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TQFT for General Lie Algebras and Applications to Open 3-Manifolds

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Abstract. We review Kohno's definition of 3-manifold invariants coming from the conformal field theory associated to a simple Lie algebra g (and a level k) and extend it to a topological quantum field theory in dimension 3. As an application, some invariants at infinity of open 3-manifolds, derived from the TQFT, are considered. Explicit computations, using mapping class group representations, are performed for a series of Whitehead manifolds. An example of an uncountable family of pairwise non-homeomorphic contractible open 3-manifolds is given.

Contents

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1. Introduction

In [48] Whitehead gave the first example of a contractible open 3-manifold not homeomorphic to $\mathbf{R}^3$. There are further examples by McMillan (see [33]) which gives an uncountable family of contractible open subsets of $\mathbf{R}^3$ no two of them homeomorphic. Other recent work in [36] prove that most of such examples are not covering spaces. This is related to a well-known conjecture which states that the universal covering of a compact irreducible 3-manifold with infinite fundamental group is $\mathbf{R}^3$.

The aim of this paper is to study some invariants at infinity for open 3-manifolds arising in the framework of topological quantum field theories (see [4]). This paper is a substantially revision of [18] correcting some mistakes from the previous version (see also the revised version of [17]).

The start point was the question raised to me by Valentin Poénaru about the relationship between the behavior of Jones polynomial for the iterates of Whitehead link and the non-simply-connectedness at infinity of Whitehead's manifold. Remark that it is precisely this last property which prevents Whitehead manifolds be homeomorphic to $\mathbf{R}^3$, despite their contractibility. Some other related results showing that the skein theory could bring some new insight in the topology of open manifolds were previously obtained by Hoste and Przytycki [21].

A TQFT in dimension 3 is a functor Z from the category of (rigid) cobordisms into that of vector spaces. This means that to a surface S we associate a vector space $Z(S)$ depending only on the topological class of S. The quantum character of the theory is reflected in the rules

$$Z(\cup_i S_i) = \bigotimes_i Z(S_i),\ Z(\emptyset) = \mathbf{C},$$

which make the difference with the usual functors encountered in the algebraic topology. Finally, to each cobordism M between the surfaces S and T we have a morphism $Z(M) : Z(S) \longrightarrow Z(T)$, satisfying the natural compatibility relations between composition of cobordisms and composition of morphisms. If we are working with oriented manifolds (as it will be always the case in what follows) it will be supposed that the vector spaces are endowed with some (compatible) hermitian structures.

Examples of non-trivial TQFTs were firstly constructed by Reshetikhin and Turaev [41] (using the quantum SU(2)), by Witten ([49] (using the path integral formalism), by Kohno [27] (from the conformal field theory coming from $sl_2(\mathbf{C})$) and many others (see for a non-exhausting list [7, 10, 35, 11, 2, 44, 45, 46, 47]). Considerable efforts have been done to work out the case of a general gauge group G according to Witten's prescriptions.

We follow here Kohno's approach in the case of a general simple Lie algebra g and extend his invariants for closed 3-manifolds to a TQFT. Notice that there is a fairly general equivalence between conformal field theories and TQFTs (see [16]).

In order to preserve the self-contained character of the paper, we will give in section 2 a description of the so-called Wess-Zumino-Witten model, following the results from [42, 43]. This way the conformal blocks, braiding and fusing matrices are constructed. Further, we outline the definition of links and 3-manifold invariants and their extension to a TQFT. The two approaches, via Dehn surgeries on framed links and via representations of mapping class groups produce in fact the same invariants, extending previous results by Piunikhin [38, 37].

In section 4 we apply this formalism to open 3-manifolds. If we regard an open manifold as an infinite composition of cobordisms, a TQFT will naturally associate an inductive system of vector spaces whose limit is a topological invariant. An example was given in [17]. The results of the previous sections provide a simple expression for that invariant, in the case when the manifold has periodic ends, in terms of colored link invariants.

It is simply to check that abelian TQFT invariants do not distinguish among contractible manifolds. Nevertheless explicit computations for $g = sl_2(\mathbf{C})$ prove that the non 1-connectedness at infinity can be detected this way.

The other goal of this paper is to give concrete recipes for computations of the TQFT invariants for cobordisms. We use the method outlined in [16] via Heegaard splittings into compression bodies and the analysis of cut systems. The main data which we need is the representation of the mapping class group, which may be in fact recovered from the invariants of closed 3-manifolds. This would be an alternative to overcome the difficulties in computing colored tangle invariants. The main idea is the homogeneity of mapping class group representations: if we have a manifold M and we cut it along a surface of some genus g, then the representation of the mapping class group of genus g associated to this slice does not depend on the position of the particular surface which was chosen. Practically, this means that one may use in computations not only Heegaard splittings but arbitrary decompositions into compression bodies. Notice that for a compression body the associated morphism depends only on the conformal blocks structures so their algebraic description is straightforward.

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2. Review of the Wess-Zumino-Witten Model

(2.1) **The KZ equation**. Let g be a finite dimensional complex simple algebra and let fix once for all some positive integer k, called the level of the theory. Denote by P_+ the set of dominant weights for g and

$$P_+(k) = \{\lambda \in P_+; 0 \leq < \lambda, \theta > \leq k\},$$

where $<,>$ is the Killing form of g and θ is the maximal root, of length $\sqrt{2}$. Choose an orthonormal basis $\{X_i\}$ of g and set

$$\Omega_{st} = \sum_i \pi_s(X_i)\pi_t(X_i) \in End(V_{\lambda_1} \otimes V_{\lambda_2} \otimes ... \otimes V_{\lambda_n}),$$

where π_s, π_t stand for the operations on the s-th and t-th components of the tensor product, V_λ is the irreducible g-module of highest weight λ and $\lambda_1, \lambda_2, ..., \lambda_n \in P_+$.

The Knizhnik-Zamolodchikov (KZ) equation is by definition

$$\frac{\partial \varphi}{\partial z_i} = (k + h^*) \sum_{j \neq i} \frac{\Omega_{ij}}{z_i - z_j} \varphi,$$

$1 \leq i \leq n$, for $\varphi : X_n = \{(z) \in \mathbf{C}^n; z_i \neq z_j \text{ if } i \neq j\} \longrightarrow V_{\lambda_1} \otimes V_{\lambda_2} \otimes ... \otimes V_{\lambda_n}$, and h^* being the dual Coxeter number of g.

A solution of the KZ equation is a flat section of the integrable connection

$$\omega = \frac{1}{k+h^*} \sum_{1 \le i < j \le n} \Omega_{ij} d \log(z_i - z_j),$$

for the trivial bundle over X_n with fibre $V_{\lambda_1} \otimes V_{\lambda_2} \otimes \ldots \otimes V_{\lambda_n}$. Remark that the integrability of ω is equivalent to the infinitesimal pure braid relations [26]:

$$[\Omega_{ij}, \Omega_{kl}] = 0, \text{ if } \{i,j\} \cap \{k,l\} = \emptyset,$$

$$[\Omega_{ij} + \Omega_{jk}, \Omega_{ik}] = 0.$$

(2.2) **Vertex operators**. Let $\hat{g}$ denote the affine Lie algebra

$$\hat{g} = g \otimes \mathbf{C}[t, t^{-1}] \oplus \mathbf{C}e,$$

with central e and canonical Lie structure given by

$$[X \otimes f, Y \otimes g] = [X, Y] \otimes fg + < X, Y > Res_{t=0} g(t) df(t) e.$$

According to Kac ([22]) we can associate to $\lambda \in P_+(k)$ a unique irreducible $\hat{g}$-module H_λ (the integrable highest weight $\hat{g}$-module) on which e acts as $k \cdot 1_{H_\lambda}$, it contains V_λ and is in some sense minimally generated. The Segal-Sugawara construction produces the Virasoro operators L_n acting on H_λ

$$L_n = \frac{1}{2(k+h^*)} \sum_{m,i} \bullet X_i \otimes t^m X_i \otimes t^{n-m} \bullet,$$

where the normal ordering is given by:

$$\bullet X \otimes t^m Y \otimes t^n \bullet = \begin{cases} X \otimes t^m Y \otimes t^n, & \text{if } n < m \\ \frac{1}{2}(X \otimes t^m Y \otimes t^n + Y \otimes t^n X \otimes t^m), & \text{if } n = m \\ Y \otimes t^n X \otimes t^n, & \text{if } n > m \end{cases}$$

Further, $\{L_n\}$ generate a Virasoro algebra, satisfying the identities:

$$[L_n, L_m] = (n-m) L_{n+m} + \frac{c}{12}(n^3 - n) \delta_{n+m,0},$$

where $c = \frac{k \dim g}{k+h^*}$ is the central charge. The space H_λ decomposes as a direct sum $H_\lambda = \oplus_{d \in \mathbf{Z}_+} H_{\lambda,d}$, where $H_{\lambda,d}$ is the proper eigenspace for L_0 corresponding to the eigenvalue $\Delta_\lambda + d$, $\Delta_\lambda = \frac{<\lambda, \lambda+2\rho>}{2(k+h^*)}$ is the conformal weight, and ρ is half of the sum of positive roots. Denote by $\hat{H}_\lambda$ the direct product $\prod_d H_{\lambda,d}$, and $H = \bigoplus_{\lambda\in P_+(k)} H_\lambda$, $\hat{H} = \bigoplus_{\lambda\in P_+(k)} \hat{H}_\lambda$.

Now, a *primary field* of type $\mu \in P_+(k)$, is a family of operators $\Phi(z) : H\otimes V_\mu \longrightarrow \hat{H}$, holomorphic with respect to $z \in \mathbf{C} - \{0\}$ and satisfying

$$[L_m, \Phi(z)(X\otimes v)] = z^m \left(z\frac{\partial}{\partial z} + (m+1)\Delta_\mu \right) \Phi(z)(X\otimes v),$$

$$[Y\otimes t^m, \Phi(z)(X\otimes v)] = z^m \Phi(z)(X\otimes Yv).$$

A component of the primary field sending $H_\lambda \otimes V_\mu$ to $\hat{H}_\nu$ is called a (chiral) *vertex operator* of type $(\lambda, \mu; \nu)$ and the vector space of all such operators is denoted by $W^\nu_{\lambda\mu}$. Notice that there is a natural embedding

$$W^\nu_{\lambda\mu} \hookrightarrow Hom_g(V_\lambda \otimes V_\mu, V_\nu).$$

We will use now some current graphical rules of associating vector spaces to graphs: consider an oriented 3-valent graph whose edges are labeled. Each internal vertex has two incoming edges and one outgoing edge. We have also a natural cyclic order among the incident edges: in the planar picture is considered that the clockwise orientation of the plane induces the cyclic order around each vertex. Now, once we labeled the edges by elements of $P_+(k)$, there is a non-ambiguous way to associate to each internal vertex a vector space $W^\nu_{\lambda\mu}$, such that ν is the label of the outgoing edge and (λ, μ, ν) is cyclically ordered. We associate to the whole labeled graph the tensor product of all those spaces associated to vertices. Eventually, if the graph has some of its edges labeled, take the sum of all the spaces obtained by the above construction, over all possible labelings of the remaining edges. For example, if the edges are labeled as in Fig. 1, then the associated space is

$$W^\nu_{\lambda_1\lambda_2\ldots\lambda_n} = \bigoplus_{\mu_1,\mu_2,\ldots,\mu_{n-1}} W^{\mu_1}_{0\lambda_1} \otimes W^{\mu_2}_{\mu_1\lambda_2} \otimes \ldots \otimes W^\nu_{\mu_{n-1}\lambda_n}.$$

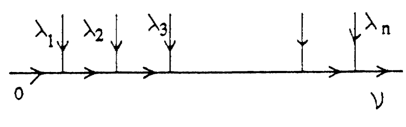

Fig. 1. An example.

Remark that these conventions make sense for an arbitrary 3-valent graph. The vector spaces constructed in this manner are called conformal blocks. We will restrict for the moment to the case when the graph is a tree, or the genus 0 case. Consider now $\Phi_j(z)$ be chiral vertex operators of type $(\mu_{j-1}, \lambda_j, \mu_j)$ where $\mu_0 = 0, \mu_n = \nu$. If $\mid 0 >\in H_0, < h_\nu \mid\in H_\nu^*$ are the highest weight vectors then the *n-point function*

$$\varphi(z_1, z_2, ..., z_n) =< h_\nu \mid \Phi_n(z_n)\Phi_{n-1}(z_{n-1})...\Phi_1(z_1) \mid 0 >$$

is a solution to the KZ equation (taking values in $W^\nu_{\lambda_1\lambda_2...\lambda_n}$). Notice that $W^\nu_{\lambda_1\lambda_2...\lambda_n}$ is naturally embedded in $Hom_g(V_{\lambda_1} \otimes V_{\lambda_2} \otimes ... \otimes V_{\lambda_n}, V_\nu)$ so the sense of "solution to the KZ equation" can be properly interpreted. Now the n-point function is holomorphic in the region $\{\mid z_1 \mid<\mid z_2 \mid< ... <\mid z_n \mid\} \subset \mathbf{C}^n$ and can be analytically continued to a multi-valued holomorphic function on X_n.

(2.3) **Monodromy of conformal blocks in genus 0**. From [43], p.467 we derive the following vanishing property for n-point functions:

$$(*) < h_\nu \mid \Phi_n(z_n)\Phi_{n-1}(z_{n-1})...(X_\theta \otimes t^{-1})^{k-<\theta,\lambda_r>+1}\Phi_r(z_r)...\Phi_1(z_1) \mid 0 > \\ = 0,$$

where $X_\theta \in g$ is corresponding to the maximal root θ. These equations translate into an algebraic systems of equations in φ, as is explicited in [42], p.331 for $g = sl_2(\mathbf{C})$. Moreover we have an intrinsic characterization of the conformal blocks:

$W^\nu_{\lambda_1\lambda_2...\lambda_n}$ *is the space of solutions of the KZ equation taking values in* $Hom_g(V_{\lambda_1} \otimes V_{\lambda_2} \otimes ... \otimes V_{\lambda_n}, V_\nu)$ *and satisfying the algebraic system (*).*

Choose now some basis $\{e^\nu_{\lambda\mu}(i)\}$ for the spaces $W^\nu_{\lambda\mu}$. A special labeling f of a tree γ (as above) consists in a labeling of its edges and the assignment of some element $e^\nu_{\lambda\mu}(i)$ to each internal vertex whose incoming edge is labeled ν and the other two by λ and μ. This way, to each tree γ and special labeling f we may associate a specified vector $C(\gamma, f) \in W^\mu_{\lambda_1\lambda_2...\lambda_n}$.

Remark that X_n is the space of configurations of n points in $\mathbf{C}$. Let us fix $n+1$ distinct points on the projective line P^1, whose coordinates satisfy $\mid z_1 \mid<\mid z_2 \mid< ... <\mid z_n \mid<\mid z_{n+1} \mid= \infty$. Consider γ be a binary tree in the complex plane, whose leaves are precisely that considered points and whose external edges $a_1, a_2, ..., a_n$, corresponding to the first n points, are out-going edges, but the last one a_{n+1} is an incoming edge. We mark the other edges of γ by the subsets of $\{1, 2, ..., n\}$ recurrently on the depth of the vertices as follows: a vertex of depth $r+1$ has two incoming edges from vertices of less depth, marked by the subsets A and B. Then the outgoing edge is marked by $A \cup B$ (see the picture 2). The convention is that $a_{n+1} = a_{12...n}$.

We consider now a sequence of blowing-ups of $\mathbf{C}^n$ whose centers are the hyperplanes $\{z_i = z_j\}$, the $(n-2)$-planes $\{z_i = z_j = z_l\}$, and so one. The several possibilities of doing this correspond to binary trees γ as above. Specifically, we can introduce the blowing-up coordinate $w_{i_1 i_2 ... i_p}$ so that the exceptional divisor $\{w_{i_1 i_2 ... i_p} = 0\}$ is the inverse image of the subspace $\{z_{i_1} = z_{i_2} = ... = z_{i_p}\}$.

