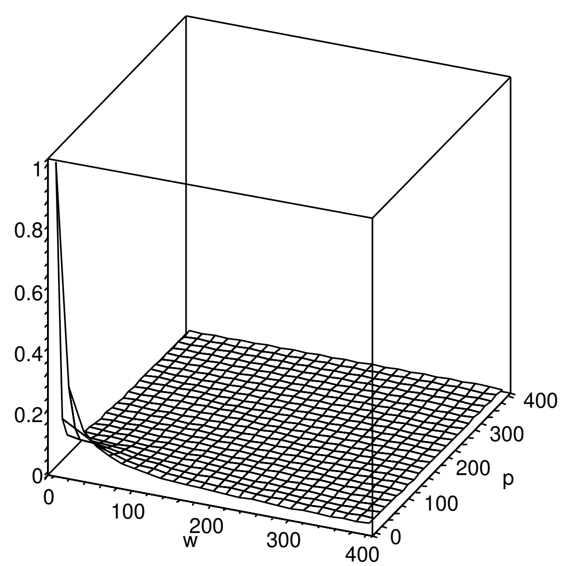

Fig. 12. Probability of collision with two randomly distanced nodes (p, w in µs).

This key assumption allows us to sidestep calculation of Eq. (8) over the many subintervals. Our calculations are now restricted to only the different cases introduced by the parameters k, m, and M. Figure 13 (left) shows the probability of successful transmissions for 1 to 200 nodes for a range of waiting times. The propagation time p is set to $100\,\mu s (20\,km)$. We can also formulate the situation where all the nodes are at an identical distance by setting $p = 0$. Figure 13 (right) shows the performance of the scheme for 1 to 200 nodes located at identical distances. Figures 14 and 15 compare the results from simulation plotted with those from our closed-form expression. Our model matches the simulation precisely except for a small range of window sizes in the uniformly random case when the number of devices is large. This error results from our assumption of the independence of two or more collision events. Our simulations show that the error introduced is negligible and is present for only a small range of window sizes. For window sizes larger than those appearing in Fig. 15, we have verified that the error diminishes rapidly.

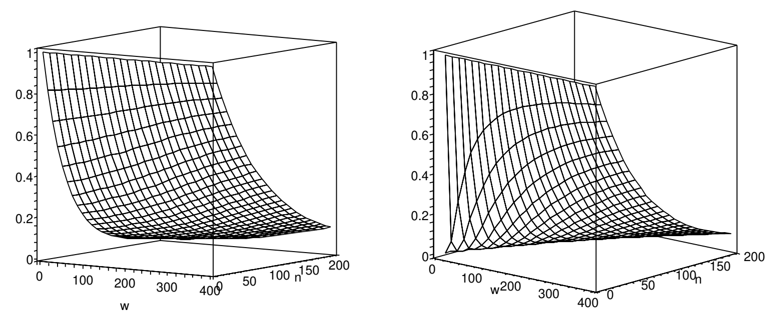

Fig. 13. Probability of successful transmission with n randomly distanced (left) and n identically distanced (right) nodes. (w is in µs.)

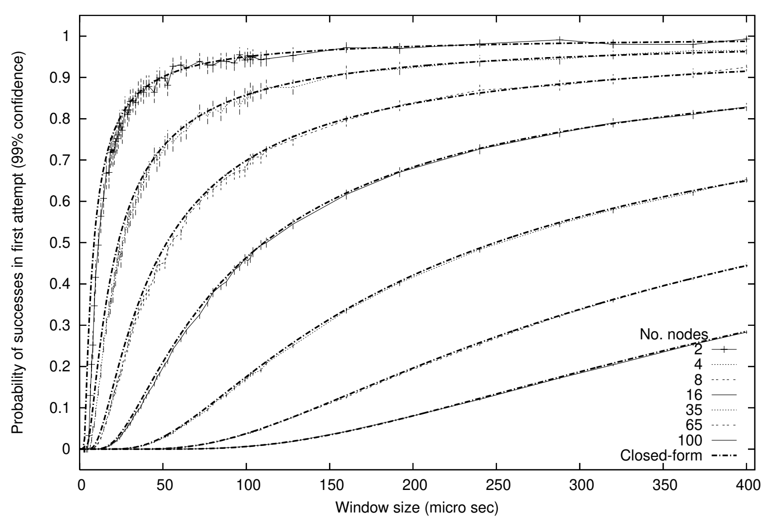

Fig. 14. Comparison of the probability of success for n identically distanced nodes obtained from simulation and closed-form expression. Thick and thin curves show the value from the closed-form expression and simulation, respectively, in each curve. Vertical lines show 99% confidence intervals.

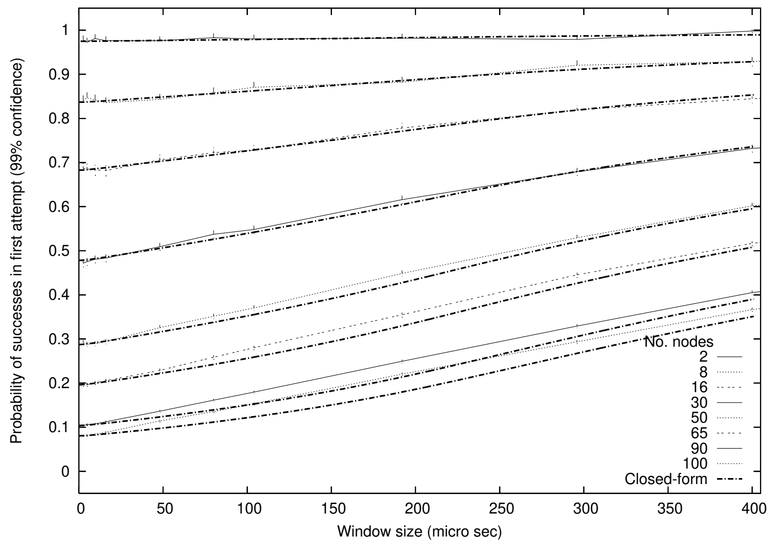

Fig. 15. Comparison of probability of success for n uniformly randomly distanced nodes obtained from simulation and closed-form expression. Thick and thin curves show the value from the closed-form expression and simulation, respectively, in each curve. Vertical lines show 99% confidence intervals.

