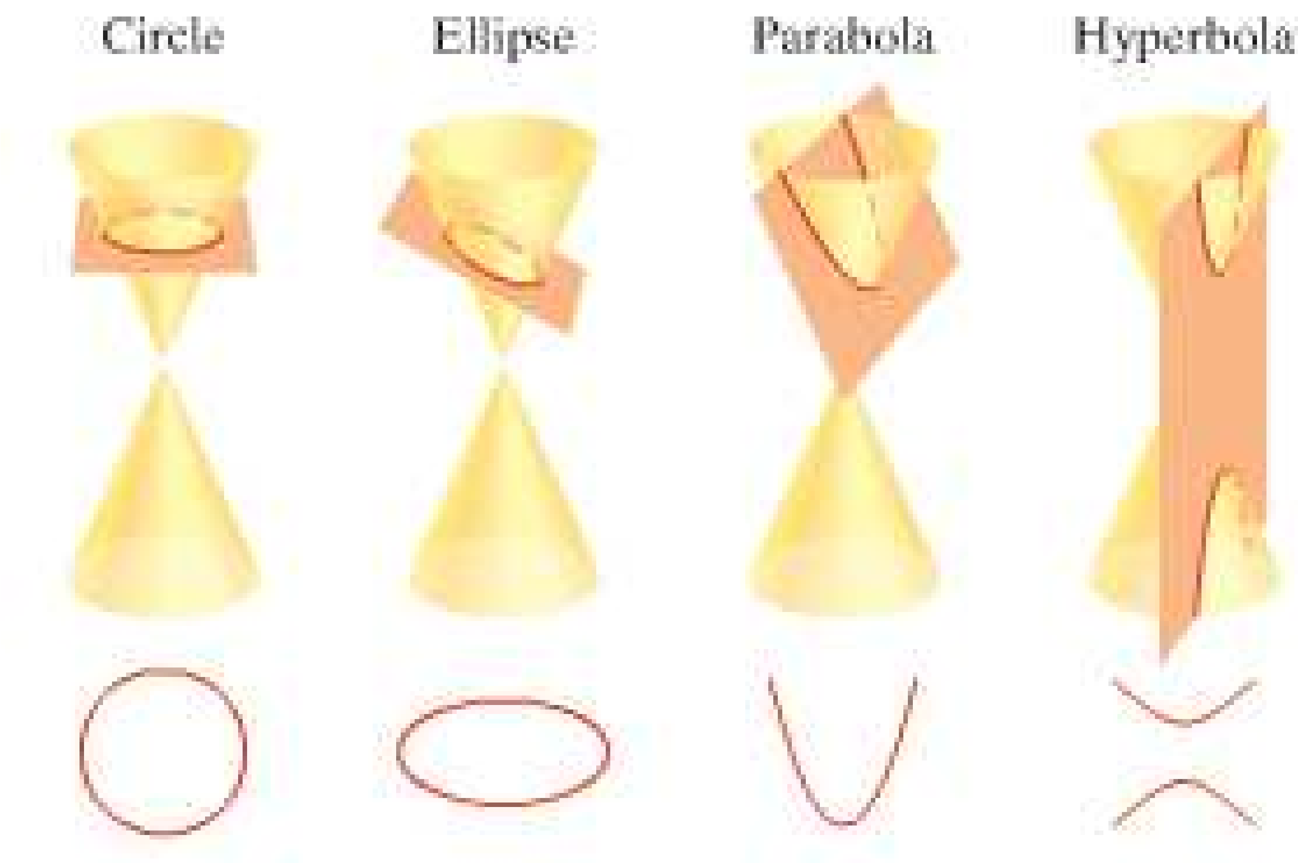
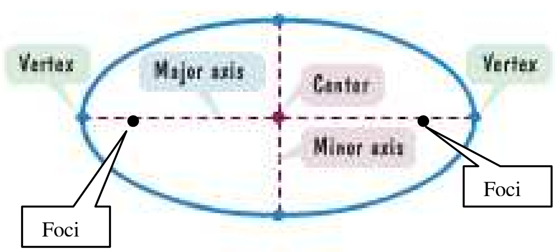
## Algebra and Trigonometry II

