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Mathematics

Explicit Plancherel theorem for ground state representation of the Heisenberg chain

(spin waves/eigenfunction expansions)

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ABSTRACT In its ground state representation, the infinite spin 1/2 Heisenberg chain provides a model for spin wave scattering that entails many features of the quantum mechanical N-body problem. Here, we give a complete eigenfunction expansion for the Hamiltonian of the chain in this representation, for *all* numbers of spin waves. Our results resolve the questions of completeness and orthogonality of the eigenfunctions given by Bethe for finite chains, in the infinite volume limit.

Introduction

Let H be the self-adjoint Hamiltonian for the ground state representation of the spin 1/2 infinite one-dimensional Heisenberg ferromagnet with nearest-neighbor interactions (1, 2). This operator provides a model for spin-wave scattering (3). Restricted to its N-spin sector, H is unitarily equivalent in a natural way to a second difference operator $-\Delta_N$ with "sticky" boundary conditions, acting on an l^2-space. In this announcement we give an explicit unitary equivalence of $-\Delta_N$ with a multiplication operator $\epsilon_N = \oplus_{\beta \in \mathcal{B}_N} \epsilon_\beta$ on $\oplus_{\beta \in \mathcal{B}_N} L^2(\hat{\Gamma}_\beta; \mu_\beta\, dz_\beta)$ in which $\mathcal{B}_N$ is the collection of N-bindings, $\hat{\Gamma}_\beta$ is essentially a torus with dimension varying with β, and $\mu_\beta(z_\beta)dz_\beta$ is the Plancherel measure on $\hat{\Gamma}_\beta$. It is important to point out that ϵ_β and $\mu_\beta(z_\beta)dz_\beta$ are given explicitly.

Notation and eigenfunctions

Let $\hat{\mathbf{Z}}^N = \{\mathbf{m} = (m_1, m_2, \ldots, m_N) \in \mathbf{Z}^N | m_1 < m_2 < \ldots < m_N\}$. Then $-\Delta_N$, acting in $l^2(\hat{\mathbf{Z}}^N)$ is defined as follows:

$$-\Delta_N f(m_1, \ldots, m_N) = -\frac{1}{2}\sum_{i=1}^{N} [f(m_1, \ldots, m_i + 1, \ldots, m_N) + f(m_1, \ldots, m_i - 1, \ldots, m_N) - 2f(m_1, \ldots, m_N)], \quad [\mathbf{1a}]$$

provided the m_is are not neighboring. If two of the m_is are neighboring—e.g., $m_{k+1} = m_k + 1$—then

$$-\Delta_N f(m_1, \ldots, m_N) = -\frac{1}{2}\sum_{i \neq k, k+1}^{N} \times [f(m_1, \ldots, m_i + 1, \ldots, m_N) + f(m_1, \ldots, m_i - 1, \ldots, m_N) - 2f(m_1, \ldots, m_N)] - \frac{1}{2} f(m_1, \ldots, m_k - 1, \ldots, m_N) - \frac{1}{2} f(m_1, \ldots, m_k, m_k + 2, \ldots, m_N) + f(m_1, \ldots, m_N). \quad [\mathbf{1b}]$$

Analogous expressions hold for the case in which more than two of the m_is are neighboring. This is the "sticky" boundary condition.

To describe the eigenfunctions we introduce some additional notation. Let $\beta = (n_1, n_2, \ldots, n_N)$ in which the n_js are non-negative integers such that $\Sigma^N_{j=1}\, jn_j = N$. The quantity β, which we call an N-binding, describes the manner in which the N spin waves combine into bound state "complexes"; n_j is the number of j-spin wave complexes. Given β, partition $\{1, \ldots, N\}$ into a disjoint set of intervals $I_{jk} = \{N_{jk} + 1, \ldots, N_{jk} + j\}$ with $N_{jk} = \Sigma_{l=1}^{j-1}\, ln_l + (k-1)j$ for $k = 1, \ldots, n_j$, $j = 1, \ldots, N$. Let S_N be the permutation group of $\{1, \ldots, N\}$ and let $\mathcal{P}_\beta = \{P \in S_N | P(N_{jk} + 1) < P(N_{jk} + 2) < \ldots < P(N_{jk} + j) \text{ for each } j,k\}$. Set $\mathbf{z} \equiv (z_1, z_2, \ldots, z_N) \in C^N$, $\mathbf{z}_\beta \equiv (z_{11}, z_{12}, \ldots, z_{ln_1}, z_{21}, \ldots, z_{Nn_N})$ in which $z_{jk} = z_{N_{jk}+j}$ (the variable z_{jk} is suppressed in $\mathbf{z}_\beta$ if $n_j = 0$), and $\mathbf{z}^{\mathbf{m}_P} \equiv z_1{}^{m_{P(1)}} z_2{}^{m_{P(2)}} z_N{}^{m_{P(N)}}$. Next, define the sets $\Gamma_j = \{z \in C | |jz - (j-1)| = 1\}$ and $\hat{\Gamma}_\beta = \{\mathbf{z}_\beta | z_{jk} \in \Gamma_j \text{ with } 0 \leq arg[jz_{jk} - (j-1)] \leq arg[jz_{jk'} - (j-1)] \leq 2\pi \text{ if } k < k'\}$. Define the fractional linear transformation