Now, the residue of the connection ω along the exceptional divisors is a diagonal matrix on the subspace $W^{\nu}_{\lambda_1 \lambda_2 ... \lambda_n}$, and is given by

$$(Res_{w_{i_1 i_2 ... i_p}} \omega) C(\gamma, f) = \Delta(i_1 i_2 ... i_p) C(\gamma, f),$$

where

$$\Delta(i_1 i_2 ... i_p) = \Delta_{f(a_{i_1 i_2 ... i_p})} - \sum_{j=1}^{p} \Delta_{f(a_{i_j})},$$

in terms of the conformal edges associated to the labeling f of the edges of the graph. Remark that the residue does not depend on the choice of the labeling of internal vertices.

This implies that, for each labeled tree (γ, f), we have a solution $\phi_{(\gamma,f)}$ to the KZ equation whose expansion near the exceptional divisors has the form

$$\phi_{(\gamma,f)} = \prod w_{i_1 i_2 ... i_p}^{\Delta(i_1 i_2 ... i_p)} \left(C(\gamma, f) + \text{ higher order holomorphic terms} \right).$$

(Compare to [12]).

In the case of the two graphs depicted in the Fig. 3 we have two basis of such normalized solutions $\{\phi_{(\gamma_1, f)}\}$ and $\{\phi_{(\gamma_2, f)}\}$. By analytic continuation we find an isomorphism relating them

$$F \left[\begin{array}{cc} \lambda_2 & \lambda_3 \\ \lambda_1 & \lambda_4 \end{array} \right] : \bigoplus_{\mu} W^{\mu}_{\lambda_1 \lambda_2} \otimes W^{\lambda_4}_{\mu \lambda_3} \longrightarrow \bigoplus_{\mu} W^{\mu}_{\lambda_2 \lambda_3} \otimes W^{\lambda_4}_{\lambda_1 \mu},$$

called the *fusing isomorphism*.

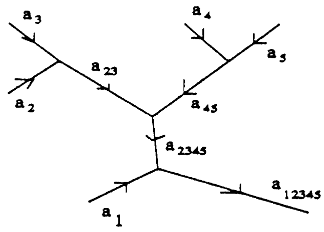

Fig. 2. The marked graph of blowing-ups.

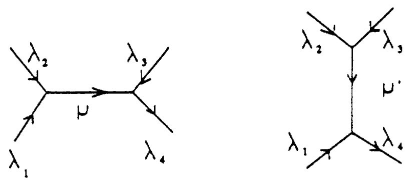

Fig. 3. The fusing graphs.

In a similar way the half monodromy can be obtained starting with the basis $\{\phi_{(\gamma_1,f)}\}$, which is defined in $\{\mid z_1 \mid<\mid z_2 \mid<\mid z_3 \mid\}$, and by analytic continuation along the curve represented by the braid σ from the Fig. 4 we get a new basis $\{\sigma^* \phi_{(\gamma_1,f)}\}$ of solutions, this time defined on $\{\mid z_1 \mid<\mid z_3 \mid<\mid z_2 \mid\}$. The last one is a matrix times the normalized solution in this region, so we derive an isomorphism

$$B\left[\begin{array}{cc}\lambda_2 & \lambda_3 \\ \lambda_1 & \lambda_4\end{array}\right]: \bigoplus_{\mu} W^{\mu}_{\lambda_1\lambda_2} \otimes W^{\lambda_4}_{\mu\lambda_3} \longrightarrow \bigoplus_{\mu} W^{\mu}_{\lambda_1\lambda_3} \otimes W^{\lambda_4}_{\lambda_2\mu},$$

called the *braiding isomorphism.*

(2.4) **The genus 1 case**. The other pieces from which the conformal field theory is made up come from the modular properties of the characters of the integrable highest weight modules. The character of H_λ defined on

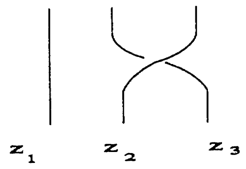

Fig. 4. The braid σ.

the upper half-plane is given by

$$\chi_\lambda(\tau) = Tr_{H_\lambda} q^{L_0 - \frac{c}{24}}, \text{ where } q = exp(2\pi\sqrt{-1}\tau),\ Im\,\tau > 0.$$

Kac [23] determined the behavior under the Möbius transformations of the plane:

$$\chi_\lambda(-\frac{1}{\tau}) = \sum_{\mu \in P_+(k)} S_{\lambda\mu}\chi_\mu(\tau),$$

$$\chi_\lambda(\tau+1) = exp(2\pi\sqrt{-1}\left(\Delta_\lambda - \frac{c}{24}\right)\chi_\lambda(\tau),$$

where

$$S_{\lambda\mu} = \alpha_{g,k} \sum_{w\in W} det(w) exp\left(-\frac{2\pi\sqrt{-1}}{k+h^*} < w(\lambda+\rho), \lambda+\rho >\right),$$

$$\alpha_{g,k} = \frac{(\sqrt{-1})^{|\Delta_+|}}{(k+h^*)^{l/2}} \left(\frac{vol(\Lambda^w)}{vol(\Lambda^r)}\right)^{\frac{1}{2}},$$

Δ_+ is the set of positive roots, l is the rank of g, Λ^w, Λ^r are the weight and coroot lattices respectively, W is the Weyl group of g, det is the usual alternate character on W, vol means volume and $|\cdot|$ the cardinal.

Usually one introduce also the diagonal matrix

$$T_{\lambda\mu} = \delta_{\lambda\mu} exp(2\pi\sqrt{-1}\left(\Delta_\lambda - \frac{c}{24}\right)).$$

Then S, T are unitary and symmetric and satisfy

$$(ST)^3 = S^2 = (\delta_{\lambda\mu^*}),$$

where $\lambda^* = -w(\lambda)$ for the longest element $w \in W$ and it is the highest weight for the dual representation V_λ^*.

This way a unitary representation of $SL_2(\mathbf{Z})$ in $W_1 = \bigoplus_{\lambda \in P_+(k)} W^0_{\lambda\lambda^*}$, which is called the 1-loop monodromy, is obtained.

How fits W_1 with the spaces associated to graphs? We consider the graph from picture 5 which is a spine for the torus. The associated space is $\bigoplus_\lambda W^\lambda_{0\lambda}$. There is an identification of $W^\lambda_{0\lambda}$ with $W^0_{\lambda\lambda^*}$ (and both are $\mathbf{C}$), however this identification is not canonical. The easiest way is to consider that S sends actually the space W_1 into $\bigoplus_\lambda W^\lambda_{0\lambda}$, which is a space isomorphic to W_1. At the same time the last space may be considered to be the space associated to the graph from picture 6, with the following convention: since there are two outgoing edges, we change the label of one outgoing edge from λ to λ^* and consider it as an incoming edge, in the picture. With this observation, the S matrix corresponds to a change in the graph. If we look at the two pants decompositions of the torus (with a little disk removed in order to make this possible) their dual graphs are both abstractly isomorphic to the graph from picture 5. However S corresponds to a homeomorphism of the torus changing one pants decompositions into the other. This explains why

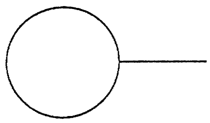

Fig. 5. A genus 1 graph.

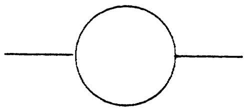

Fig. 6. Another genus 1 graph.

the conformal blocks W_1 have two non-canonically isomorphic descriptions in terms of oriented graphs. In fact the conformal blocks are associated to pants decompositions of surfaces, not only to their dual graphs. Some other related projective representations of $SL_2(\mathbf{Z})$ which complete the conformal field theory picture in genus 1 are given by the higher S-matrices, computed by Li and Yu [30]:

$$S(\nu)_{\lambda\mu} = \sum_{\xi} exp(2\pi\sqrt{-1}(\Delta_\xi - \Delta_\lambda - \Delta_\mu))S_{0\xi}B_{\lambda\mu}\left[\begin{array}{cc} \nu & \xi \\ \mu & \lambda \end{array}\right],$$

These matrices verify

$$(S(\nu)T)^3 = S(\nu)^2 = (exp(-\pi\sqrt{-1}\Delta_\lambda)\delta_{\lambda\mu^*}),$$

and furnish a projective representation into $W_{1,\nu} = \bigoplus_\lambda W^\nu_{\lambda\lambda^*}$.

(2.5) **Higher genera conformal blocks and Moore-Seiberg equations**. The main features of these complicated fusing and braiding operators are the pentagon and hexagon equations which they satisfy. Usually stated as (part of) the Moore-Seiberg equations ([34]) these equations have been discussed in a different context by Drinfeld [12, 5].

The Pentagon Condition. We have the following identity:

$$F_{\lambda_{234}\lambda_{123}}\left[\begin{array}{cc} \lambda_{23} & \lambda_4 \\ \lambda_1 & \lambda_5 \end{array}\right] F_{\lambda_{23}\lambda_{12}}\left[\begin{array}{cc} \lambda_2 & \lambda_3 \\ \lambda_1 & \lambda_{123} \end{array}\right]$$
$$= \sum_{\lambda_{34}} F_{\lambda_{23}\lambda_{34}}\left[\begin{array}{cc} \lambda_2 & \lambda_3 \\ \lambda_{234} & \lambda_4 \end{array}\right] F_{\lambda_{234}\lambda_{12}}\left[\begin{array}{cc} \lambda_2 & \lambda_{34} \\ \lambda_1 & \lambda_5 \end{array}\right] F_{\lambda_{34}\lambda_{123}}\left[\begin{array}{cc} \lambda_3 & \lambda_4 \\ \lambda_{12} & \lambda_5 \end{array}\right].$$

The proof is immediate: There are five ways to do the blowing-ups corresponding to the trees from picture 7. To each tree we have associated a basis of solutions of the KZ equation satisfying the algebraic constraints (*). Since our connection ω is integrable the parallel transport is independent on the choice of the path inside a homotopy class. When translated algebraically this is the pentagon condition.

Remark again this is a fairly particular case of the pentagon condition for the Drinfeld's associator ([12, 5]). Using the results of [5, 39, 1, 29] we can derive the equation for F by interpreting it in terms of weight systems for Vassiliev invariants.

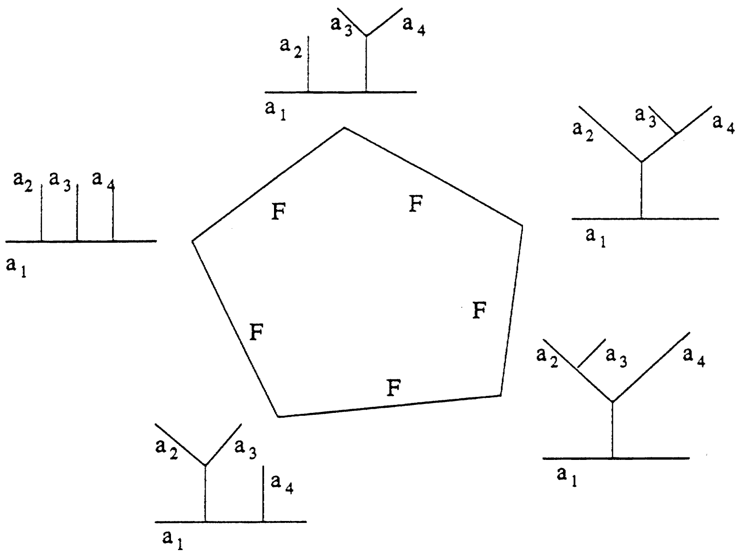

Fig. 7. The Pentagon Condition.

The Braiding-Fusing equation. We have the identities:

$$B_{\lambda\lambda_{12}}\begin{bmatrix}\lambda_2 & \lambda_{34}\\ \lambda_1 & \lambda_5\end{bmatrix} F_{\lambda_{34}\lambda_{123}}\begin{bmatrix}\lambda_3 & \lambda_4\\ \lambda_{12} & \lambda_5\end{bmatrix}$$
$$= \sum_\mu F_{\lambda_{34}\mu}\begin{bmatrix}\lambda_3 & \lambda_4\\ \lambda_1 & \lambda\end{bmatrix} B_{\lambda\lambda_{123}}\begin{bmatrix}\lambda_2 & \lambda_4\\ \mu & \lambda_5\end{bmatrix} B_{\mu\lambda_{12}}\begin{bmatrix}\lambda_2 & \lambda_3\\ \lambda_1 & \lambda_{123}\end{bmatrix}.$$

Proof. We consider the blowing-ups using the trees from picture 8. They are related by braiding and fusing transformations as marked on the picture. Again by the integrability of ω we may conclude. □

There is also an hexagonal relation equivalent to the Yang-Baxter equation in the braiding operator (see e.g. [34]).

These equations are sufficient for a rigorous construction of conformal blocks in higher genera. We start with a closed (oriented) surface Σ_g of

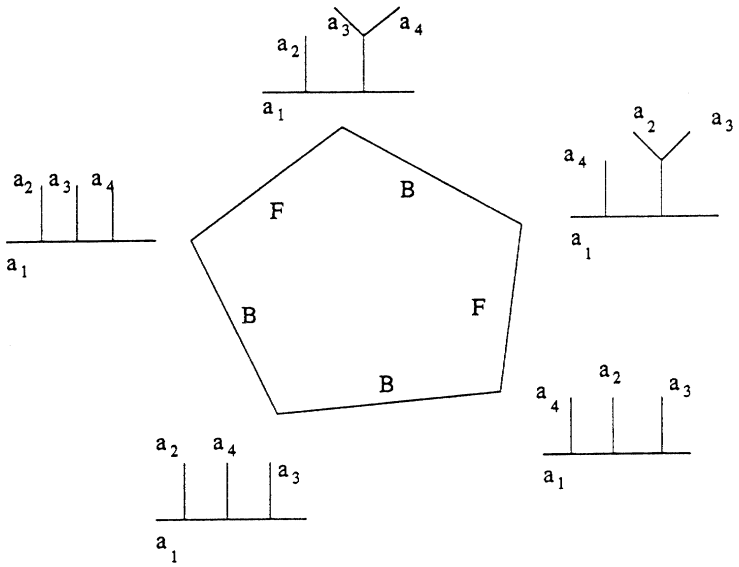

Fig. 8. The Braiding-Fusing condition.

genus $g > 1$. We fix a cut-system $c = \{c_1, c_2, ..., c_g\}$ in the sense of Hatcher and Thurston [20], to get a $2g$-holed sphere. We set

$$W(c) = \bigoplus_{\lambda_i \in P_+(k)} W^0_{\lambda_1\lambda_1^*\lambda_2\lambda_2^*...\lambda_g\lambda_g^*}.$$

Then the isomorphism class of $W(c)$ does not depend on the choice of cut system and is abstractly the conformal block W_g associated to the surface of genus g. Moreover, the different spaces $W(c)$ are related by canonical isomorphisms (see below). Notice that, in order to fix a basis of this vector space we need some more data which would be corresponding to a rigid structure on the surface (see (3.4)).

Alternatively, we can consider a pants decomposition $d = \{e_1, e_2, ..., e_{3g-3}\}$ of the surface, where the e_i are the $3g-3$ circles whose complementary consists in $2g-2$ trinions. The dual graph Γ_d associated to d is a 3-valent graph of genus g. Choose an orientation of its edges and a cyclic order around each vertex as in the case of trees. Next we associate a space, say $W(\Gamma_d)$, using the various labelings of the internal edges.

We may further extends this definition to generalized pants decompositions, where some other bounding circles may be added. The convention is that the edges incident to the leaves (corresponding to the disks bounded by these new circles) should be automatically labeled by 0. This way we have an unified description including the genus 1 situation.

It will become clear soon that all these definitions agree. We claim that:

For any two pants decompositions the associated spaces are canonically isomorphic.

