A Hexagon Result and its Generalization via Proof

Michael de Villiers[1]
University of KwaZulu-Natal, South Africa

Abstract

This paper presents the discovery of a hexagon result on Geometer's Sketchpad and its generalization via proof for any 2n-gon. The result is : If ABCDEF is a hexagon with opposite sides parallel (not necessarily equal), then the respective centroids G, H, I, J, K and L of triangles ABC, BCD, CDE, DEF, EFA and FAB, form a hexagon with opposite sides both equal and parallel.

Keywords: Discovery; Geometer's sketchpad; Hexagons; Generalization; Mathematical experimentation; 2n-gons; Proof

"*The object of mathematical rigour has been only to sanction and legitimatize the conquests of intuition.*" – Jacques Hadamard about 1900 (in Kline, 1980:318)

1. Introduction

The above quote represents a fairly common myth, namely, that mathematics is mainly a product of intuition and experimentation, and that the only role of proof is to sanction these empirical discoveries. In the majority of textbooks at high school and university, the purpose of proof in mathematics is still presented almost exclusively as that of *verification*; i.e. only as a means of obtaining certainty and to eliminate doubt. Quite often the approach followed is to allow students to experimentally discover the results, and then to try and cast a little doubt on the process of experimentation as a general means of validation. Proof is then presented as a means of "*making absolutely sure*".

However, proving is not just about making sure. Particularly, given the very high level of conviction one can nowadays obtain through many different computer programs, proof may instead serve the purpose of a logical *explanation* of *why* a certain result is true (see De Villiers, 2003). Moreover, since a proof often provides valuable insight into why a result is true, it often immediately enables one to generalise or vary the result in different ways. Usually this happens during the "*looking back*" or "*reflective*" stage of Polya's famous model of problem solving

[1] University of KwaZulu-Natal, School of Science, Mathematics & Technology Education, Edgewood Campus, Ashwood 3605, South Africa. E-mail: profmd@mweb.co.za

(Polya 1945), and illustrates what I have called a "*discovery*" function of proof that is seldom emphasised in textbooks or teaching.

The purpose of this article is to give one example of a recent problem I worked on that illustrates this "*discovery*" function very well. The example should be well within reach of talented high school or under-graduate students, and can also be a good training problem for a Mathematics Olympiad. Other examples of this "*discovery*" function are given in De Villiers (1997 & 2003). *Sketchpad 4* sketches in zipped format (Winzip) of the problem and its generalisation can be downloaded directly from: http://mysite.mweb.co.za/residents/profmd/hexcentroids.zip

2. The Problem

I was recently exploring some properties of hexagons with opposite sides parallel with the aid of *Sketchpad* and discovered the following interesting result:

If *ABCDEF* is a hexagon with opposite sides parallel (not necessarily equal), then the respective centroids *G*, *H*, *I*, *J*, *K* and *L* of triangles *ABC*, *BCD*, *CDE*, *DEF*, *EFA* and *FAB*, form a hexagon with opposite sides both equal and parallel (see Figure 1).