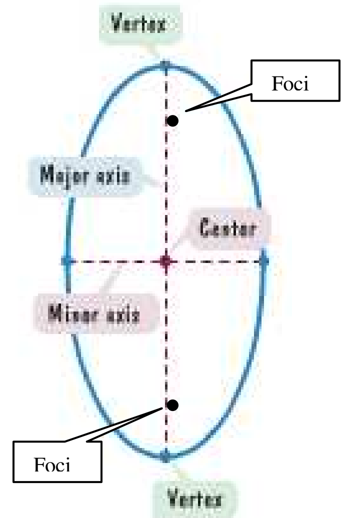
### 9.1 – The Ellipse

# Obtaining Conic Sections by Intersecting a Plane and a Cone

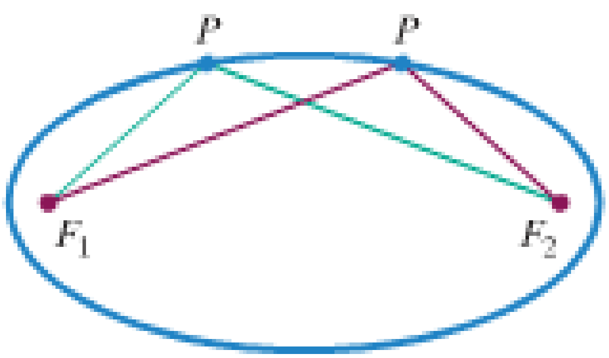


## Definition of an Ellipse




An Ellipse can be elongated in any direction. We will limit our discussions to ellipses elongated vertically and horizontally.




Sum of Red line = Sum of Blue Line

**Definition of an Ellipse**

An **ellipse** is the set of all points, $P$, in a plane the sum of whose distances from two fixed points, $F_1$ and $F_2$, is constant (see Figure 7.3). These two fixed points are called the **foci** (plural of **focus**). The midpoint of the segment connecting the foci is the **center** of the ellipse.

**The line segment that joins the vertices is the major axis; the midpoint of the major axis is the center of the ellipse. The line segment whose endpoints are on the ellipse and that is perpendicular to the major axis is called the minor axis.**

# Standard Form of the Equation of an Ellipse

**Standard Forms of the Equations of an Ellipse**

The **standard form of the equation of an ellipse** with center at the origin, and major and minor axes of lengths $2a$ and $2b$ (where $a$ and $b$ are positive, and $a^2 > b^2$) is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{or} \quad \frac{x^2}{b^2} + \frac{y^2}{a^2} = 1.$$

Figure 7.6 illustrates that the vertices are on the major axis, $a$ units from the center. The foci are on the major axis, $c$ units from the center. For both equations, $b^2 = a^2 - c^2$. Equivalently, $c^2 = a^2 - b^2$.

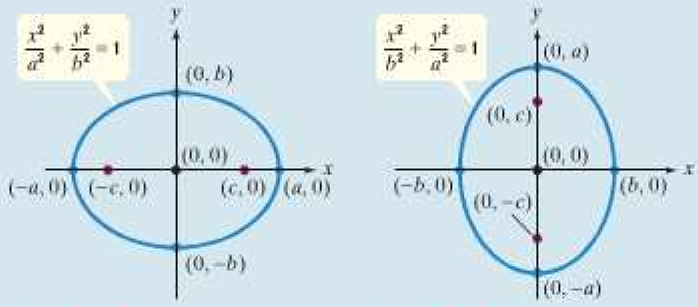



**Figure 7.6(a)** Major axis is horizontal with length $2a$.

**Figure 7.6(b)** Major axis is vertical with length $2a$.

*"a" is half the length of the major axis*
*"b" is half the length of the minor axis*

$b^2 = a^2 - c^2$ or $c^2 = a^2 - b^2$

$\pm c$" is the ycoordinate of the foci if the major axis is vertical
$\pm c$" is the xcoordinate of the foci if the major axis is horizontal

X-Intercepts: Set y=0

$$\frac{x^2}{a^2} = 1$$
$$x^2 = a^2$$
$$x = \pm a$$

Y-Intercepts: Set x=0

$$\frac{y^2}{b^2} = 1$$
$$y^2 = b^2$$
$$y = \pm b$$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## Using the Standard Form of the Equation of an Ellipse

### Technology

We graph $\frac{x^2}{9} + \frac{y^2}{4} = 1$ with a graphing utility by solving for $y$.