In fact, it was noted by Hatcher and Thurston that two pants decompositions are obtainable one from the other by some sequence of elementary moves. The elementary moves are given in picture 9. It is only the first move which changes the dual graph of the decomposition, acting as a fusing move. Consider now Y_g be the 2-dimensional complex whose vertices correspond to 3-valent graphs of genus g, whose edges corresponds to elementary fusing in the graphs and whose 2-cells are attached on the pentagons from picture 9. By [27] the complex Y_g is connected and simply connected. There is therefore a path connecting in Y_g any two given vertices. By the Pentagon Condition the composition of the fusing operators corresponding to the elementary fusing of graphs does not depend upon the choice of the path. This way a canonical isomorphism between the associated spaces is

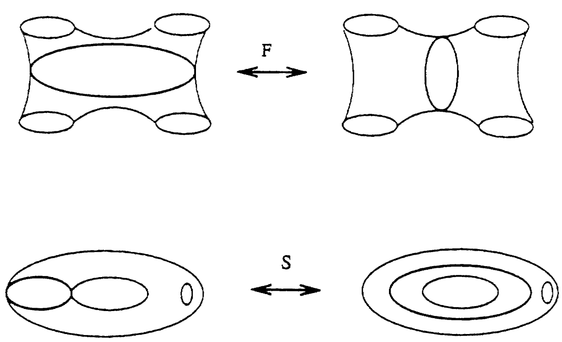

Fig. 9. Hatcher-Thurston moves.

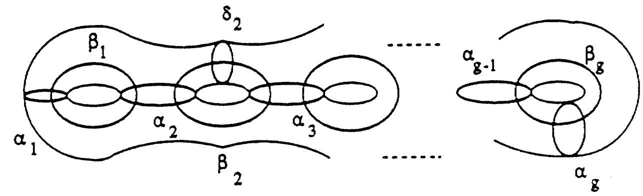

Fig. 10. Generators of $\mathcal{M}_g$.

obtained. In what concerns the introduction of bounding circles, it suffices to observe that $W^i_{0j} \cong \delta_{ij}\mathbf{C}$.

Thus we may denote by W_g the isomorphism class of $W(\Gamma_d)$, for any choice of a pants decomposition of the surface of genus g. The extension to n-holed surfaces with labeled boundaries it is immediate (see [27]).

To get a complete picture of the WZW-model we must note that W_g comes with a natural projective unitary representation of the mapping class group $\mathcal{M}_g$ in genus g. Consider $\alpha_1, \alpha_2, ..., \alpha_g, \beta_1, \beta_2, ..., \beta_g, \delta_2$ the usual generators of $\mathcal{M}_g$. They are the Dehn twists around the curves depicted in picture 10, and denoted by the same letters.

Then the following formulas provide a projective representation of the mapping class group:

$$\rho_g(\alpha_1) = T_1^{-1} \tag{1}$$

$$\begin{aligned}
&\rho_g(\alpha_l) = \\
&= T^{-1}_{i_{l-1}}(B^-_{j_{l-1}}\begin{bmatrix} i_{l-1} & i_l \\ k_{l-1} & k_l \end{bmatrix} B^-_{j_{l-1}}\begin{bmatrix} i_l & i_{l-1} \\ k_l & k_{l-1} \end{bmatrix})T^{-1}_{i_l} = \\
&= F_{j_{l-1}}\begin{bmatrix} i_{l-1} & i_l \\ k_{l-1} & k_l \end{bmatrix} T_{j_{l-1}} F^{-1}_{j_{l-1}}\begin{bmatrix} i_l & i_{l-1} \\ k_l & k_{l-1} \end{bmatrix}
\end{aligned}$$

for $l > 1$.

$$\rho_g(\beta_l) = T_{k_l} S_{k_l i_l}(j_{l-1}, j_l) T_{k_l}. \tag{2}$$

$$(3) \qquad \rho_g(\delta_2) = T_{i_2}.$$

The indices on the linear transformations tell us on which of the subspace it acts, and

$$S(j,l) = F^{-1}(\oplus_\mu S(\mu))F.$$

We used the pants decomposition whose dual graph is that from picture 11.

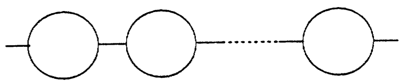

Fig. 11. Standard dual graph.

Now ρ_g is a projective representation corresponding to the signature cocycle (see [4, 7, 15]) so that:

$$\rho_g(xy) = \rho_g(x)\rho_g(y)\xi(x,y),$$

where $\xi : \mathcal{M}_g \times \mathcal{M}_g \longrightarrow R_k$ is given by

$$\xi(x,y) = exp(\pi\sqrt{-1}\frac{c}{8})^{\sigma(x,y)},$$

σ is the signature defect of the 4-manifold bounding the fibrations over the circle whose monodromies are x, y and xy, and R_k is the subgroup of roots of unity generated by $C = exp(\pi\sqrt{-1}\frac{c}{8})$.

3. Topological Invariants for Links and 3-Manifolds

We follow closely the approaches of Kohno ([27, 28]) and the extension to TQFTs from [16].

(3.1) **Colored links in S^3.** Let T be an oriented framed (m,n) tangle which is supposed to be colored, with colors from $P_+(k)$, as in Fig. 12 (see [24, 41]). We suppose the framing is the blackboard one, when drawn by a planar diagram. It is well-known that such a tangle may be decomposed into elementary tangles from the list given in picture 13. Using the fusing operators suitably normalized one define the annihilation and creation operators,

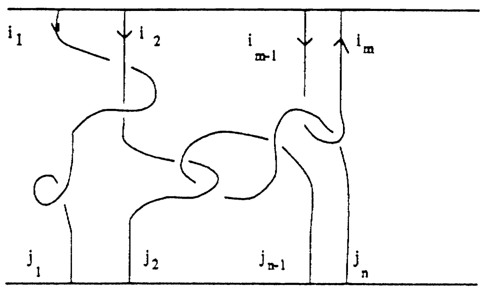

Fig. 12. A colored tangle.

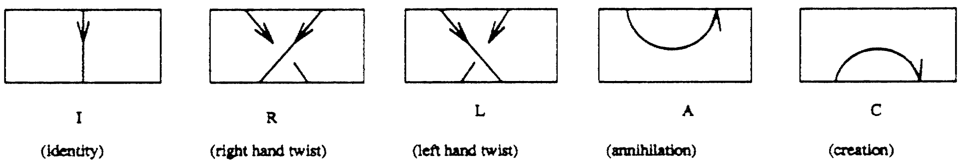

Fig. 13. The list of elementary tangles.

associated to the respective tangles as in picture 14. On the other hand, to each braiding tangle we associate the braiding operator. These assignments for elementary tangles furnish a linear map obtained by composition and tensor product from the elementary ones (according to the pattern of the tangle),

$$J(T) : W_{\lambda_1\lambda_2\ldots\lambda_m} \longrightarrow W_{\mu_1\mu_2\ldots\mu_n}.$$

The Reidemester moves for tangles leave invariant the map $J(T)$ due to the equations from the precedent section (see [27, 10, 16]). In particular, to every link L with m components and coloring $\lambda : \{1, 2, ..., m\} \longrightarrow P_+(k)$ we can associate a link invariant $J(L, \lambda) \in \mathbf{C}$. Notice that this construction may be further extended to 3-valent graphs as is done in [10, 16].

(3.2) **Closed oriented 3-manifolds: The surgical approach.** There are essentially two ways of constructing 3-manifold invariants: start-

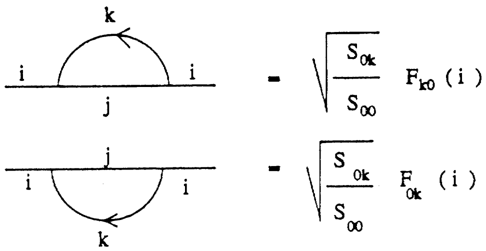

Fig. 14. The annihilation and creation operators.

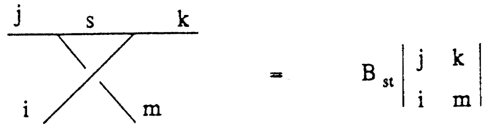

Fig. 15. The braiding operator.

ing from a presentation of the manifold M, as the result of the Dehn surgery on the link L, or else using Heegaard decompositions (there is also the Turaev-Viro method using triangulations and 6-j symbols which we don't discuss here (see [46, 44, 45])). Both methods lead to the same invariant (see [38, 37, 16]).

The first method assigns to the (oriented, closed) 3-manifold M the expression

$$Z_{g,k}(M) = C^{\sigma(L)} \sum_{\lambda} S_{0\lambda(1)} S_{0\lambda(2)} \ldots S_{0\lambda(m)} J(L, \lambda) \in \mathbf{C}/R_k,$$

where $\sigma(L)$ is the signature of the linking matrix of L, λ is a coloring of the components of L by elements of $P_+(k)$ and the sum is made over all such colorings. We remember that R_k denotes the group of roots of unity generated by C. The invariant does not depend upon the choice of the orientation of L.

There is a method to remove the phase ambiguity of the invariant by using Atiyah's framing ([3]) or a p_1-structure as in [7]. Let outline briefly how the things go on. Firstly a p_1-structure up to homotopy is the analogue of a spin structure, and it is equivalent to a 2-framing in the terminology of [3]. For the reader familiar with algebraic topology technics the exact definition is the following: set X for the homotopy fiber of the map $p_1 : BO \longrightarrow K(\mathbf{Z}, 4)$, corresponding to the first Pontryagin class of the universal stable bundle γ over the classifying space BO. Let γ_X be the pull-back of γ to X. Then a p_1-*structure* on the manifold M is a fiber map from the stable tangent bundle of M into γ_X.

Now any closed oriented 3-manifold M bounds a 4-manifold W, $\partial W = M$. If α is a p_1-structure on M, let $p_1(W, \alpha) \in H^4(W, M; \mathbf{Z})$ be the obstruction to extend it over W. Consider then

$$\sigma(\alpha) = 3\, signature(W) - < p_1(W, \alpha), [W] > \in \mathbf{Z}.$$

Then, by Hirzebruch signature theorem, $\sigma(\alpha)$ is independent on the particular choice of W, and it is 3 times Atiyah's σ from [3]. Furthermore the set of homotopy classes of p_1-structures on M is affinely isomorphic to $\mathbf{Z}$ via σ. Now the invariant from above extends to a $\mathbf{C}$-valued invariant for closed manifolds endowed with a homotopy class of a p_1-structure. When $\sigma(\alpha)$ increases by an unit, the invariant is multiplied by C. So it suffices to explain this for the case when the p_1-structure is the canonical one α, namely that satisfying $\sigma(\alpha) = 0$. Firstly the framing of L induces a p_1-structure β on $M = M_L$, starting from the canonical structure on S^3. Then the regularized invariant for M (endowed with the canonical canonical p_1-structure α) is:

$$Z_{g,k}^{p_1}(M) = C^{\sigma(L)+\sigma(\beta)-\sigma(\alpha)} \sum_{\lambda} S_{0\lambda(1)} S_{0\lambda(2)} ... S_{0\lambda(m)} J(L, \lambda) \in \mathbf{C},$$

An alternative way to see this, in the language of 2-framings, may be found in [13], where the factor $\varphi_L = \sigma(\beta) - \sigma(\alpha)$ is computed explicitly for any rational surgery presentation. Roughly speaking, fixing the 2-framing on the manifold M, which was obtained by surgery on L (explicitly, by a p_i/q_i-surgery on each component L_i of L), is the same thing as giving all entries of the gluing matrix $\begin{bmatrix} p_i & r_i \\ q_i & s_i \end{bmatrix} \in SL_2(\mathbf{Z})$. Then, from ([13], (2.7), p.93)

$$\varphi_L = -3\sigma(W_L) + \sum_i p_i,$$

where W_L is a 4-manifold bounding M, obtained by adding 2-handles on L.

Remark that $Z^{p_1}_{g,k}(M)$ is no more multiplicative under connected sum.

The proof of the invariance of $Z_{g,K}$ under the Kirby moves (which yields the topological invariance) is by now standard and we refer for more general results to [16].

(3.3) **Closed oriented 3-manifolds: Heegaard splittings**. Re member that each 3-manifold may be decomposed as the union of two handlebodies (of some genus g) glued together along their boundary surface by a homeomorphism φ and any two such decompositions become equivalent under connected sum with the standard decomposition of S^3 into two solid tori (stable equivalence).

Consider now a pants decomposition d of the surface of genus g as in (2.4). We have therefore a distinguished vector $w_g \in W(\Gamma_d)$ corresponding to the 1-dimensional subspace $W^0_{00} \otimes W^0_{00} \otimes \ldots \otimes W^0_{00}$. This vector is well defined up to a scalar by the condition that

$$w_{g+h} = w_g \otimes w_h \in W_g \otimes W_h \hookrightarrow W_{g+h}.$$

Let $w_g^* \in W_g^*$ be its dual and denote by the same letter φ the class of the homeomorphism φ in $\mathcal{M}_g$. We set

$$Z^H_{g,k}(M) = S_{00}^{-g} < w_g^*, \rho_g(\varphi) w_g > \in \mathbf{C}/R_k.$$

Notice the independence on the choice of the representative of $\rho_g(\varphi)$ in $GL(W_g)$, defined up to the scalar multiplication by an element of R_k.

The equivalence of the two definitions and *a fortiori* the topological invariance of the quantity $Z^H_{g,k}$ are proved in a more general context in [16]. We include a self-contained proof in the 4-th section.

(3.4) **Rigid cobordisms and the TQFT**. A *rigid structure* on the surface Σ of genus g consists in the choice of a parametrization

$$F : \Sigma \longrightarrow \Sigma_g,$$

into a fixed, say standard, surface of genus g, considered up to an isotopy.

Usually we choose a 3-valent graph Γ of genus g lying in the plane (as for example the graph from picture 11) and consider $\Sigma(\Gamma)$ be the boundary of the tubular neighborhood $N(\Gamma)$ of Γ in $\mathbf{R}^3$. Then a parametrization $F : \Sigma \longrightarrow \Sigma(\Gamma)$ is determined by the isotopy class of the framed graph

$F^{-1}(\Gamma) \subset \Sigma$. The framing here is a tubular neighborhood of the graph into Σ. This way we have a preferred identification of the conformal block W_g with $W(\Gamma)$.

Further, a rigid cobordism M between the surfaces Σ and Σ' (supposed to be connected for the sake of simplicity) is an oriented cobordism with $\partial M = \Sigma \cup -\Sigma'$, where the - sign denotes the opposite orientation, endowed with rigid structures on Σ and Σ', given by the graphs $\Gamma \subset \Sigma, \Gamma' \subset \Sigma'$.

In the sequel a TQFT will mean a representation Z of the category of rigid cobordisms into that of vector spaces. We shall relax also the multiplication property

$$Z(M \circ N) = Z(M) \circ Z(N)$$

(where the left hand side $\circ$ is the composition of cobordisms) to hold only up to the multiplication by a scalar from a group of roots of unity R. This is usually called a TQFT with R-anomaly (see [45]).

At the two dimensional level we set for a rigid surface:

$$Z_{g,k}(\Sigma, \Gamma) = W(\Gamma).$$

We have seen before that the isomorphism class of the vector space does not depend upon the choice of the rigid structure.

It remains to find the value of $Z(M) : W(\Gamma) \longrightarrow W(\Gamma')$ for a rigid cobordism M as above. Again there are two methods parallelizing the precedent discussion: we think at M either as the result of the Dehn surgery on a special framed link, or else as an union of two compression bodies (see [9]).