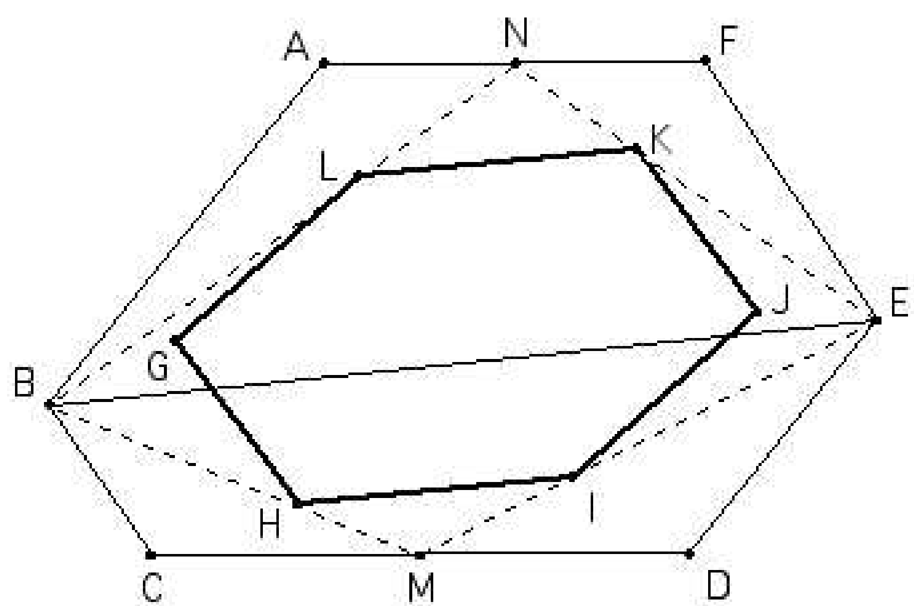

Figure 1

2.1. Proof

A problem like this may at first glance look quite challenging. Where does one start? However, a useful problem solving strategy is to find a way of relating or reducing the problem to results that are well known. One way of doing this is to start making some constructions by adding points and lines. Though this is no guarantee, and one may have to spend a little time experimenting, it allows one to get a better grip on the problem.

In this case, by drawing the diagonal *BE*, midpoints *N* and *M* respectively of *AF and CD*, and the medians *BN*, *BM*, *EN* and *EM*, a proof immediately pops out. For example, since *L* and *K* are centroids, we have $NL = \frac{1}{3}NB$ and $NK = \frac{1}{3}NE$. From a well known high school theorem, it therefore follows in triangle *NBE* that *LK* // *BE* and $LK = \frac{1}{3}BE$. Similarly, it follows that *HI* // *BE* and $HI = \frac{1}{3}BE$. Thus, *LK* is equal and parallel to *HI*. In the same way, the two other pairs of opposite sides of *GHIJKL* can be shown to be equal and parallel, and completes the proof.

2.2. Looking back

Looking back over this proof, one can immediately see that nowhere is the result dependent on *ABCDEF* having opposite sides parallel. Thus, the result immediately generalises to ANY hexagon, i.e. the centroids of ANY hexagon form a hexagon with opposite sides equal and parallel!

2.3. Comment

Unfortunately the typical textbook or classroom teacher or lecturer is likely to just present the final hexagon generalisation and its proof above, thus missing an excellent pedagogical opportunity for teaching learners or students not only the value of "*looking back*", but also that proof has a very useful "*discovery*" function.

3. Further generalization

It seems natural to ask: Can the result perhaps be generalised further to perhaps other even sided polygons, for example, octagons, decagons, etc.?

Maybe just on the basis of intuition, one will perhaps try looking and testing with dynamic geometry software whether the centroids of triangles, *ABC*, *BCD*, etc. of an octagon *ABCDEFGH* also form an octagon with opposite sides parallel and equal. And perhaps at this point, readers should pause and first try it for themselves?

Unfortunately it does not work, as the reader would've found out by checking. So is it just a case of a result that just works for hexagons, or is there more to it?

Not on the basis of first experimenting, but on the basis of my knowledge of a related theorem and its proof (theorem given further down), I immediately anticipated that the hexagon result should further generalise to an octagon *ABCDEFGH* where the centroids of the 8 quadrilaterals *ABCD*, *BCDE*, *CDEF*, etc. form an octagon with opposite sides equal and parallel (see Figure 2). So this is a far more advanced example of the "*discovery*" function of a proof where one anticipates a result on the basis of related results and proof techniques, but another example nonetheless!

In fact, the result holds generally for a $2n$-gon, $A_1A_2A_3...A_{2n}$ ($n \geq 2$), that the centroids of the n-gons, $A_1A_2A_3...A_n$, $A_2A_3A_4...A_{n+1}$, etc. sub-dividing it, form a $2n$-gon with opposite sides equal and parallel. (Note that in the trivial case of a quadrilateral, the centroids of the n-gon become the centroids of the sides, and we obtain the Varignon parallelogram. So this result is really a generalisation of the Varignon parallelogram result).