$$\frac{y^2}{4} = 1 - \frac{x^2}{9}$$

$$y^2 = 4\left(1 - \frac{x^2}{9}\right)$$

$$y = \pm 2\sqrt{1 - \frac{x^2}{9}}$$

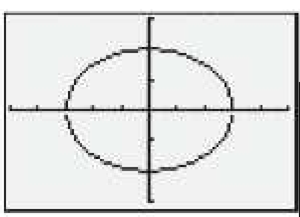
Notice that the square root property requires us to define two functions. Enter

$y_1 = 2$ [√] ( 1 [−] $x$ [^] 2 [÷] 9 )

and

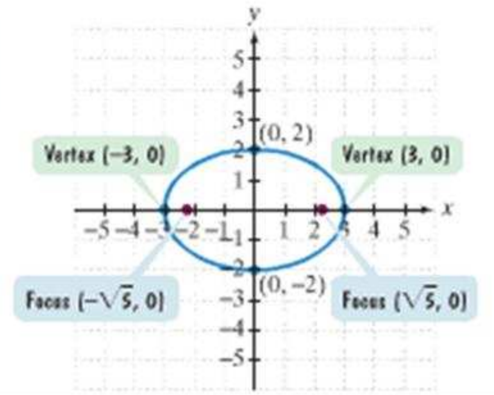
$y_2 = -y_1$.

To see the true shape of the ellipse, use the [ZOOM SQUARE] feature so that one unit on the $y$-axis is the same length as one unit on the $x$-axis.



[−5, 5, 1] by [−3, 3, 1]

### Ellipse with the Major Axis on the X-Axis




$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

$a^2 = 9$. This is the larger of the two denominators.

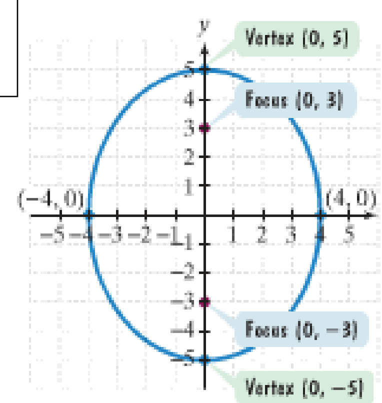
$b^2 = 4$. This is the smaller of the two denominators.

$$\frac{x^2}{3^2} + \frac{y^2}{2^2} = 1$$

Endpoints of the major axis are 3 units to the right and left of the center.

Endpoints of the minor axis are 2 units up and down from the center.

Ellipse with the Major Axis on the Y-axis




$$\frac{x^2}{16} + \frac{y^2}{25} = 1$$

$b^2 = 16$. This is the smaller of the two denominators.

$a^2 = 25$. This is the larger of the two denominators.

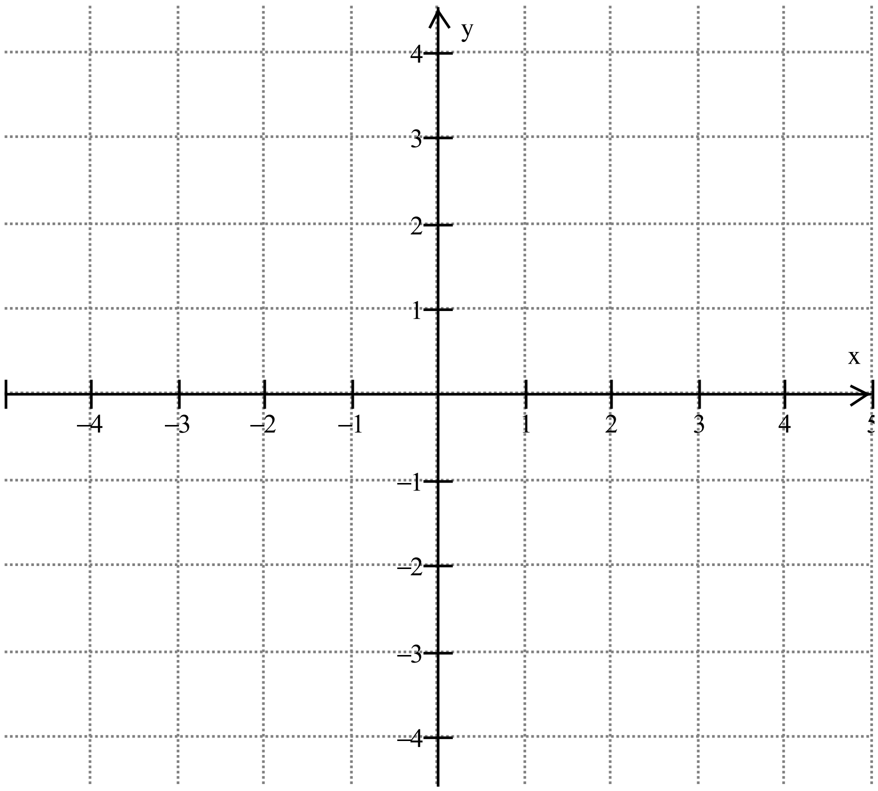
$$\frac{x^2}{4^2} + \frac{y^2}{5^2} = 1$$

Endpoints of the minor axis are 4 units to the right and left of the center.