5. Efficiency of the Contention Window

In the IEEE EPON registration scheme, the head end must reserve the communication link every time it needs to allow new nodes to register. Thus, a valuable portion of the available bandwidth is used at every such discovery cycle. The head end must reserve the channel for a duration of $2p+w$ (as is clear from Fig. 1), where $p = 100\,\mu\text{s}\ (20\,\text{km})$ as specified by the IEEE standard. It is desirable to minimize this duration when the channel is exclusively used for discovering new devices—regular traffic cannot be transmitted. To take this criterion into account, we can define a measure for the efficiency of a particular contention window size as the ratio of the average number of successful registrations to the size of the duration of the reservation [6]. Thus, the efficiency is

$$\rho = \frac{nP_s(n)}{2p_{\max}+w}. \tag{17}$$

We use our n-node model to relate efficiency to window size and node number. Figure 16 shows the variation of efficiency with the window size and node number for the identically distanced (left) and the uniformly randomly distanced (right) cases. For the identically distanced case, Table 1 shows the most efficient window size for a given number of nodes, i.e., the smallest window size that maximizes the success probability. Due to the shape of the surface in Fig. 16 (right) equivalent maxima cannot be obtained for the uniformly random case. However, Fig. 17 shows the most efficient number of nodes that can be serviced by a contention window of a given size.

6. Summary and Future Work

We derived the probability of message collision in the 802.3ah EPON registration scheme. We derived an approximating closed-form expression for the probability of message collision in the 802.3ah EPON registration scheme. We compared the probability computed by the expression with simulation results and obtained a reasonably precise match. Further,

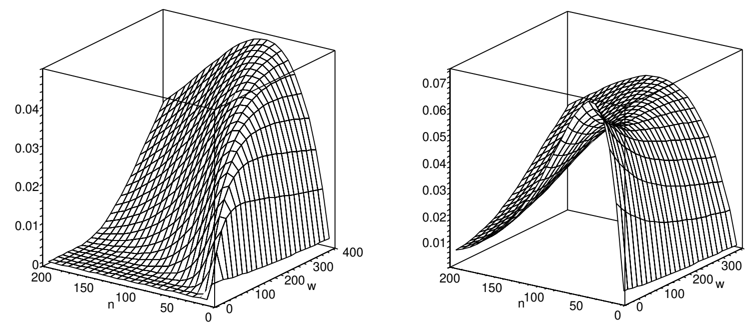

Fig. 16. Contention window efficiency for n identically distanced (left) and randomly distanced (right) nodes. (w is in μs.)

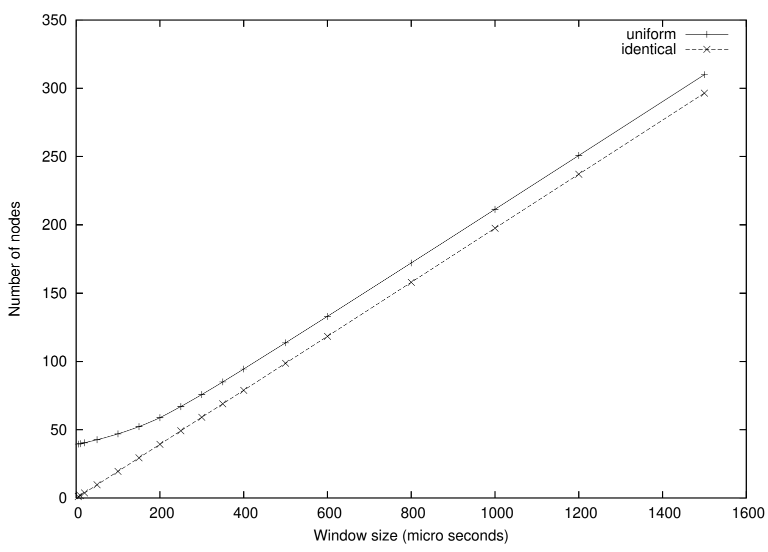

Fig. 17. Number of nodes served by a window size with maximum efficiency.

Table 1. Most Efficient Contention Window Size For Identically Distanced Nodes

Number of Devices ($p = 0$)	Most EfficientWindow Size (μs)
2	35.82
4	64.63
8	105.20
10	122.39
16	168.43
32	273.77
50	380.49
64	459.65
100	655.74
200	1179.31

we used our model to compute the most efficient contention window sizes for identically and randomly distributed nodes.

We are currently working on an exact solution of Eq. (8). Approximate collision probabilities for smaller clusters of nodes and other distributions of nodes can be computed using the current model by setting the appropriate value for parameter p. However, multiple clusters cannot be modeled with the current setup. Now that the average number of successful registrations is known, the model can be extended to evaluate the multistep performance of the registration scheme. Specifically, we can now model our scheme proposed in Ref. [5] and evaluate its efficacy.

Acknowledgments

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