(3.5) **Surgery on special links**. A special framed link $\mathcal{L}$ consists in two framed 3-valent graphs Γ, Γ' linked together with a framed link L. Although the graphs Γ, Γ' are supposed to be planar, their framings may be not planar. Let $N(\Gamma)$ and $N(\Gamma')$ be tubular neighborhoods of the graphs in $\mathbf{R}^3$. We push the graphs along their framings to get copies of them Γ and Γ' on $\partial N(\Gamma)$ and $\partial N(\Gamma')$ respectively. We perform Dehn surgery on L away from $N(\Gamma)$ and $N(\Gamma')$ and remove further the interiors of $N(\Gamma)$ and $N(\Gamma')$. The cobordism obtained this way, together with the rigid structures induced by the graphs on the boundary, is a rigid cobordism which we call $D(\mathcal{L})$.

We may assume that:

$$\Gamma \subset \mathbf{R}^2 \times [\frac{2}{3}, 1] \cap \{u = 0\},\ \Gamma' \subset \mathbf{R}^2 \times [0, \frac{1}{3}] \cap \{u = 0\},\ L \subset \mathbf{R}^2 \times [\frac{1}{6}, \frac{5}{6}].$$

Assume also that Γ (Γ') has genus g (respectively g').

The generic projection of $\Gamma \cup \Gamma' \cup L$ onto the plane $\{u = 0\}$ is a framed tangle of type (g, g') with bottom and lower capping off which defines an usual tangle $T(\mathcal{L})$.

We identify $W(\Gamma)$ with $\bigoplus_{\lambda_i} W_{\lambda_1\lambda_1^*\lambda_2\lambda_2^*\ldots\lambda_g\lambda_g^*}$ and $W(\Gamma')$ with $\bigoplus_{\nu_i} W_{\nu_1\nu_1^*\nu_2\nu_2^*\ldots\nu_g\nu_g^*}$.

Now, for $u \in W_{\lambda_1\lambda_1^*\lambda_2\lambda_2^*\ldots\lambda_g\lambda_g^*}$, $v \in (W_{\nu_1\nu_1^*\nu_2\nu_2^*\ldots\nu_g\nu_g^*})^*$ and a coloring μ of L we have an induced coloring (λ, μ, ν) of $T(\mathcal{L})$.

We set therefore:

$$\begin{aligned}(4)\qquad & Z_{g,k}(D(\mathcal{L}))(u \otimes v) \\ & = C^{\sigma(L)} \sqrt{\prod_{i=1}^{g} S_{0\lambda_i}} \sqrt{\prod_{i=1}^{g'} S_{0\nu_i}} \sum_{\mu} \prod_{j} S_{0\mu(j)} J(T(\mathcal{L}), (\lambda, \mu, \nu))(u \otimes v).\end{aligned}$$

Written in this form $Z_{g,k}(D(\mathcal{L}))$ is an element of $W(\Gamma) \otimes W(\Gamma')^*$.

(3.6) **Compression bodies**. There is an analogue for Heegaard decompositions for cobordisms, replacing handlebodies by compression bodies. A *compression body* B is the result of attaching 2-handles to a thickened surface along some circles which are bounding in the corresponding handlebody.

Assume that the compression body B is obtained from $\Sigma \times [0, 1]$ and the attaching circles are $c_1, c_2, ..., c_s$. There exist generalized pants decompositions d of the surface Σ (considered as one component of the boundary of B) such that $d \supset \{c_1, c_2, ..., c_s\}$. We pick one such decomposition d. Therefore, there is a canonical way to transport this decomposition to the other boundary of B: the circles of d which are not among the c_i's are pushed using the local product product structure of B in a neighborhood of them. Next, around each attaching circle we have two parallel copies of it bounding the attaching area. So a generalized decomposition which we denote by $d(B)$ is obtained on $\partial B - \Sigma$. At the dual graph level $\Gamma_{d(B)}$ is obtained from Γ_d by making some cuts (hence introducing new leaves which will be therefore labeled automatically by 0) on the edges corresponding to the attaching circles.

On the other hand we have a natural embedding

$$W(\Gamma_{d(B)}) \hookrightarrow W(\Gamma_d),$$

since the right-hand space is the same sum of spaces but over a larger set of labelings. We are ready now to define the TQFT morphisms for compression bodies. Remark before that Γ_d and $\Gamma_{d(B)}$ induce rigid structures on the corresponding surfaces, and denote by β the rigid compression body they define. Consider that β is a cobordism from Σ to the other boundary.

We put

$$Z^H_{g,k}(\beta) : W(\Gamma_d) \longrightarrow W(\Gamma_{d(B)}),$$

be the canonical projection.

Consider further a cobordism $M = B \cup_\varphi -B'$ which splits into two compression bodies β and β' glued together by a homeomorphism $\varphi : \Sigma \longrightarrow \Sigma$. We choose some pants decomposition d and d' of Σ so that they contain the attaching circles of their respective compression bodies and $d' = \varphi(d)$. This can always be done by changing φ by left and right multiplication by some homeomorphisms extending to the handlebody. We have therefore a rigid structure on M given by $\Gamma_{d(B)}$ and $\Gamma_{d'(B')}$, which we denote by $\mathcal{M}$.

We set finally

$$Z^H_{g,k}(\mathcal{M}) = Z^H_{g,k}(\beta) \circ \rho_g(\varphi) \circ Z^H_{g,k}(\beta')^*.$$

If we wish to compute $Z^H_{g,k}$ using another rigid structures on the boundaries it suffices to note that the mapping class group acts transitively on the set of rigid structures. The change of a rigid structure (on the left or right) by ψ amounts to compose (on the left or right respectively) by $\rho_*(\psi)$, with $*$ the appropriate genus.

Now we are able to state the main result of the first part:

Theorem 3.1. *The two formulas for $Z_{g,k}$ and $Z^H_{g,k}$ are equivalent and define a TQFT in dimension 3 with R_k-anomaly.*

Proof. We divide the proof into several steps:

(3.7) **Passing from Heegaard splittings to Dehn surgeries**. We prove first that the two definitions for $Z_{g,k}$ are equivalent. Before to proceed we need a method to convert gluings of 3-manifolds along surfaces into operations on special links, via Dehn surgery.

Consider $\varphi \in \mathcal{M}_g$ and let $\tilde{\varphi}$ be a lift of φ as a homeomorphism of Σ_g, whose mapping cylinder is denoted by $cyl(\tilde{\varphi})$. Choose a pants decomposition d of Σ_g inducing the rigid structure Γ_d on $\Sigma_g \times \{0\}$. Further, $\tilde{\varphi}(d)$ is a pants decomposition giving the rigid structure $\Gamma_{\tilde{\varphi}} \subset \Sigma_g \times \{1\}$ and we have a rigid cobordism $C(\tilde{\varphi}) = (cyl(\tilde{\varphi}), \Gamma_d, \Gamma_{\tilde{\varphi}})$. Choose for simplicity Γ_d be that from the Fig. 11. It is simply to check that $C(1) = D(L(1))$, where $L(1)$ is the special link drawn in picture 16.

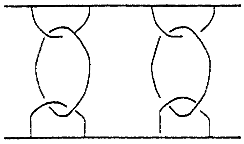

Fig. 16. $L(1)$.

LEMMA 3.1.1. *We have $C(\tilde{\varphi}) = D(L(\varphi))$, for each φ in the set $G = \{\alpha_1, ..., \alpha_g, \beta_1, ..., \beta_g, \delta_2\}$ of usual generators of $\mathcal{M}_g$, where the various $L(\varphi)$ are those from picture 17.*

The proof is straightforward.

Now we can use the fact that mapping cylinders may be composed in the obvious way. We write

$$\varphi = x_1 x_2 ... x_p, \text{ with } x_j \text{ or } x_j^{-1} \text{ in } G.$$

We make the assignments from picture 18 and consider them as framed tangles.

LEMMA 3.2. *We have $C(\tilde{\varphi}) = D(L(\varphi))$ where $L(\varphi)$ is constructed explicitly in picture 19, in terms of the expansion of φ.*

The proof follows from the functoriality of mapping cylinder's composition.

(3.8) **The unitary projective representation revisited**. We derive from 3.3 a mapping $Z : \mathcal{M}_g \longrightarrow End(W_g)$ given by $\varphi \longrightarrow Z_{g,k}(C(\tilde{\varphi}))$.

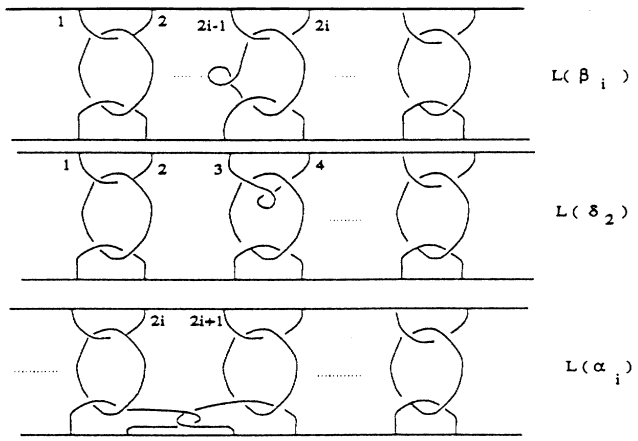

Fig. 17. $L(\varphi)$ for the generating Dehn twists.

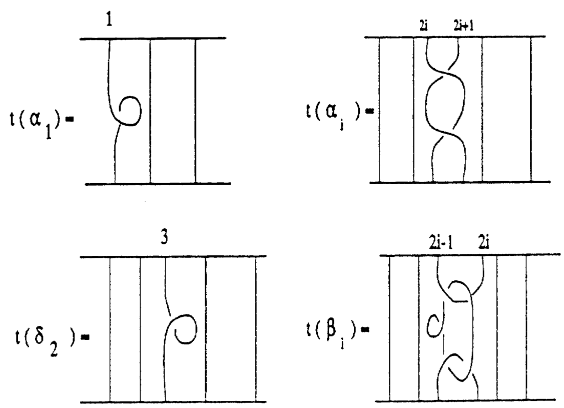

Fig. 18. The basic framed tangles.

LEMMA 3.3. *The map Z is an unitary projective representation of $\mathcal{M}_g$ whose associated cocycle is ξ (see 2.5) and coincide with the representation ρ_g.*

PROOF. Consider two lifts $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ of φ. We have two special framed

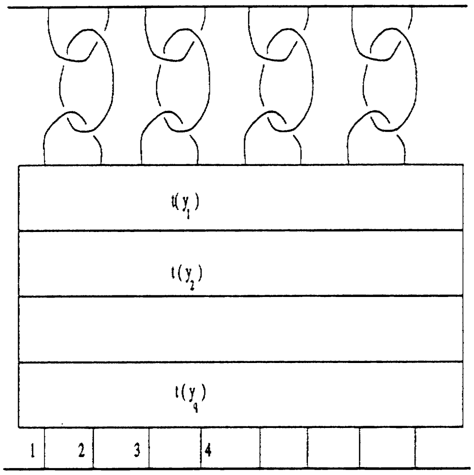

Fig. 19. $L(\varphi)$.

links associated L_1 and L_2 which yield the same cobordism by Dehn surgery. By the cobordism version of Kirby's theorem there is a sequence of Kirby moves which transforms L_1 into L_2. But $Z_{g,k}$ is invariant to Kirby moves, and this proves that the map Z is in fact well-defined on $\mathcal{M}_g$.

In order to get the functoriality we need to know the behavior of J for connected sum of links.

SUBLEMMA 3.4. *Assume that $L_1\sharp L_2$ denotes the connected sum of the colored links (L_1, μ_1) and (L_2, μ_2) obtained by using only one component from each link colored in both by λ, and $\mu_1\sharp\mu_2$ is the resulted coloring. Then the following holds:*

$$J(L_1\sharp L_2, \mu_1\sharp\mu_2) = \frac{S_{00}}{S_{0\lambda}} J(L_1, \mu_1) J(L_2, \mu_2).$$

The proof is a standard computation.

Now, using this sublemma and computing the terms in the expression of $Z(\varphi_1\varphi_2)$, we find

$$Z(\varphi_1\varphi_2) = \xi(\varphi_1, \varphi_2) Z(\varphi_1) Z(\varphi_2).$$

We used the fact that $\sigma(x, y)$ can be expressed in terms of the linking matrices of the links associated to the tangles corresponding to xy, x and y. Specifically,

$$\sigma(x, y) = \sigma(L(xy)) - \sigma(L(x)) - \sigma(L(y)),$$

holds, where the σ on the right hand side is the signature of the linking matrices (see also [35]).

Eventually, since ρ_g and Z are both multiplicative, it suffices to prove that they agree on the generators. But ρ_g's definition is exactly $J(L())$ in the corresponding basis from 3.1.

Remark that in [43] the authors constructed the conformal blocks W_g not only locally, but also as a family of sheaves over the moduli spaces of curves. These sheaves are holomorphic and they support some natural projectively flat connections. The rules allowing to recover a conformal block in genus g in terms of the conformal blocks associated to the 3 punctured sphere permit to recover also the monodromy representations associated to these flat connections, in terms of some building blocks. These building blocks are the braiding, fusing, S and T matrices. By example, this splitting procedure is carried out for TQFT's in [16]. The explicit formulas in (1-3) identify the representations ρ_g with the monodromy representation. But the last one is a unitary representations because the considered sheaves are holomorphic. Therefore ρ_g is unitary. Remark that for $g = 1$ it is already clear from the definition (see [22, 23]) of S and T. The case of the punctured sphere and the associated braid group representations was studied explicitly in [42] for $sl_2(\mathbf{C})$. Otherwise, the unitarity might be checked directly on the generators of the mapping class groups. For α_i and δ_j it is known in conformal field theory that there is a normalization of chiral vertex operators such that braiding and fusing are unitary. For β_i this follows from $\rho_g(\beta_i^{-1}) = \rho_g(\beta_i)^*$. We don't enter in the details because we don't make use of the unitarity in the sequel. This ends the proof of the lemma. $\square$

As a side remark, the unitarity of the theory is the main ingredient for the TQFT could be used to compute approximations of the Heegaard genus of closed 3-manifold (see [19]) and tunnel numbers of knots (see [28]).

COROLLARY 3.5. *For closed 3-manifolds the two definitions of $Z_{g,k}$ agree.*

(3.9) **Compression bodies**. It follows now that $Z_{g,k}$ is an invariant for rigid cobordisms by a standard argument using Kirby moves for cobordisms. Since gluing two cobordisms along a common boundary is equivalent to insert the mapping cylinder of the gluing homeomorphism we find that

$$Z_{g,k}(M \cup_{\varphi} N) = Z_{g,k}(M) \circ \rho_g(\varphi) \circ Z_{g,k}(N),$$

so $Z_{g,k}$ is a TQFT in the sense of Atiyah.

From the previous lemma it suffices now to find the values of $Z_{g,k}$ and $Z^H_{g,k}$ for compression bodies in order to conclude. A link presentation of a compression body B has the form depicted in Fig. 20. This implies that $Z_{g,k}(B) = Z^H_{g,k}(B)$ for any compression body, hence from 3.15, it will hold for all cobordisms. This ends the proof of the theorem 3.1. $\square$

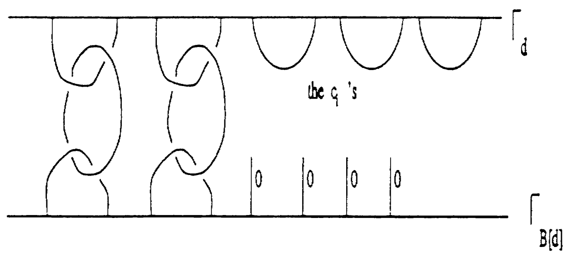

Fig. 20. Special link for a compression body.