$$t^l(z) = \frac{(l+1)z - l}{lz - (l-1)}\quad z \in C,\, l \in \mathbf{Z} \qquad [\mathbf{2}]$$

and the function

$$e^{-i\varphi_P}(\mathbf{z}) = \prod_{\substack{i<j \text{ with} \\ P(i) > P(j)}} \left(-\frac{z_i z_j - 2z_j + 1}{z_i z_j - 2z_i + 1}\right) \qquad [\mathbf{3}]$$

for $N \geq 2$. For each fixed $\mathbf{z}_p$, $\psi_\beta(\mathbf{z}_\beta, \mathbf{m})$ is a generalized eigenfunction of $-\Delta_N$ in which

$$\psi_\beta(\mathbf{z}_\beta, \mathbf{m}) = \sum_{P \in \mathcal{P}_\beta} \mathbf{z}^{\mathbf{m}_P} e^{-i\varphi_P},\quad \mathbf{z}_\beta \in \hat{\Gamma}_\beta \qquad [\mathbf{4}]$$

and it is understood that, if $i \in I_{jk}$ with $i = N_{jk} + j - l$, then $z_i = t^l(z_{jk})$. The eigenvalue corresponding to $\psi_\beta(\mathbf{z}_\beta, \mathbf{m})$ is given by

$$\epsilon_\beta(\mathbf{z}_\beta) = -\sum_{j=1}^{n}\sum_{k=1}^{n_j} \frac{j(z_{jk} - 1)^2}{2(jz_{jk} - j + 1)},\quad \mathbf{z}_\beta \in \hat{\Gamma}_\beta. \qquad [\mathbf{5}]$$

The eigenvalue $\epsilon_\beta(z_\beta)$ is non-negative for $\mathbf{z}_\beta \in \hat{\Gamma}_\beta$. The eigenfunctions are the infinite volume limit of the finite volume eigenfunctions described by Bethe (1), rewritten in a form in which they are rational functions (cf. 2).

Associated with each binding β there is a Plancherel measure $\mu_\beta(z_\beta)dz_\beta$ on $\hat{\Gamma}_\beta$ in which:

$$\mu_\beta(\mathbf{z}_\beta) = \prod_j \prod_{k=1}^{n_j} \left(\frac{(-1)^{j-1}}{2\pi i}[(j-1)!]^2 \times \frac{j}{(jz_{jk} - j + 1)} \prod_{l=1}^{j-1} \left(\frac{z_{jk} - 1}{lz_{jk} - l + 1}\right)^2\right). \qquad [\mathbf{6}]$$

One may verify that, under the substitution of variables $z_\beta \to \kappa_\beta$ with $e^{i\kappa_{jk}} = jz_{jk} - j + 1$, the quantity $\mu_\beta(z_\beta)|\partial z_\beta/\partial\kappa_\beta|$ is real and positive.

Results

Let $\mathcal{H}_N = \oplus_{\beta\in\mathcal{B}_N} L^2(\hat{\Gamma}_\beta;\mu_\beta dz_\beta)$ and ϵ_N, the multiplication operator on $\mathcal{H}_N$, be defined by $\oplus_{\beta\in\mathcal{B}_N}\epsilon_\beta$. Our main result is summarized as follows:

THEOREM. *The mapping* $f \to \Sigma_m \psi_\beta(z_\beta, m) f(m)$ *defines a unitary mapping* U *from* $l^2(Z^N)$ *onto* $\mathcal{H}_N$ *such that* $U(-\Delta_N)U^{-1} = \epsilon_N$.