Endpoints of the major axis are 5 units up and down from the center.

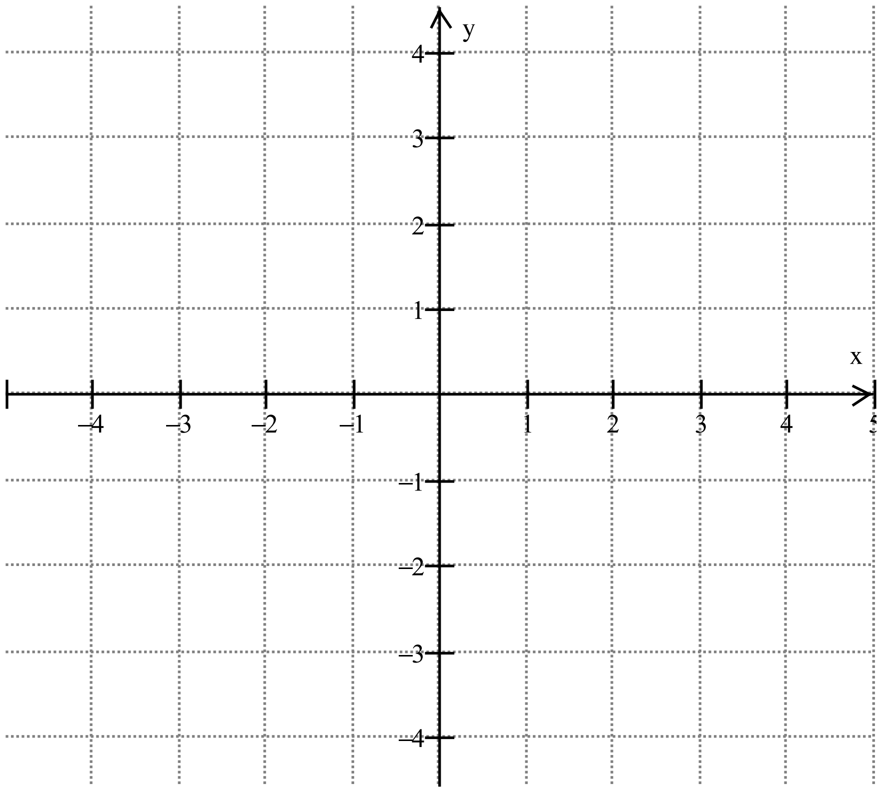
Example – Graph and locate the foci.

$9x^2 + 4y^2 = 36$

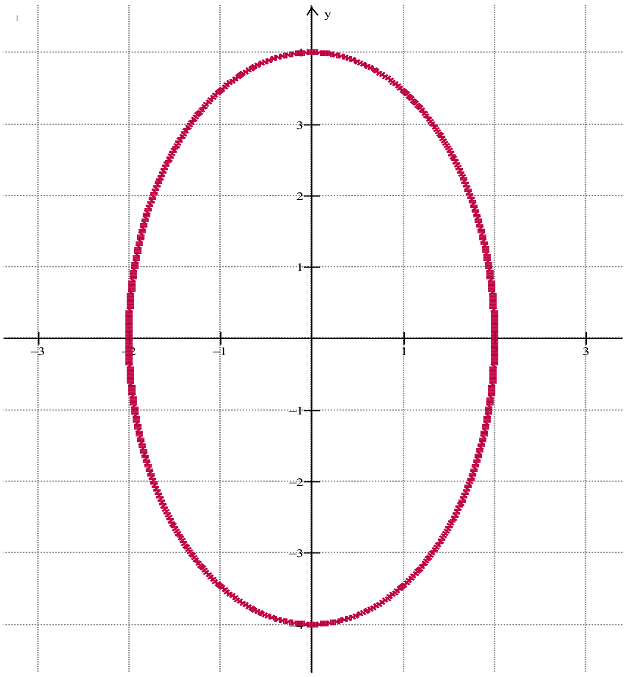


Graph and locate the foci.