4. Open 3-Manifolds

(4.1) **General TQFT invariants at infinity**. Let Z be an anomaly free TQFT in dimension 3 coming from a conformal field theory, as for example $Z_{g,k}$ for particular values of the central charge. There is a general result that (almost) all TQFT come from conformal field theories (see [16]) but we don't enter in the details here. Also, by a recent result of Sawin, it suffices to restrict our attention to the study of those unitary TQFT which arise this way, because all of the topological information is carried by them. Let W be an open oriented 3-manifold without boundary. We choose an ascending sequence of compact submanifolds $\{K_n\}$ fulfilling $K_j \subset int(K_{j+1})$, for all j, and $W = \cup_n K_n$.

Then $V_i = cl(K_{i+1} - K_i)$ are oriented cobordisms from ∂K_i to ∂K_{i+1}. We may choose arbitrary rigid structures on ∂K_i making V_i rigid cobordisms. We derive this way a sequence of morphisms

$$Z(V_i) : Z(\partial K_i) \longrightarrow Z(\partial K_{i+1}),$$

which form an inductive system of vector spaces. We set therefore $Z_\infty(W)$ for the limit of this inductive system.

For each n we have a distinguished vector $Z(K_n) \in Z(\partial K_n)$. If π_n denotes the canonical projection $Z(\partial K_n) \longrightarrow Z_\infty(W)$ we set also

$$Z_f(W) = \pi_n(Z(K_n)) \in Z_\infty(W).$$

It is easy to see that $Z_f(W)$ is independent of n, and both $Z_f(W)$ and $Z_\infty(W)$ do not depend on the choices of intermediary rigid structures.

Definition-Lemma 4.1. *The assignment $W \to (Z_f(W) \in Z_\infty(W))$ is functorial and topologically invariant i.e. any homeomorphism $h : W \longrightarrow W'$ induces an action $h_* : Z_\infty(W) \longrightarrow Z_\infty(W')$ which is an isomorphism of vector spaces so that*

$$h_*(Z_f(W)) = Z_f(W').$$

Proof. Any two exhaustions by compact submanifolds K_n and K'_n as above have sub-families which are mutually disjoint and further have a common refinement inducing an isomorphism at the limit level. □

Remarks 4.2.

- The vector space $Z_\infty(W)$ depends only on the structure at infinity of W: if W' is another manifold so that $W - U$ and $W' - U'$ are homeomorphic for some open subsets U, U' having compact closure, then $Z_\infty(W)$ and $Z_\infty(W')$ are isomorphic.
- By contrast $Z_f(W)$ encodes topological information at finite distance. However it is useless to distinguish non-homeomorphic open 3-

manifolds having the same structure at infinity. There is an interesting exception: we replace W by a new manifold $W' = (W - K) \cup K'$ by doing modifications at finite distance, with K and K' are compact. We wish to test whether the new manifold W' is homeomorphic to W by a homeomorphism h satisfying:

(*) h is isotopic to the identity when viewed as a self map of the complementary of a sufficiently large compact.

This condition makes sense since W and W' coincide at long distance. But now (*) implies that $h_* = 1$ so a necessary condition is that $Z_f(W') = Z_f(W)$, which may be effectively tested.

- If the functor Z is a TQFT with an anomaly in $R \subset U(1)$, R being some group of roots of unity, then the morphisms $Z(V_i)$ are defined up to scalar multiplication by a root of unity. Nevertheless the vector space Z_∞ is well-defined and the vector $Z_f(W)$ is precised up to an element of R. This is the case for all examples of quantum group invariants $Z_{g,k}$ from above. So we can proceed to computations without making difference between anomaly free and TQFTs with anomaly.

Definition 4.3. The open 3-manifold is homology 1-connected at infinity (abbrev. h-1-connected) if for each compact $K \subset W$ there exists a compact submanifold $Y \subset W$ whose interior contains K and $H_1(Y) = 0$.

This is a slightly weaker condition than the 1-connectedness at infinity. The interest is that this condition can be tested using the TQFTs. We say that a TQFT is reduced provided that $Z(S^2) \cong \mathbf{C}$. This is the case for all $Z_{g,k}$ and in fact for the most of TQFTs (see [16]).

Proposition 4.4. *1) If W is homeomorphic to $\mathbf{R}^3$ then $Z_\infty(W) \cong \mathbf{C}$ for any reduced TQFT.*

2) If W is h-1-connected at infinity then $\dim Z_\infty(W) \leq 1$ *for all reduced TQFTs.*

Proof. The first part is immediate. For the second one remark that a compact 3-manifold Y with $H_1(Y) = 0$ has the boundary an union of spheres S^2, from an Euler characteristic argument. Choose now an exhaustion K_n of

W. Then each K_n may be engulfed in some compact submanifold Y_n whose boundary is an union of spheres. By compactness each Y_n is contained in some $K_{r(n)}$, with $r(n) >> n$. Then the morphism $Z(cl(K_{r(n)} - K_n)$ factors through $Z(\partial Y_n) \cong \mathbf{C}$ and we are done. □

(4.2) **Manifolds with periodic ends**. Assume now that the open 3-manifold has periodic ends i.e. it has an exhaustion K_n as above such that the associated cobordisms V_n are pairwise homeomorphic for n greater than some n_0. Then the inductive system associated has a very simple form: it consists of iterates of the linear map $Z(V_n)$. For a linear map A we set $R(A) = \bigcup_{n>0} \ker A^n$.

PROPOSITION 4.5. *For an open 3-manifold W with periodic ends we have*

$$Z_\infty(W) \cong Z(\partial K_n)/R(Z(V_n)), \text{ for } n >> 0.$$

The proof follows from the fact that $\lim_\to(C^m, A) = C^m/R(A)$.

Suppose now that all cobordisms V_n belong to a finite set of cobordisms $C = \{C_1, C_2, ..., C_q\}$. The order in which these are composed corresponds to some $r \in (0,1)$ written in base q and the open manifold we get we denote by $W(r, C)$. We may assume that r is not rational because this way a manifold with periodic ends is obtained, and further that each C_i appears infinitely many times, otherwise we could restrict to a smaller C. If $A_i = Z(C_i) : \mathbf{C}^{m_i} \longrightarrow \mathbf{C}^{n_i}$, let $B_i : \mathbf{C}^n \longrightarrow \mathbf{C}^n$ be obtained from A_i by boarding it with zeroes for some n greater than all n_i, m_i.

PROPOSITION 4.6. *We have the surjections*

$$\mathbf{C}^n/\sum_{i=1}^q \ker B_i \longrightarrow Z_\infty(W(r,C)) \longrightarrow \mathbf{C}^n/\sum_{i=1}^q R(B_i)$$

The proof follows from:

$$\lim_{\rightarrow}(\mathbf{C}^m, A_i) = \mathbf{C}^m/\bigcap_{i>0}\bigcup_{p\geq 0} \ker(A_iA_{i+1}...A_{i+p}).$$

The corollary of this computation is that there exist infinitely many distinct r for which $Z_\infty(W(r,C))$ are the same, for all g, k. Among these, and for specific C we can find uncountably many pairwise non-homeomorphic open manifolds as in [36, 8]. In conclusion we may find many non-homeomorphic open 3-manifolds with the same quantum invariants at infinity.

However, there exists another interesting application: to test whether the open 3-manifold may be tamely compactified i.e. there exists a closed 3-manifold M and a finite simplicial complex Y which is PL immersed in M, such that W is homeomorphic to $M - Y$. In such situations Y has an unique (up to isotopy) embedded regular neighborhood $N_M(Y)$ in M whose closure is a compact manifold with boundary.

PROPOSITION 4.7. *If W may be tamely compactified then $Z_\infty(W) \cong Z(\partial cl(N_M(Y))$. Under this isomorphism $Z_f(W)$ corresponds to $Z(M - N_M(Y))$.*

In fact we have a sequence of homeomorphic regular neighborhoods of Y in M approaching Y and the associated cobordisms are trivial.

Example 4.8. Consider $B_{3,2}$ be the compression body obtained from a surface of genus 2 by adding one 1-handle. Choose a homeomorphism φ of Σ_3 whose action on $H_1(\Sigma_3)$ be given by the symplectic matrix

$$\begin{pmatrix} 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & -2 & 0 & 0 & 0 \end{pmatrix}.$$

Consider now the cobordism $C = B_{3,2} \cup_\varphi -B_{3,2}$. We claim that an open 3-manifold with periodic ends modeled on C whose gluing maps between consecutive boundary surfaces are from the Torelli group, could not be tamely compactified.

We check this by computing $Z(C)$ for the abelian TQFT from [14, 35], in the case when the level $k = 6$ (the algebra g is trivial). Therefore

$$W_g \cong \mathbf{C} < e_m; m \in (\mathbf{Z}/6\mathbf{Z})^g >,$$

and the inclusion $W_g \hookrightarrow W_{g+1}$ is the natural one. On the other hand the mapping class group representation ρ_g factors through the symplectic group, and was computed in [14]. We derive that the matrix of $Z(C)$ is

$$Z(C) : W_2 \longrightarrow W_2,\ Z(C)e_m = e_{Am},\ \text{where } A = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}.$$

It follows that $R(Z(C)) = \ker Z(C)^2 = \mathbf{C} < e_{00}, e_{30}, e_{03}, e_{33} >$. The previous results imply that

$$\dim Z_\infty(W) = 32$$

for any open 3-manifold W whose ends are modeled on C like in the statement.

On the other hand if W would be tamely compactified we must have $Z_\infty(W) \cong Z($ union of surfaces$)$. But $\dim W_g = 6^g$ hence $\dim Z_\infty(W)$ should be a power of 6, which is false by the above computation.

(4.3) **Whitehead manifolds of genus 1**. We start with a homotopically trivial knot $K \subset T_0$ in the standard handlebody of genus 1 denoted by T_0, knot which is trivial in S^3. Then a regular neighborhood of K, say $T_1 = N_{T_0}(K)$, is another solid torus and there exists an homeomorphism h of S^3 so that $h(T_1) = T_0$. We put $Wh(K) = \bigcup_{n\geq 0} h^n(T_0)$. This manifold is contractible since K is homotopically trivial. If the winding number of K is non-trivial (see [48, 36, 8]) then $Wh(K)$ is not homeomorphic to $\mathbf{R}^3$ since it fails to be 1-connected at infinity. The original example of Whitehead ([48]) is for K chosen as in picture 21. Now $Wh(K)$ is an open 3-manifold with periodic ends, and the associated cobordism is $V = cl(T_0 - T_1)$. By

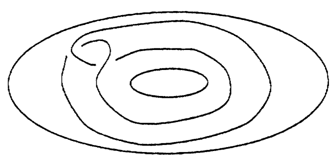

Fig. 21. Whitehead link.

the results of the previous sections

$$Z_{g,k}(\partial T_0) \cong \mathbf{C}^{P_+(k)},$$

and it remains to compute the morphism $Z_{g,k}(V)$.

Let U be the unknotted circle in $S^3 - T_0$ corresponding to the rotation axis of the torus and $K^*(\lambda, \mu)$ be the link K^* with two components K and U colored by λ and μ respectively.

PROPOSITION 4.9. *The morphism $Z_{g,k}$ in the canonical basis has the matrix*

$$A = (J(K(\lambda, \mu))_{\lambda,\mu}.$$

PROOF. We can find a simple data for the surgery on a special link yielding V: consider K^* viewed as a tangle in plane as may be seen on picture 22. Then the claim follows from 3.3. □

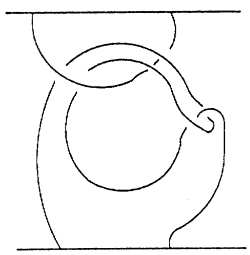

Fig. 22. Special link presentation for V_n.

Now the meaning of $\dim Z_\infty(W) \leq 1$ may be translated in terms of quantum invariants of the link K^*. The first entry on the left column of the matrix A is always 1, hence we must have $\dim Z_\infty(W) = 1$. Now the matrix A^n corresponds to the n^{th} iterate of the cobordism V. This is the same to iterate n times the link K. We denote this iterated link by K_n. We set $K_n^* = K_n \cup U$ be the link with two components constructed as above. There exists some n so that $\dim coker(A^n) = 1$. Since we may remove components colored by 0 in the computations of J an alternative condition is

$$J(K_n^*(\lambda,\mu)) = J(K_n,\lambda)J(U,\mu) = J(U,\lambda)J(U,\mu),$$

for all colors $\lambda,\mu \in P_+(k)$. Notice that for $g = sl_2(\mathbf{C})$ this implies that some values of Jones polynomial for K_n^* are trivial, so:

COROLLARY 4.10. *A necessary condition that $Wh(K)$ be h-1-connected at infinity is that for each k there is some n so that the Jones polynomial of K_n^* evaluated at k-th roots of unity be trivial.*

(4.6) **General open contractible 3-manifolds**. In a more general setting each irreducible open contractible 3-manifold is an ascending union (see [32]) of handlebodies H_n of genera g_n so that $H_n \subset int(H_{n+1})$, and the map $\pi_1 H_n \longrightarrow \pi_1 H_{n+1}$ is zero. The manifold is considered irreducible, in order to avoid the Poincaré conjecture. Now we have a similar procedure computing $Z(cl(H_{n+1}-H_n))$. Since handlebody embeddings are completely described by their spines (as knotted 3-valent graphs) one can figure out a special link presentation for $V_n = cl(H_{n+1} - H_n)$ as in Fig. 23. The graph Γ_+ is the spine of $S^3 - H_{n+1}$ and Γ_- is the spine of H_n. Now the invariants $Z_{g,k}$ canonically extend to 3-valent graphs and singular tangles (see [10, 16]). It is simply to identify now

$$Z(V_n) \text{ and } \bigoplus_{\lambda,\mu} J(\Gamma_+ \cup \Gamma_-(\lambda,\mu)),$$

where λ,μ are the colorings of the 3-valent graphs.

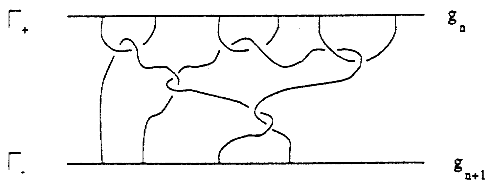

Fig. 23. The special link.

5. Some Examples

(5.1) **Statements and definitions**. The genus at infinity of an open 3-manifold W is the least $n = g(W)$ so that there exists some compact exhaustion of the manifold by submanifolds whose boundary have genus n. Remark now that for any open manifold W we have

$$\dim Z_\infty(W) \leq \dim Z(\Sigma_{g(W)}).$$

In particular arbitrary infinite connected sums of compact manifolds (using an arbitrary infinite graph) have trivial invariants at infinity. A natural question is whether some other examples exist. We will provide such an example whose genus at infinity is 1 but all its invariants at infinity are trivial.

Consider $S^2 \times S^1 \sharp S^2 \times S^1$ written as the union of two handlebodies H_- and H_+ of genus 2. Choose the solid tori $A \subset H_-$ and $B \subset H_+$ like in the picture 24 and form the manifold with boundary $X = S^2 \times S^1 \sharp S^2 \times S^1 - (int(A) \cup int(B))$. One may think at X as being a cobordism between the two boundary tori ∂A and ∂B. We consider the open manifold U obtained by iterating the cobordism X. Specifically,

$$U = A \cup X \cup X \cup ...,$$

where the gluing maps between consecutive boundaries of X are arbitrary.

Theorem 5.1. *The manifold U has all invariants at infinity trivial but its genus at infinity is 1 for some choices of the gluing maps.*

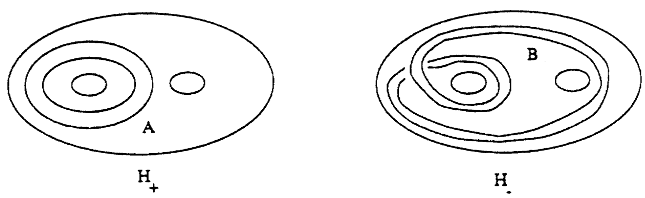

Fig. 24. The embedded tori.