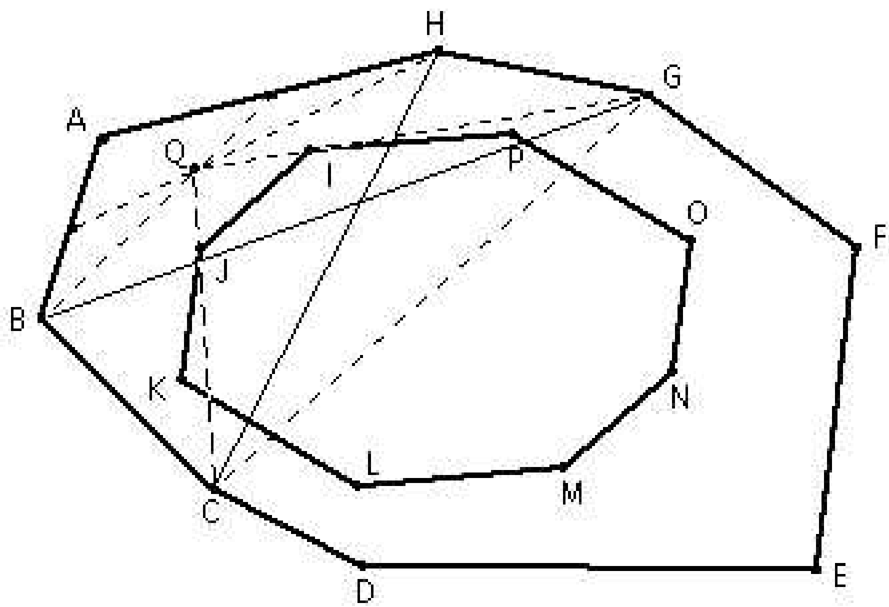

Figure 2

3.1. Proof

The general result depends on the following general theorem referred to above, and given and proved in De Villiers (1999) as well as Yaglom (1968): "Given a n-gon $A_1A_2A_3...A_n$ ($n > 3$)…, then the centroids of the (n-1)-gons, $A_1A_2A_3...A_{n-1}$, $A_2A_3A_4...A_n$, etc. that subdivide it, form a n-gon similar to the original n-gon with a scale factor of $\frac{1}{n-1}$, while the centre of similarity is the centroid of the original n-gon."

For example, for the given octagon in Figure 2, draw diagonal CG. Then from the above theorem $CJ = 3JQ$ because J is the centroid of quadrilateral $ABCH$ and Q is the centroid of triangle ABH. Similarly, $GI = 3IQ$. Thus, JI // = $\frac{1}{3}CG$. Similarly, MN //= $\frac{1}{3}$ CG. Thus, JI //= MN. In the same way, the other pairs of opposite sides can be shown to be parallel and equal. It is also obvious that in exactly the same way using the above-mentioned theorem, the result can be proved for a decagon, duodecagon, etc.

3.2. Corollary

Due to the half-turn symmetry of the "inner" $2n$-gons formed by the centroids, it also immediately follows that the diagonals connecting opposite vertices are concurrent at the centroid of the original $2n$-gon.

Though this hexagon result and its generalisation are probably not original, I've not yet seen them in the literature available to me. However, I believe this interesting result can be used in much the same way as presented here to give students some appreciation for the discovery function of proof.

More generally, this example shows that mathematics is not just discovered via experimentation (or just deduction for that matter), but often involves a symbiotic interaction between the two processes as argued in De Villiers (2004). For example, sometimes experimentation leads to new results, but proving them can sometimes lead to further avenues of research and discoveries.

References

De Villiers, M. (1997). The Role of Proof in Investigative, Computer-based Geometry: Some personal reflections. Chapter in Schattschneider, D. & King, J. *Geometry Turned On!* Washington: MAA, pp. 15-24.

De Villiers, M. (1999). Generalizations involving maltitudes. *International Journal for Mathematical Education in Science & Technology*, 30(4), pp. 541-548.

De Villiers, M. (2003) *Rethinking Proof with Sketchpad 4*. Emeryville, CA: Key Curriculum Press.

De Villiers, M. (2004). The role and function of quasi-empirical methods in mathematics, *Canadian Journal of Science, Mathematics & Technology Education*, 4(3), Nov, pp. 397-418. (A copy of this article can be downloaded from: http://mysite.mweb.co.za/residents/profmd/experiment2.pdf

Kline, M. (1980). *Mathematics: The loss of certainty.* Oxford: Oxford University Press.

Polya, G. (1945) *How to solve it.* Princeton: Princeton University Press.

Yaglom, I.M., 1968, *Geometric Transformations II*, (Washington, DC: MAA), pp.24; 108-109.

Endnotes

One of my students recently found the hexagon centroid result listed at the Wolfram MathWorld site under the heading Centroid Hexagon at http://mathworld.wolfram.com/CentroidHexagon.html , but no proof is given nor mention of any further generalization is made. However, it is possible that the one of the references at this site contains a proof of the hexagon result & perhaps even the above-mentioned generalization to any 2n-gon. See:

Cadwell, J. H.(1966). *Topics in Recreational Mathematics.* Cambridge, England: Cambridge University Press, 1966.

Wells, D. (1991). *The Penguin Dictionary of Curious and Interesting Geometry.* London: Penguin, pp. 53-54.