Example – $\frac{x^2}{16}+\frac{y^2}{4}=1$

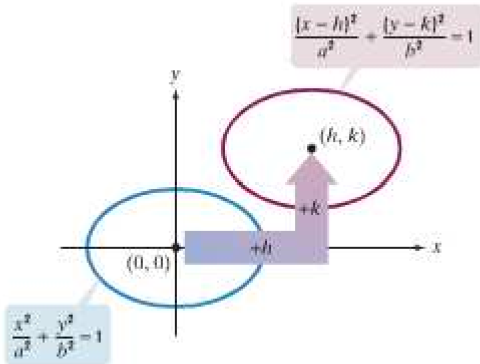


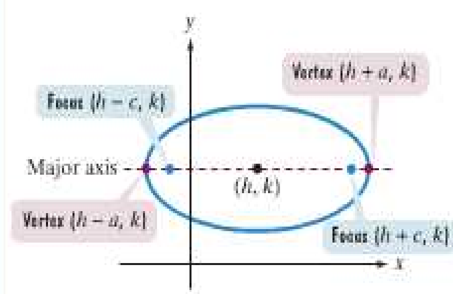
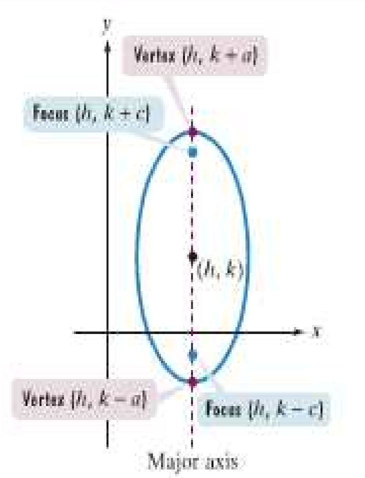
Example – Write the standard equation of the ellipse pictured.



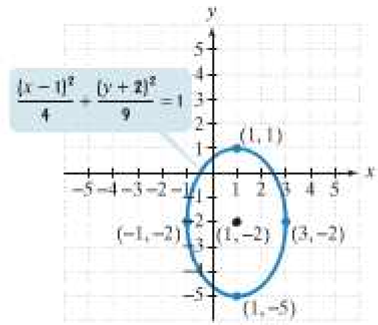
# Translations of Ellipses

# Standard form of Equations of Ellipses Centered at h,k




| Equation | Center | Major Axis | Vertices | Graph |
|---|---|---|---|---|
| $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$<br>Endpoints of major axis are $a$ units right and $a$ units left of center.<br>$a^2 > b^2$<br>Foci are $c$ units right and $c$ units left of center, where $c^2 = a^2 - b^2$. | $(h, k)$ | Parallel to the $x$-axis, horizontal | $(h-a, k)$<br>$(h+a, k)$ |  |
| $\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$<br>$a^2 > b^2$<br>Endpoints of the major axis are $a$ units above and $a$ units below the center.<br>Foci are $c$ units above and $c$ units below the center, where $c^2 = a^2 - b^2$. | $(h, k)$ | Parallel to the $y$-axis, vertical | $(h, k-a)$<br>$(h, k+a)$ |  |

# How to Use the Center to find the Endpoints of the Axes




3 units above and below center

| For a Vertical Major Axis with Center (1, −2) | |
|---|---|
| Vertices | Endpoints of Minor Axis |
| (1, −2 + 3) = (1, 1) | (1 + 2, −2) = (3, −2) |
| (1, −2 − 3) = (1, −5) | (1 − 2, −2) = (−1, −2) |