So the most optimistic expectations would be that for contractible open 3-manifolds the invariants at infinity could determine the genus at infinity. We are seeking now for concrete examples, in order to see that these invariants are, in general, not trivial. We are able to do that in the case of the classical Whitehead manifold Wh for $sl_2(\mathbf{C})$ theories in level 2 and 3. More generally we may consider the twisted Whitehead manifolds $Wh(n,k)$ obtained from the link $L(n,k)$ (see the picture 31). We can state:

THEOREM 5.2. *The $sl_2(\mathbf{C})$ invariants for $Wh(n,k)$ are trivial in level 2. They are not trivial in level 3 provided that $n \neq -1 (mod\ 5)$.*

The rest of this section is devoted to the proof of these two results.

(5.2) **Proof of Theorem 1**. Let's compute first the invariant at infinity associated to X and the simplest non-abelian TQFT based on $sl_2(\mathbf{C})$ and in level 2. This gives an idea about how things go on.

LEMMA 5.3. *The associated cobordism X is $B_0 \cup B_1$, where B_i are compression bodies: B_0 is obtained from a torus by attaching a 1-handle, and B_1 is obtained from the genus 2 surface by attaching a 2-handle on the circle c figured in picture 25. The compression bodies are glued along their genus 2 boundary surfaces via the identity.*

We must include now c in a pants decomposition from which the projection associated to B_1 by the method of (3.6) may be read. On the other hand, we have the pants decomposition associated to B_0 and we have to

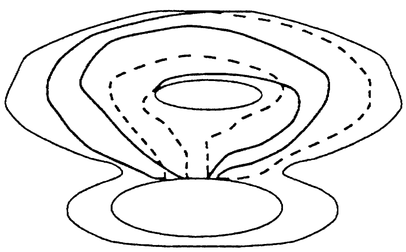

Fig. 25. The attaching circle.

identify the homeomorphism of the surface of genus 2 carrying one into the other. It is simply to check that the required homeomorphism is the Dehn twist T_γ around the curve γ from picture 26. We need now to compute the Dehn twist T_γ in terms of standard generators of $\mathcal{M}_2$.

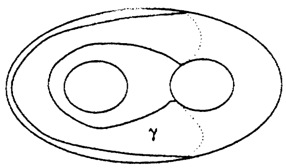

Fig. 26. The curve γ.

LEMMA 5.4. *We have the formula*

$$\begin{aligned} T_\gamma &= \alpha_1\beta_1\alpha_2\beta_1^{-1}\alpha_1^{-2}\beta_1^{-1}\alpha_2^{-1}\alpha_1^{-1}\beta_1\alpha_1^2\beta_1\alpha_1^{-1}\delta_2^{-2}\alpha_2^{-2}\beta_1^{-1}\alpha_1^{-1} \\ &= \alpha_1\beta_1\alpha_2\alpha_1^2\beta_1\alpha_1^2\beta_1^2\alpha_1^2\beta_1\alpha_1^2\alpha_2^{-1}\beta_1^{-1}\alpha_1^{-1} \end{aligned}$$

PROOF. One may see on the Fig. 27 that

$$\alpha_2^{-1}\beta_1^{-1}\alpha_1^{-1}(\gamma) = v.$$

Also it is well-known that the Dehn twist around the curve $h(\lambda)$, for an arbitrary homeomorphism h, is the conjugate of the Dehn twist around λ by h, i.e. $T_{h(\lambda)} = hT_\lambda h^{-1}$. Then, using the previous formula, we obtain T_γ in terms of T_v.

In order to compute T_v one can use the lantern relation (applied on the 4-holed sphere obtained from the 2-torus by removing the handles) which yields

$$T_v = T_w^{-1}\alpha_1^{-2}\delta_2^{-2}\alpha_2^{-1},$$

where w is figured in 28. Finally, one can check that $w = \beta_1^{-1}\alpha_1^{-1}\alpha_2\beta_1(\alpha_1)$ (see again the picture 28). From these three relations we derive the first equality of the lemma. For the second one we could use the relations in

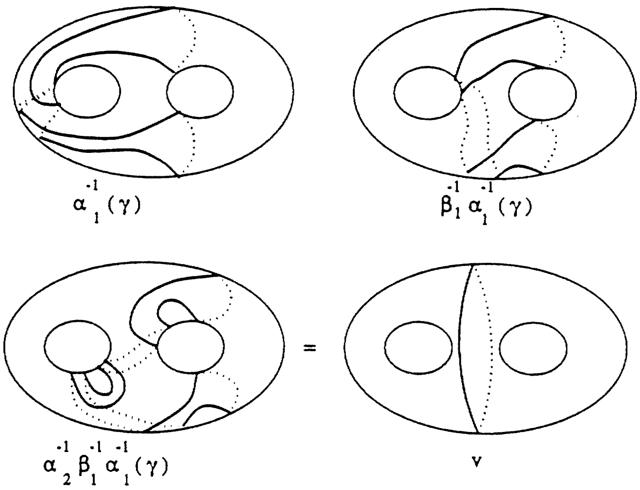

Fig. 27. Computing T_γ.

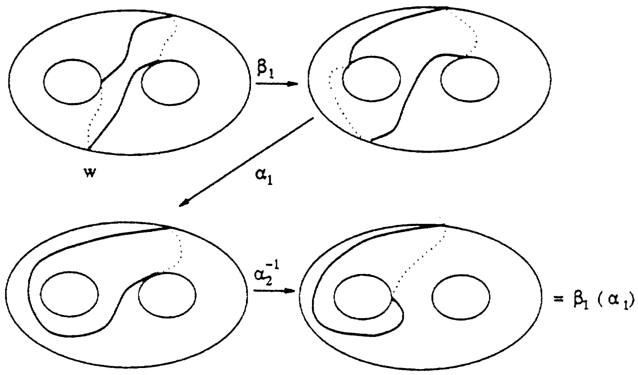

Fig. 28. Computation of T_v.

$\mathcal{M}_2$ (see [6]) to simplify the word we obtained in the generators. Otherwise, making use of the formula given by Lickorish in [31], p.772 (Lemma 3), we find that

$$T_v = \alpha_1^2\beta_1\alpha_1^2\beta_1^2\alpha_1^2\beta_1\alpha_1^2$$

and this ends the proof of the lemma. □

(5.3) **Numerical processing**. Let us write now the representation of $\mathcal{M}_2$ given by the formulas (1-3) from section 2.

Recall from [27] that $P_+(2) = \{0, \frac{1}{2}, 1\}$ and the non-trivial spaces of interwinners are

$$W_{00}^0 = \mathbf{C}f_0,\ W_{\frac{1}{2}\frac{1}{2}}^0 = \mathbf{C}f_{\frac{1}{2}},\ W_{11}^0 = \mathbf{C}f_1,\ W_{1\frac{1}{2}}^{\frac{1}{2}} = \mathbf{C}e,$$

and those obtained by permutations of indices, the others being zero.

The dual graphs corresponding to the pants decompositions are δ_1 and δ_2, hence the associated conformal blocks are

$$W_1 = W(\delta_1) = \mathbf{C}f_0 \oplus \mathbf{C}f_{\frac{1}{2}} \oplus \mathbf{C}f_1 = H,$$

$$W_2 = W(\delta_2) = H \otimes H \oplus \mathbf{C}e \otimes e.$$

The only nontrivial fusing matrices are:

$$\left(F_{ij}\left[\begin{array}{cc} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{array}\right]\right)_{ij=0,1} = \frac{1}{\sqrt{2}}\left(\begin{array}{cc} -1 & 1 \\ 1 & 1 \end{array}\right).$$

Further $S(0) : \mathbf{C}f_0 \oplus \mathbf{C}f_{\frac{1}{2}} \oplus \mathbf{C}f_1 \longrightarrow \mathbf{C}f_0 \oplus \mathbf{C}f_{\frac{1}{2}} \oplus \mathbf{C}f_1$ is given by

$$S(0) = \frac{1}{2}\left(\begin{array}{ccc} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{array}\right)$$

and $S(1) : \mathbf{C}e \otimes e \longrightarrow \mathbf{C}e \otimes e$ is the multiplication by $exp\left(\frac{-3\pi\sqrt{-1}}{4}\right)$.

Now the pants decompositions change like in the picture 29. The first two lines are coming from the trivial inclusion of a torus in the in-boundary of the compression body B_0, composed with several rearrangements of the cut system (the S and F transformations respectively). The compression body B_1 induces the projection map from W_2 onto a subspace isomorphic to W_1 and the relative position of the last one is completely determined by the dual graphs inclusion. The two cut systems on Σ_2 which are compatible with B_0 and B_1 respectively are related by the homeomorphism T_γ. So now we need only to compute the matrix associated to this Dehn twist. We remark

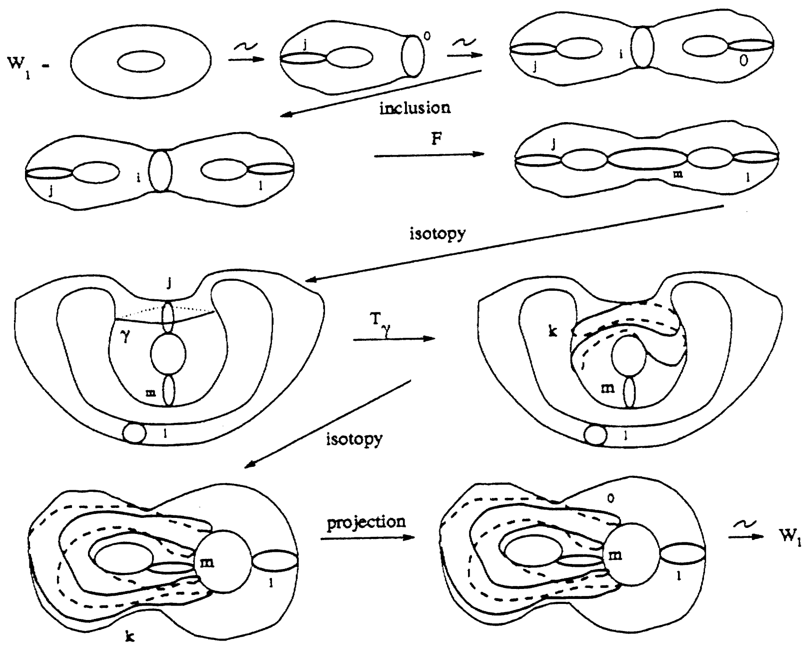

Fig. 29. The sequence of pants decompositions.

first that, once we fixed a cut system (and so a dual graph), we have already an explicit description of the conformal blocks. Since in many cases all spaces of interwinners are 1-dimensional we have an explicit description of a basis for the conformal blocks. We will freely use in this sequel the term *basis* for the decompositions in one dimensional spaces which we obtain. However, we must notice that, actually, a basis should mean actually more. In fact, we need to have also a specific embedding of the dual graph corresponding to the rigid structure of the surface (see (3.4)). However, the ambiguities of left and right multiplication by unitary matrices do not affect the invariants at infinity. If we follow the changes of dual graphs corresponding to the pants decompositions we get the picture 30. In particular the projection given by B_1 (in the vector spaces basis corresponding to the pictured cut systems) is the projection of W_2 onto the subspace spanned by the three vectors $f_j \otimes f_j$. The last one is identified to W_1 via the isomorphism $f_j \otimes f_j \to f_j$. The (projective up to a 16-th root of unity factor) representation

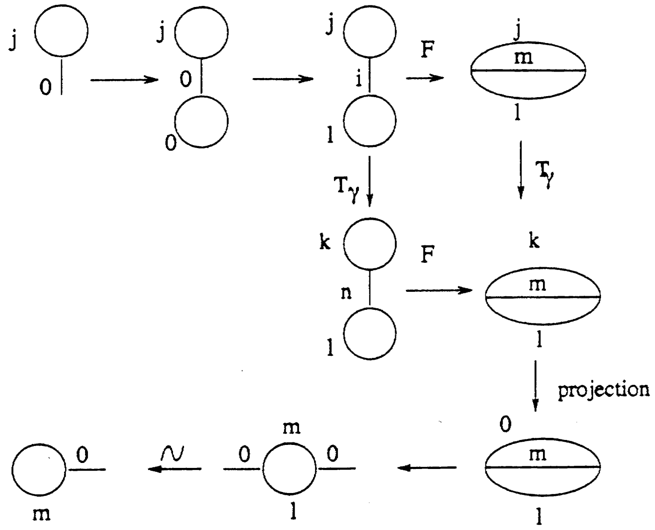

Fig. 30. How dual graphs change.

of $\mathcal{M}_2$ was explicitly stated by Kohno [27]: let $x = exp\left(\frac{3\pi\sqrt{-1}}{8}\right)$, and $T = diag(1, x, -1)$, $U = TST$ (notice that the correct U is

$$U = \frac{1}{2}\begin{pmatrix} 1 & x\sqrt{2} & -1 \\ x\sqrt{2} & 0 & x\sqrt{2} \\ -1 & x\sqrt{2} & 1 \end{pmatrix}$$

correcting a misprint from [27]). The cut system giving the preferred basis is assumed to be $\{\alpha_1, v, \delta_2\}$. Therefore the following

$$\rho_2(\alpha_1) = (T^{-1} \otimes 1) \oplus x^{-1},$$

$$\rho_2(\beta_1) = (U \otimes 1) \oplus \sqrt{-1}x,$$

$$\rho_2(\delta_2) = (1 \otimes T^{-1}) \oplus x^{-1},$$

$$\rho_2(\beta_2) = (1 \otimes U) \oplus \sqrt{-1}x,$$

$$\rho_2(\alpha_2) = M \oplus N,$$

hold, where $M = diag(1, x, -1, x)$, N is the 6-by-6 matrix obtained from $diag(x, -1, x, 1)$ by adding one entry on the left lower case and one entry on the right bottom case, both equal to -1, and then boarding by zeroes (the matrix written in [27] is misprinted, is missing one column). In this arrangement of line and columns the 5-th and 10-th lines and columns are corresponding to $f_{\frac{1}{2}} \otimes f_{\frac{1}{2}}$ and $e \otimes e$ respectively. Therefore one may compute T_γ in this basis. Observe first that

$$T_v = \mathbf{1}_{W_1 \otimes W_1} \oplus (-1),$$

and thus

$$\alpha_2 T_v \alpha_2^{-1} = \mathbf{1}_{\mathbf{C} < f_j \otimes f_k, (j,k) \neq (\frac{1}{2}, \frac{1}{2}) >} \oplus (-1)_{\mathbf{C} f_{\frac{1}{2}} \otimes f_{\frac{1}{2}}} \oplus (1).$$

It results that T_γ has $W_1 \otimes W_1$ as an invariant subspace, and

$$T_\gamma \mid_{W_1 \otimes W_1} (f_j \otimes f_k) = \begin{cases} f_0 \otimes f_{\frac{1}{2}} & \text{if} \qquad (j,k) = (1, \frac{1}{2}) \\ f_1 \otimes f_{\frac{1}{2}} & \text{if} \qquad (j,k) = (0, \frac{1}{2}) \\ f_j \otimes f_k & \text{elsewhere} \end{cases}.$$

Now the basis which we need is induced (see picture 29) from the cut system $\{\alpha_1, \alpha_2, \delta_2\}$. The change of basis (from the previous one to the actual) is a fusion F which acts trivially, except for the subspace spanned by $f_{\frac{1}{2}} \otimes f_{\frac{1}{2}}$ and $e \otimes e$, where it has the entries given above. So T_γ has the matrix FAF^{-1} in this new basis. The composition which gives $Z(V)$ is therefore

$$\begin{aligned} \mathbf{C}^3 &\cong \mathbf{C} < f_j \otimes f_o; j = 0, \frac{1}{2}, 1 > \\ &\subset W_2 \xrightarrow{FA} W_2 \xrightarrow{projection} \mathbf{C} < f_j \otimes f_j; j = 0, \frac{1}{2}, 1 > \cong \mathbf{C}^3 \end{aligned}$$

One notice that F acts trivially means that the matrix coefficients are trivial but it could interchange some one dimensional subspaces of W_2. For example $f_k \otimes f_0$ spans the source vector space for the fusing block $F \begin{bmatrix} k & 0 \\ k & 0 \end{bmatrix}$ so its image under the fusing F is the vector $f_k \otimes f_k$ (spanning the target vector space). Remark however that the label of the edge (in the dual graph of the pants decomposition) which transforms into 0 by the projection is k. In order to see this we remark that F and T_γ commutes in the following sense:

T_γ is the same linear application but is computed in two basis related by a fusion. Now T_γ (as shown before) leaves the labeled graph corresponding to $f_j \otimes f_0$ invariant. If it is followed by a fusing, which should necessarily be $F\begin{bmatrix} j & 0 \\ j & 0 \end{bmatrix}$, it transforms into the graph whose labels are $k = l = j$ and $m = 0$. This may lead to confusions since the corresponding space is already spanned by $f_j \otimes f_j$, but in another basis. So, the projection identifies k with 0 since it is the label of the curve which is killed by adding a handle (not the other label already 0). Therefore, the only non-trivial component of the projection is given by $k = 0$. Now it is easy to see that in the basis we choosed, (and up to a 16-th root of unity) the morphism associated is

$$Z_{sl_2(\mathbf{C}),2}(X) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We derive that

$$(Z_{sl_2(\mathbf{C}),2})_\infty(U) \cong \mathbf{C}.$$

One may generalize a little bit: consider the open manifolds $X(n,k)$ obtained from the knotted solid torus $L(n,k)$ from picture 31, replacing $B \subset H_-$, which we regard embedded in the genus 2 handlebody by sliding the right hand strand through the new 1-handle. Here $n \neq 0$ and k are integers.