Remark: This theorem thus resolves the questions of completeness and orthogonality for Bethe's eigenfunctions, *in the infinite volume limit.*

The basic idea in proving that U is an isometry is to show that

$$\sum_\beta \frac{1}{\beta!}\int_{\Gamma_\beta} \psi_\beta(z_\beta,m)\overline{\psi}_\beta(z_\beta,m')\mu_\beta(z_\beta)\, dz_\beta = \delta m\, m' \qquad [7]$$

in the special case when $m_2 \le m'_2$, and $\beta! = n_1!n_2!\ldots n_N!$ and $\Gamma_\beta = \{z_\beta | z_{jk} \in \Gamma_j\}$. The path of integration is written in the more symmetrical form to exploit contour integration. ($\overline{\psi}_\beta(z_\beta,m)$ can be written in terms of z_β in such a way as to extend off Γ_β to a rational function in z_β.) Let $K(\beta) = \{P \in \mathcal{P}_\beta | P(N_{1k}+1) = 1$ for some $k = 1,\ldots,n_1\}$ and $L(\beta) = \mathcal{P}_\beta - K(\beta)$. Set

$$\psi_{K(\beta)}(z_\beta, m) = \sum_{P\in K_\beta} z^{m_P} e^{-i\varphi_P},$$

$$\psi_{L(\beta)}(z_\beta, m) = \sum_{P\in L(\beta)} z^{m_P} e^{-i\varphi_P}. \qquad [8]$$

We show that

$$\frac{1}{\beta!}\int_{\Gamma_\beta} \psi_{K(\beta)}(z_\beta,m)\overline{\psi}_\beta(z_\beta,m')\mu_\beta(z_\beta)dz_\beta = \frac{n_1}{\beta!}\int_{\Gamma_\beta}\sum_{\substack{P\\P(1)=1}} z^{m_P}e^{-i\varphi_P}\overline{\psi}_\beta(z_\beta,m')\mu_\beta(z_\beta)dz_\beta \qquad [9]$$

and that the z_1 integration may be performed in the integral on the right-hand side to give

$$-\sum_j{}' \frac{1}{\beta_j!}\int_{\Gamma_{\beta_j}} \psi_{L(\beta_j)}(z_{\beta_j}, m)\overline{\psi}_{\beta_j}(z_{\beta_j}, m')\mu_{\beta_j}(z_{\beta_j})dz_{\beta_j} + \frac{1}{\hat{\beta}!}\delta m_1 m_1' \int_{\Gamma_{\hat\beta}} \psi_{\hat\beta}(z_{\hat\beta},\hat{m})\overline{\psi}_{\hat\beta}(z_{\hat\beta},\hat{m}')\mu_{\hat\beta}(z_{\hat\beta})dz_{\hat\beta}, \qquad [10]$$

in which β_j, $j \ge 2$, is the N-binding $(n_1 - 1,\ldots,n_j - 1, n_{j+1} + 1,\ldots,n_N)$, $\beta_1 = (n_1 - 2, n_2 + 1, n_2,\ldots)$, $\hat\beta$ is the $(N-1)$-binding $(n_1 - 1, n_2,\ldots,n_N)$, $\hat{m} = (m_2,\ldots,m_N)$, $\hat{m}' = (m_2',\ldots,m'_N)$, and the sum extends over all $j \ge 2$ and $j = 1$ if $n_j \ge 2$. We remark that the j term in the sum corresponds to poles in $\overline{\psi}_\beta$ regarded as a function of z_1 at $z_1 = t^j(z_{jk})$, $k = 1,\ldots,n_j$. The resulting relationship from Eq. **9** and Eq. **10** and a modest induction argument on N give Eq. **7**.

A major part of our proof involves describing the singularities of $\psi_\beta(z_\beta,m)$ regarded as a function of z_β. This task is complicated by the fact that individual terms in the sum for ψ_β, Eq. **4**, have poles on Γ_β. But, by considering terms in the sum collectively, we show that $\psi_\beta(z_\beta,m)$ is bounded and therefore integrable. Similarly, we show that $\psi_{K(\beta)}$, $\psi_{L(\beta)}$ are bounded, so that each term in Eq. **9** and Eq. **10** is well defined. The proof of boundedness for these functions utilizes an elementary Sobolev inequality and the fact that the functions are quotients of polynomials. Details of the proof will be published elsewhere.

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