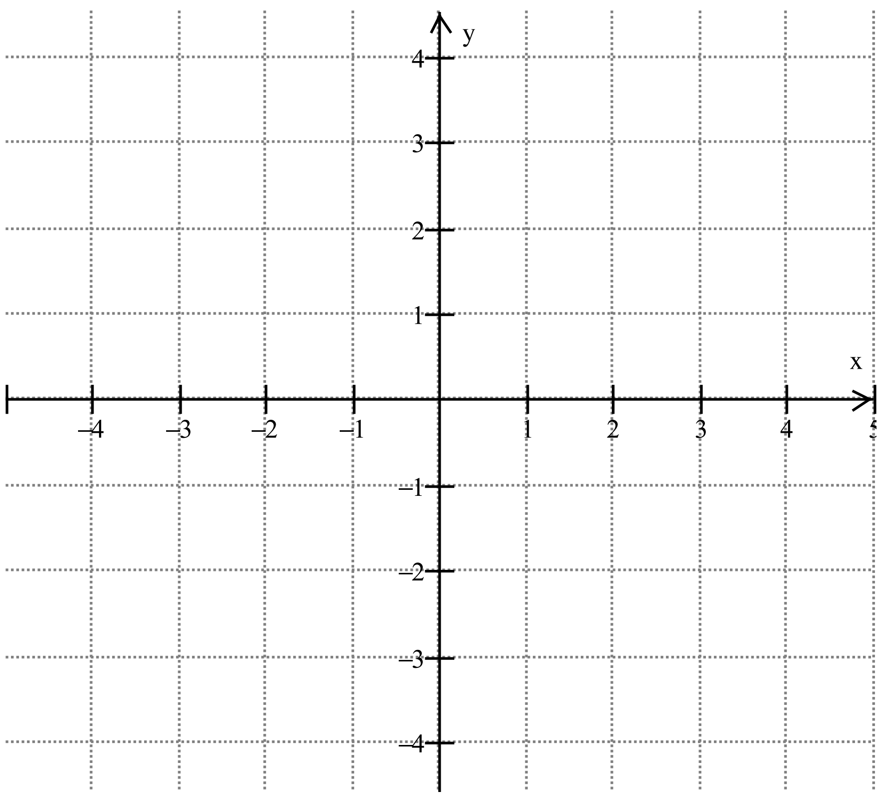
2 units right and left of center

Example – Graph $\frac{(x+2)^2}{9} + \frac{(y-1)^2}{4} = 1$

Where are the foci?

Where are the vertices, and

endpoints of the minor axis.



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**Example –** Write the equation of an ellipse given the following information.
Find the center of the ellipse, and the endpoints of the minor axis.
Vertices: (-1,-3), (-1,5)  Foci: (-1,-1),(-1,3)

## Completing the Square to find the Standard Form of the Ellipse

- $x$-terms are arranged in descending order.
- $y$-terms are arranged in descending order.
- the constant term appears on the right.

$$9x^2 + 4y^2 - 18x + 16y - 11 = 0$$ This is the given equation.

$$(9x^2 - 18x) + (4y^2 + 16y) = 11$$ Group terms and add 11 to both sides.

$$9(x^2 - 2x + \square) + 4(y^2 + 4y + \square) = 11$$ To complete the square, coefficients of $x^2$ and $y^2$ must be 1. Factor out 9 and 4, respectively.

We added 9 · 1, or 9, to the left side.
We also added 4 · 4, or 16, to the left side.

$$9(x^2 - 2x + 1) + 4(y^2 + 4y + 4) = 11 + 9 + 16$$ Complete each square by adding the square of half the coefficient of $x$ and $y$, respectively.

9 and 16, added on the left side, must also be added on the right side.

$$9(x-1)^2 + 4(y+2)^2 = 36$$ Factor.

$$\frac{9(x-1)^2}{36} + \frac{4(y+2)^2}{36} = \frac{36}{36}$$ Divide both sides by 36.

$$\frac{(x-1)^2}{4} + \frac{(y+2)^2}{9} = 1$$ Simplify.


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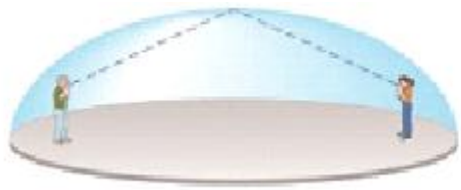
**Example –** Convert the equation into standard form by completing the square. Where is the center, the vertices, foci and the endpoints of the minor axis.

$4x^2 + 9y^2 + 16x - 18y = 11$

## Applications – Ellipses Have Many Applications

Whispering in an elliptical dome


Disintegrating kidney stones

*Example:* The Victoria and Albert restaurant in the Grand Floridian Hotel at Disney World is an example of an elliptical room. Thus if one person whispers at one focus, another person can hear what is spoken if they are seated at the other focus. If the restaurant is 40 ft long and the ceiling is 10 ft high, find the standard form of the equation of the conic section. How far from the center of the room is the focus located?