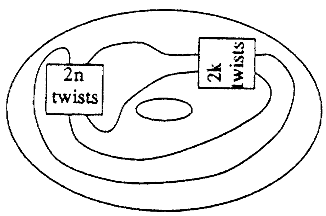

Fig. 31. $L(n,k)$.

LEMMA 5.5. *The cobordism associated to the manifold $X(n,k)$ has an Heegaard splitting $B_0 \cup B_1$ where B_0 is obtained from the torus by adding an 1-handle, and B_1 is obtained by adding a 2-handle on the circle $\alpha_1^k T_\gamma^n(\alpha_1)$.*

PROOF. It suffices to see that the spine of the outer boundary of B_1 is obtained from the standard Whitehead knot by twisting it using T_γ^n and α_1^k and this way we get $L(n,k)$. $\square$

Let $X_{n,k}$ denote the intermediary cobordism corresponding to the complement of a tubular neighborhood of $A \cup L(n,k)$ in $S^2 \times S^1 \sharp S^2 \times S^1$. We derive as above the matrix of the TQFT morphism be:

$$Z_{sl_2(\mathbf{C}),2}(X_{n,k}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

As an immediate corollary all the manifolds $U(n,k)$, which are obtained from the cobordisms $X_{n,k}$ in place of X, have trivial invariants at infinity for the TQFT $Z_{sl_2(\mathbf{C}),2}$.

(5.4) **The general case**. The general case follows the pattern from above. We consider an arbitrary Lie TQFT, or more generally a TQFT based on the RCFT (see [16]). The sequence of maps we are computing is therefore the same as in the picture 29. As we observed before one may invert the order of T_γ and the fusing F. This is useful since the matrix of T_γ is easier to compute using the cut system $\{\alpha_1, v, \delta_2\}$. Since at the target $k = 0$ (and consequently $m = l$) we need the fusing $F \begin{bmatrix} 0 & m \\ 0 & m \end{bmatrix}$ at the precedent stage. This forces $n = 0$; also $F \begin{bmatrix} 0 & n \\ 0 & n \end{bmatrix}$ is trivial since it is an 1-dimensional isomorphism.

We denote by f_j the vector spanning W_{jj}^0. Therefore W_1 is spanned by the f_j's. As before the space $W_1 \otimes W_1 \cong \bigoplus_{j,k} W_{jj}^0 \otimes W_{kk}^0 \subset W_2$. Denote by π the projection of W_2 onto $W_{00}^0 \otimes W_1$. From above it suffice to compute the values of πT_γ on $W_1 \otimes W_1$.

The representation of $\mathcal{M}_2$ (in the particular basis of W_2 induced by the cut system $\{\alpha_1, v, \delta_2\}$) is simply to describe:

$$\alpha_1(f_u \otimes f_v) = \lambda_u f_u \otimes f_v,$$

$$\beta_1(f_u \otimes f_v) = U_{us} f_s \otimes f_v,$$

where $U = T^{-1}ST^{-1}$, and $T = diag(\lambda_u)_u$. The Dehn twist T_v is also diagonal and

$$T_v \mid W^r_{jj} \otimes W^r_{kk} = \lambda_r \mathbf{1}.$$

Further α_2 is acting diagonally too in the basis induced by the cut system $\{\alpha_1, \alpha_2, \delta_2\}$. The last basis is obtained from the first one by performing a fusing F. Observe also that α_1 and β_1 split with respect to $W_1 \otimes W_1$. In fact

$$\alpha_1(W^r_{jj} \otimes W^r_{kk}) = W^r_{jj} \otimes W^r_{kk},$$

$$\beta_1(W^r_{jj} \otimes W^r_{kk}) = W^r_{jj} \otimes W^r_{kk}.$$

Therefore

$$\pi T_\gamma(f_u \otimes f_v) = \pi \alpha_1 \beta_1 \tilde{\pi} \alpha_2 T_v \alpha_2^{-1} \beta_1^{-1} \alpha_1^{-1} (f_u \otimes f_v)$$

where $\tilde{\pi}$ is the projection of W_2 onto $W_1 \otimes W_1$.

It is simple to check now that $\tilde{\pi}\alpha_2 T_v \alpha_2^{-1}$ acts diagonally:

$$\tilde{\pi}\alpha_2 T_v \alpha_2^{-1}(f_u \otimes f_v) = \Gamma_{uv} f_u \otimes f_v$$

where

$$\begin{aligned} \Gamma_{uv} &= \sum_{k,n,m} \lambda_k \lambda_n \lambda_m^{-1} F_{0m} \begin{bmatrix} u & v \\ u & v \end{bmatrix} F^{-1}_{mk} \begin{bmatrix} u & v \\ u & v \end{bmatrix} \\ & \quad \cdot F_{kn} \begin{bmatrix} u & v \\ u & v \end{bmatrix} F^{-1}_{n0} \begin{bmatrix} u & v \\ u & v \end{bmatrix}. \end{aligned}$$

We obtain

$$\pi T_\gamma(f_u \otimes f_v) = \lambda_0 \lambda_u^{-1} \sum_s U^{-1}_{us} U_{s0} \Gamma_{sv} f_0 \otimes f_v,$$

which permits to compute $Z(X)$. We take the basis f_j for W_1 at one boundary of V and identify the other basis to $f_0 \otimes f_j$ (according to our precedent discussion, this is equivalent via a trivial fusing to the natural basis of the other boundary torus). Therefore we have

$$Z(X) f_j = \pi T_\gamma(f_j \otimes f_0) = \lambda_0 \lambda_j^{-1} (\sum_s U^{-1}_{js} U_{s0} \Gamma_{s0}) f_0 \otimes f_0$$

Observe that $\Gamma_{0u} = 0$ since all fusings $F\begin{bmatrix} 0 & v \\ 0 & v \end{bmatrix}$ are trivial since they are 1-dimensional isomorphisms. We obtain

$$Z(X)f_j = \delta_{0j} f_0 \otimes f_0.$$

Furthermore

$$Z_\infty(U) \cong \mathbf{C}.$$

Now it is simple to check that the genus at infinity of U is precisely 1 for some particular gluing maps, as for the reversing orientation. In fact no embedded sphere can separate A and an arbitrary far boundary of X in this case. This ends the proof of theorem 1.

(5.5) **Proof of Theorem 2**. We consider first the case of Whitehead manifold for $g = sl_2(\mathbf{C})$ and the level $k = 2$. Computations using the proposition 4.9 and the Kirby-Melvin formula [24] were made in [17] but contain an error which we correct now (see also the revised version of [17]) . Our aim is to use the mapping class group representations this time.

LEMMA 5.6. *The associated cobordism V is $B_0 \cup B_1$, where B_i are compression bodies: B_0 is obtained from a torus by attaching a 1-handle, and B_1 is obtained from the genus 2 surface by attaching a 2-handle on the circle c figured in picture 25. The gluing map is the gluing map of the standard genus 2 Heegaard decomposition of the sphere.*

PROOF. Let consider the cobordism V has a 2-handle added like in the Fig. 32a. By a sequence of Whitehead dilatations we may pass from 32.a to 32.b. These moves preserve the homeomorphism class of the complement. Further one may add a 1-handle over the circle c surrounding the hole in 25.b and get a trivial cobordism. So it suffices to draw the image of the curve c in the original picture, and to identify it with the circle from 24. □

We suppose for instant the TQFT is an arbitrary Lie TQFT, or more generally a TQFT based on the RCFT (see [16]). The sequence of maps we are computing is therefore the same as in the picture 29 with an $S \otimes S$ inserted at the beginning. From above we derive that

$$Z(V)f_j = \sum_t \lambda_0 S_{0t} (\sum_k S_{jk} \lambda_k^{-1} (\sum_s U_{ks}^{-1} U_{s0} \Gamma_s t)) f_0 \otimes f_t.$$

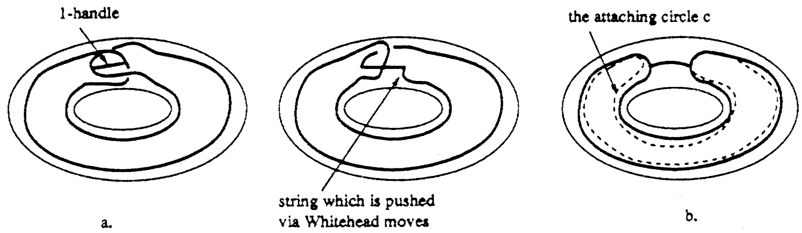

Fig. 32. Whitehead dilatations.

Using the expression for U and the fact that T is diagonal we reduce the matrix L of $Z(V)$ in the considered basis to the simple form

$$L_{jk} = S_{j0} S_{0k} \Gamma_{jk}.$$

(5.6) **Computations for $sl_2(\mathbf{C})$ at level 2**. Assume the TQFT is that given by $sl_2(\mathbf{C})$ at level 2 (called also the Ising model). Then all but one of the fusing matrices are trivial. The only non-trivial entry of Γ is $\Gamma_{\frac{1}{2}\frac{1}{2}}$. A simple computation gives $\Gamma_{\frac{1}{2}\frac{1}{2}} = -1$ (using the fusing matrices or directly the form of $\alpha_2 T_v \alpha_2^{-1}$ from (5.3)) and therefore

$$L = \frac{1}{4} \begin{bmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & -2 & \sqrt{2} \\ 1 & \sqrt{2} & 1 \end{bmatrix}.$$

Remark that the two basis in which we computed the linear map $Z(X)$ are not the same, if we identify the corresponding vector spaces. This should be essential if we wish to compute the rank of iterates. Otherwise, we could use these computations but we need to insert another term corresponding to the gluing between two cobordisms. Actually this map is not the identity but it changes the orientation of the torus (i.e. the gluing map of the standard genus 1 Heegaard splitting of the sphere).

In fact a basis of the vector space associated to the torus corresponds to a curve on the torus. Usually one consider a framing of the core of that torus. But this curve, transported on the outer boundary of the cobordism is no more a longitude but becomes a meridian. Therefore we need to change this meridian into a longitude. On the torus level this amounts to consider the map $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in SL_2(\mathbf{Z})$. The action of this element is nothing but the S-matrix of the theory.

It follows that the matrix of the linear map in the right basis is

$$(Z_{sl_2(\mathbf{C}),2})(V) = SL = \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix}.$$

Now we can iterate this formula and find that the rank of $(SL)^2$ is 1. Therefore $(Z_{sl_2(\mathbf{C}),2})_\infty(Wh) \cong \mathbf{C}$.

Consider now the cobordism $W_{n,k}$ obtained when we replace the Whitehead link by the twisted version $L(n,k)$. There is a twisted Whitehead manifold $Wh(n,k)$ associated. For $n = 1$ in our description we need to replace T_γ by $\alpha_1^k T_\gamma$. Now it is clear that α_1 acts trivially on those vectors which are not killed by the projection π. So the matrices of the associated linear endomorphisms coincide with L. Notice however that the induced morphisms for the natural rigid structures on the tori are different, for distinct k. In fact it suffices to see how a longitude of the torus (or framing) is transported to the other boundary. It may be checked that altering the power of α_1 corresponds to alter the framing by a Dehn twist of the solid torus. The reason is that after twisting the two rigid cobordisms become equivalent because the underlying manifolds are homeomorphic. We don't know if the associated open manifold are homeomorphic. Actually, if the real basis is not specified, then we are computing only the morphism modulo left and right multiplication by unitary matrices, as was already pointed before.

Consider now the case $k = 1$, which corresponds to T_γ^n. Since $T_\gamma(W_1 \otimes W_1) = W_1 \otimes W_1$, it suffices to consider only $T_\gamma^n \mid_{W_1\otimes W_1}$. From (5.3) we derive

$$T_\gamma^2 \mid_{W_1\otimes W_1} = \mathbf{1},$$

so that only $n(\mathrm{mod}\ 2)$ does matter. We find therefore the matrix of

$(Z_{sl_2(\mathbf{C}),2})_\infty(W_{2n,1})$ be:

$$\frac{1}{4}\begin{bmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 2 & \sqrt{2} \\ 1 & \sqrt{2} & 1 \end{bmatrix},$$

which has rank 1. If we modify the basis as above we get always a matrix of rank 1:

$$(Z_{sl_2(\mathbf{C}),2})(W_{2n,1}) = \frac{1}{2}\begin{bmatrix} 1 & \sqrt{2} & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We can summarize that by saying that $(Z_{sl_2(\mathbf{C}),2})_\infty$ does not distinguish any of twisted Whitehead manifolds $Wh(n,k)$ from $\mathbf{R}^3$.

(5.7) **Computations for $sl_2(\mathbf{C})$ at level 3.** We describe first the combinatorial data.

- the set of colors $C = \{0, \frac{1}{2}, 1, \frac{3}{2}\}$; set $D = \{\frac{1}{2}, 1\}$.
- $W_1 = \bigoplus_{j\in C} W^0_{jj}$; $W_2 = \bigoplus_{j,k\in C} W^0_{jj} \otimes W^0_{kk} \bigoplus_{j,k\in D} W^1_{jj} \otimes W^1_{kk}$.
- The S and T matrices are known from (2.3).
- In order to compute the representation of $\mathcal{M}_2$ we need the fusing matrices $F\begin{bmatrix} u & v \\ u & v \end{bmatrix}$. General arguments give

$$F^{-1}\begin{bmatrix} u & v \\ u & v \end{bmatrix} = F\begin{bmatrix} u & u \\ v & v \end{bmatrix},$$

$$F\begin{bmatrix} u & v \\ u & v \end{bmatrix} = F\begin{bmatrix} v & u \\ v & u \end{bmatrix}.$$

- Also $F\begin{bmatrix} 0 & u \\ 0 & u \end{bmatrix}$ are trivial since they are scalars; the same is true for all $F\begin{bmatrix} k & n \\ k & n \end{bmatrix}$ with k or n lying in $C - D$.

- The only non-trivial fusings are those for which $k, n \in D$. In [27] the entries of the fusing matrices are identified to the q-6j-symbols (see [46, 25]). Set $q = exp\left(\frac{\pi\sqrt{-1}}{5}\right)$ and $[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$. Using the explicit formulas from [46, 25] we derive that

$$F_{ij}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \frac{1}{[2]^{\frac{1}{2}}}\begin{bmatrix} -\frac{1}{[2]^{\frac{1}{2}}} & 1 \\ 1 & \frac{1}{[2]^{\frac{1}{2}}} \end{bmatrix}, \text{ and } i, j \in \{0, 1\}$$

$$F_{ij}\begin{bmatrix} \frac{1}{2} & 1 \\ \frac{1}{2} & 1 \end{bmatrix} = \frac{-1}{[2]^{\frac{1}{2}}}\begin{bmatrix} -1 & \frac{1}{[2]^{\frac{1}{2}}} \\ \frac{1}{[2]^{\frac{1}{2}}} & 1 \end{bmatrix}, \text{ and } i \in \{0, 1\}, j \in \{\frac{1}{2}, \frac{3}{2}\}$$

Notice the indices i, j do not take values in the same set; also the inverse of this fusing F has the entries $-F_{ij}$.

$$F_{ij}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \frac{-1}{[2]^{\frac{1}{2}}}\begin{bmatrix} -\frac{1}{[2]^{\frac{1}{2}}} & 1 \\ 1 & \frac{1}{[2]^{\frac{1}{2}}} \end{bmatrix}, \text{ and } i, j \in \{0, 1\}$$

By a simple computation we find

$$\Gamma_{\frac{1}{2}1} = \Gamma_{1\frac{1}{2}}\Gamma_{\frac{1}{2}\frac{1}{2}} = \Gamma_{11} = 1 + \alpha = 1 + \frac{1}{8}(\cos\frac{\pi}{5})^{-3}(3e + e^{-1} - e^2 - 3),$$

$$\text{where } e = \exp\left(\frac{-4\pi\sqrt{-1}}{5}\right).$$

Therefore up to a constant the matrix L associated to V is

$$L = \begin{bmatrix} 1 & 2\cos\frac{\pi}{5} & 2\cos\frac{\pi}{5} & 1 \\ 2\cos\frac{\pi}{5} & 4\alpha\cos^2\frac{\pi}{5} & 4\alpha\cos^2\frac{\pi}{5} & 2\cos\frac{\pi}{5} \\ 2\cos\frac{\pi}{5} & 4\alpha\cos^2\frac{\pi}{5} & 4\alpha\cos^2\frac{\pi}{5} & 2\cos\frac{\pi}{5} \\ 1 & 2\cos\frac{\pi}{5} & 2\cos\frac{\pi}{5} & 1 \end{bmatrix}.$$

Using the remark from the precedent subsection, in order to iterate this map we need to modify the target basis of the vector space associated to the torus. The S-matrix in this case is obtained from (2.3). We derive

$$Z_{sl_2(\mathbf{C}),3}(V) = SL = \frac{1}{2}\begin{bmatrix} 1 + 2\cos\frac{\pi}{5} & * & * & 1 + 2\cos\frac{\pi}{5} \\ 0 & 0 & 0 & 0 \\ 0 & 4\alpha^2\cos\frac{\pi}{5} & 4\alpha^2\cos\frac{\pi}{5} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

This implies that the rank of $(SL)^n$ is 2 for all $n \neq 0$. Alternatively this shows that

$$\dim(Z_{sl_2(\mathbf{C}),3})_\infty(Wh) = 2.$$

We may compute explicitly the morphism associated to the twisted cobordisms $W_{1,k}$, corresponding to the gluing map $\alpha_1^k T_\gamma$. In fact since α_1 acts trivially on the vectors $f_0 \otimes f_k$ we find again that

$$(Z_{sl_2(\mathbf{C}),3})_\infty(Wh(1,k)) = (Z_{sl_2(\mathbf{C}),3})_\infty(Wh)$$

The computations for higher n, which could prove the non-simply connectedness of all $Wh(n,k)$, become more complicated because the T_γ action has no more $W_1 \otimes W_1$ as an invariant subspace. So we need to take into account the higher S-matrices $S(r)$. We omit the cumbersome computations which prove as above that for $n \neq -1 (\text{mod } 5)$, the corresponding matrix has rank 2 (for $n = -1 (\text{mod } 5)$ has rank 1), and that the sequence of matrices indexed by n has period 5. A simpler proof is given in the revised version of [17].

6. Cofinal Invariants

(6.1) **Cofinal invariants for some genus 1 manifolds**. We review first some topological aspects of Whitehead type manifolds after Brown [8]. Several definitions and notations are needed. Firstly (A,B) is an *unknotted pair* if $B \subset A$ are unknotted solid tori and there exists an embedding of A in S^3 so that both A and B are unknotted in the ambient space. The unwrapping number $n(A,B)$ is the minimal number of points the core of B meets an meridian disk of A. An unknotted pair is *trivial* provided that $n(A,B) < 2$: so B is contained in a ball in A (if $n(A,B) = 0$) or the boundary tori of A and B are parallel (if $n(A,B) = 1$). Notice that

$$n(A,C) = n(A,B)n(B,C)$$

if we have $C \subset B \subset A$. A similar product formula holds for the *winding* number $w(A,B)$ defined as the class of the core of B in $\pi_1(A) \cong \mathbf{Z}$.

Now a (non-trivial) unknotted pair (A,B) *factors* if there is some solid torus C, $B \subset C \subset A$, so that both of (A,C) and (C,B) are non-trivial unknotted pairs. Otherwise, the pair (A,B) is *prime*. Since n takes only integer values and is multiplicative it follows that any non-trivial pair has a finite prime factorization. Fortunately, in the context of unknotted pairs we have also the uniqueness of the prime factorization (due to Brown):*if* $B \subset E_1 \subset E_2 \subset ... \subset E_k \subset A$, *and* $B \subset E'_1 \subset E'_2 \subset ... \subset E'_m \subset A$, *are two prime factorizations of the unknotted non-trivial pair* (A, B), *then* $k = m$ *and there exists an isotopy of* A *rel* B *carrying each* E_j *into* E'_j.

Consider now an irreducible contractible open 3-manifold W of genus 1 at infinity. It is an ascending union of solid tori T_n which sit homotopically trivial each one in the forthcoming. If additionally W embeds in some compact 3-manifold, then Brown, using results of Haken, proved that one may choose the exhaustion so that (T_{n+1}, T_n) are unknotted pairs. If all pairs (T_{n+1}, T_n) are prime the sequence is called a *prime decomposition* of W. Now an easy extension of the previous uniqueness result is: *for two prime decompositions* $\{T_n\}$ *and* $\{T'_n\}$ *of an open manifold as before, there is an homeomorphism* h *and integers* k, m *so that* $h(T_{n+k}) = T'_{n+m}$ *for all positive integers* n. The integers k, m cannot be eliminated since a prime decomposition can be extended by adding arbitrary terms at the beginning. An immediate application of this description is:

COROLLARY 6.1. *For any TQFT* Z, *and prime decomposition* $\{T_n\}$ *of the irreducible open contractible 3-manifold* W, *which embeds in a compact 3-manifold, the cofinal class of the sequence of linear maps*

$$Z_{cf}(W) = ...Z(\partial T_n) \stackrel{Z(cl(T_{n+1}-T_n))}{\longrightarrow} Z(\partial T_{n+1}) \stackrel{Z(cl(T_{n+2}-T_{n+1}))}{\longrightarrow} Z(\partial T_{n+2})...$$

is a topological invariant of W, *which we call the cofinal invariant induced by* Z.

Notice that, choosing specific basis for all $Z(\partial T_n)$, we have to be careful because the matrices $Z(cl(T_{n+1} - T_n))$ do not form an invariant cofinal sequence of matrices: we need to take into account the various changes of basis. For example, we can choose Jordan normal forms for the matrices, etc.

Example 6.2. Using the previous computations from before for the morphisms associated to the cobordisms $W_{n,k}$ one can obtain an uncountable family of open contractible 3-manifold which are not pairwise homeomorphic. The various ways in which we may compose the cobordisms $V_0 = W_{2,1}$ and $V_1 = W_{1,1}$ are in one-to-one correspondence with reals $r \in \mathbf{R}$

written in the base 2 (except for the dyadic numbers). Denote by $W(r)$ the open 3-manifold we obtain this way. We claim that the contractible open 3-manifolds $W(r)$ and $W(r')$ cannot be homeomorphic unless $r - r'$ is not a rational number.

PROOF. Assume that the considered manifolds would be homeomorphic. We have therefore two prime unknotted decompositions for them, since the unwrapping numbers are 2 for all cobordisms. Then these prime decompositions should be equivalent. Consider the first position k where the digits r_k and r'_k do not coincide. We suppose k is sufficiently large and the shift of indices is trivial. By the Brown's results there is some homeomorphism h (which can be taken to be isotopic to identity on the first $k-1$ cobordisms and interchanges V_0 and V_1. The homeomorphism h induces at the TQFT level a commutative diagram; its first raw is trivial since h is isotopic to identity on one boundary. The second raw (corresponding to the action of h on the space associated to the other boundary) is a certain unitary matrix A. It remains to check that there is no unitary matrix A such that $(Z_{sl_2(\mathbf{C}),2})_\infty(V_0) = A(Z_{sl_2(\mathbf{C}),2})_\infty(V_1)$. This is a consequence of the fact that the two morphisms have different ranks. This proves our claim. □

These examples are somewhat simpler that those given by Brown and Myers [8, 36], and so the collection of all TQFT may produce a rich source of such examples. Moreover, the classical sequence of unwrapping numbers is constant 2 so that classical invariants cannot detect the non-triviality of this family.

These cofinal invariants encode some global topological information of the 3-manifold, but they are still very close to invariants at infinity. Specifically we have:

PROPOSITION 6.3. *Assume that W' is obtained from W by surgery on a proper knot. Then their cofinal invariants coincide.*

PROOF. We have two solid tori T and T' so that $W' - T'$ and $W - T$ are homeomorphic rel boundary. According to Poénaru's theorem [40] each torus can be considered as the first stage for an homotopically trivial exhaustion. Thus we have such exhaustions $\{T_n, n \geq 0\}$ and $\{T'_n, n \geq 0\}$ starting with the initial tori. One have refinements of both sequences leading to unknotted prime decompositions. Since the manifolds $W'-T'$ and $W-T$ are homeomorphic rel boundary an easy extension of Brown's argument [8] gives the cofinal equivalence between the prime refinements of $\{T_n, n \geq 1\}$ and $\{T'_n, n \geq 1\}$. This proves the claim. $\square$

7. Comments

- The multiplicativity of n and w suggests that they are very close to TQFT. Let us describe an easy extension of the winding number $w(A, B)$. We consider (A, B) a handlebody pair of genus g. Using the inclusion map $B \hookrightarrow A$ we may identify $\pi_1(B)$ with a lattice in $\pi_1(A) \cong \mathbf{Z}^g$. Let $w(A, B)$ be the volume of the lattice $\pi_1(B)$ normalized by that of $\pi_1(A)$. It is a non-negative integer and it satisfies

$$w(A, B) = w(A, C)w(C, B)$$

  for all intermediary handlebodies C. Also, in the case of genus 1 pairs, it coincides with the winding number. We don't know whether such an extension exists for the unwrapping number. However the TQFT furnish analog tools for more general pairs of handlebodies.

- The cofinal invariants may be defined for some open 3-manifolds with finite genus at infinity in a similar manner. The analogue for unknotted pairs of tori is the *unknotted pair* of handlebodies of same genus g. An unknotted pair (A, B) is *trivial* if the surfaces ∂A and ∂B are boundary parallel or it is possible to insert a smaller genus handlebody C between them: $B \subset C \subset A$. It is simply to check that prime decomposition for non-trivial pairs exists. In fact, all intermediary surfaces must be incompressible and the claim follows from Haken's finiteness theorem. The uniqueness of prime decompositions seems to hold for all unknotted pairs, and thus for those open 3-manifolds which can be built up from unknotted pairs. Again, one large class of examples is given by those contractible open 3-manifold which embed in a compact 3-manifold. The details will be considered in a further paper.

- The condition under which the invariant at infinity of a genus 1 Whitehead manifold is trivial is from (4.3):

$$J(K_n^*(\lambda, \mu)) = J(K_n, \lambda)J(U, \mu) = J(U, \lambda)J(U, \mu),$$

Assume that this hold uniformly for all g, k with the same rank n of iterating the link. This is a very strong assumption: we say that a two components link $K \subset S^3$ is *numerically split* if it consists in two trivial but linked knots and

$$J(K(\lambda, \mu)) = J(U, \lambda) J(U, \mu),$$

holds for all g, k and colors $\lambda, \mu \in P_+(k)$. We formulate the following:

Conjecture 1. *A numerically split link which is homotopically trivial is split.*

This is similar to the well-known conjecture that a knot with trivial Homfly polynomial is trivial.

A positive answer to this conjecture would imply that an irreducible contractible end periodic open 3-manifold W of genus 1 at infinity is simply connected at infinity if $Z_\infty(W) \cong \mathbf{C}$ for all TQFTs $Z_{g,k}$ and there is some uniform bound for the least n so that all n-th powers $Z_{g,k}^n$ computed for the end model have rank 1. Of course the last condition is hard to test in practice.

In fact from [32, 8] each such manifold may be written as an ascending union of solid tori T_n. Then the core of the torus T_n together with the circle surrounding once T_N is a numerically trivial link if N is sufficiently large. If this link is split there is a sphere S^2 separating T_n and ∂T_N. By a standard argument we find that W is simply connected at infinity.

One may introduce *numerically split* 3-valent graphs like in the link context. Again if the more general conjecture that *numerically split 3-valent graphs with two components are actually split* would hold, then we could derive that *an irreducible contractible open 3-manifold is simply connected at infinity if and only if all its quantum invariants at infinity are trivial.* We observed previously that, for any manifold W having finite genus at infinity and any TQFT Z we have

$$\dim Z_\infty(W) \leq \dim Z(\Sigma_{g(W)}).$$

The right meaning of the previous conjecture is to get a converse of this statement. This cannot be stated for arbitrary manifolds as the theorem 5.1 should furnish counterexamples. We are tempted to formulate a stronger variant: *For most levels k the quantum invariants at infinity compute the right genus at infinity of a contractible manifold*:

$$\dim(Z_{g,k})_{\infty}(W) = \dim Z_{g,k}(\Sigma_{g(W)}).$$

As stated, this claim fails. In fact for all k we have

$$\dim(Z_{sl_2(\mathbf{C}),k})_{\infty}(Wh) \leq \dim Z_{sl_2(\mathbf{C}),k}(S^1 \times S^1) - 1.$$

For this it suffices to see that Γ_{uv} is boarded by trivial entries. We already seen that $\Gamma_{0v} = \Gamma_{v0} = 1$ since the corresponding fusing matrices are trivial. Let $N = k/2$. Then $\Gamma_{N,v} = \Gamma_{v,N} = 1$ for all v since the associated fusing matrices are trivial too. In fact any label r (the third in a vertex whose two other edges are labeled by N) must satisfy $r + 2N \leq k$, hence $r = 0$. Other symmetry principles for arbitrary Lie algebra g lead to more general counterexamples. However, we think that a weaker form is true:

Conjecture 2. *For each open contractible manifold W the quantum invariants at infinity compute asymptotically the right genus at infinity:*

$$sup_{g,k}\frac{\dim(Z_{g,k})_{\infty}(W)}{\dim Z_{g,k}(\Sigma_{g(W)})} > 0.$$

A still weaker assertion will be that

$$sup_{g,k}\frac{\dim(Z_{g,k})_{\infty}(W)}{\dim Z_{g,k}(\Sigma_{g(W)-1})} > 1.$$

Therefore, despite the dependence on the TQFT, all invariants Z_{∞} seem to be approximations of the genus at infinity of open 3-manifolds, but there is no evidence about how good these approximations should be. Remark that in the context of closed manifolds, unitary TQFT could be used for estimations of the Heegaard genus (see [19]) and tunnel numbers for knots (see [28]).

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