Basket options and implied correlations: a closed form approach

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- **Basket option**: option whose underlying is a *basket* (i.e. a portfolio) of assets.

- Payoff of a European call basket option: $\left(B(T) - X\right)^{+}$

  $B(T)$ is the basket value at the time of maturity T,

  X is the strike price.

Commodity baskets

- Crack spreads:

  *Qu * Unleaded gasoline + Qh * Heating oil - Crude*

- Soybean crush spread:

  *Qm * Soybean meal + Qo * Soybean oil - Soybean*

- Energy company portfolios:

  *Q1 * E1 + Q2 * E2 + … + Qn En,*

  where *Qi*'s can be positive as well as negative.

Motivation:

- Commodity baskets consist of two or more assets with negative portfolio weights (crack or crush spreads), Asian-style.\
- The valuation and hedging of basket (and Asian) options is challenging because ***the sum of lognormal r.v.'s is not lognormal***.
- Such baskets can have negative values, so ***lognormal distribution cannot be used, even in approximation***.
- Most existing approaches can only deal with baskets with positive weights or spreads between two assets.
- Numerical and Monte Carlo methods are slow, do not provide closed formulae.

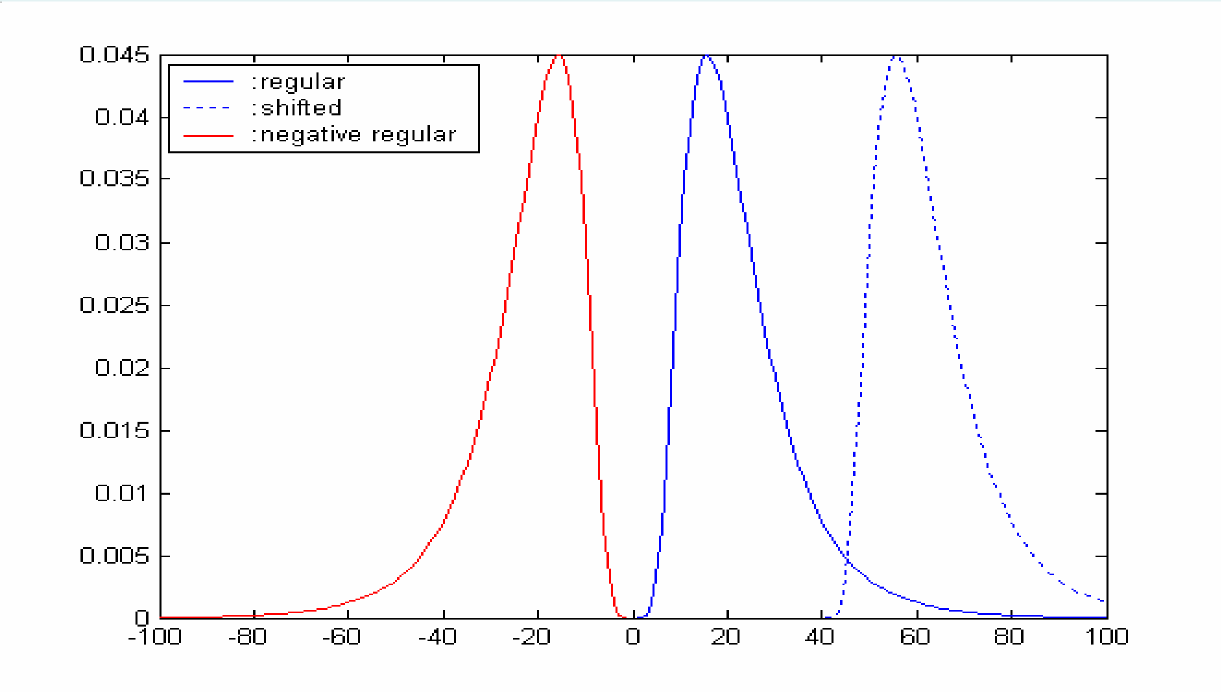
Our approach:

- Essentially a *moment-matching method*.
- Basket distribution is approximated using a *generalized family of lognormal distributions* : regular, shifted, negative regular or negative shifted.

The main attractions:

- applicable to baskets with several assets and negative weights, easily extended to Asian-style options
- allows to apply Black-Scholes formula
- provides closed form formulae for the option price and greeks

Regular lognormal, shifted lognormal and negative regular lognormal



Assumptions:

- Basket of futures on different (but related) commodities. The basket value at time of maturity *T*

$$B(T) = \sum_{i=1}^{N} a_i . F_i(T)$$

where a_i : the weight of asset (futures contract) *i*,

N : the number of assets in the portfolio,

$F_i(T)$: :the futures price *i* at the time of maturity .

- The futures in the basket and the basket option mature on the same date.

Individual assets' dynamics:

Under the risk adjusted probability measure Q, the futures prices are martingales. The stochastic differential equations for $F_i(t)$ is

$$\frac{dF_i(t)}{F_i(t)} = \sigma_i . dW^{(i)}(t), \; i = 1,2,3,\ldots,N$$

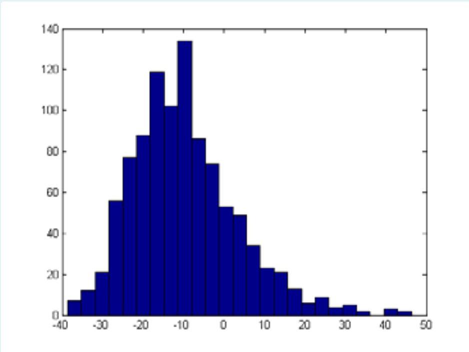
where

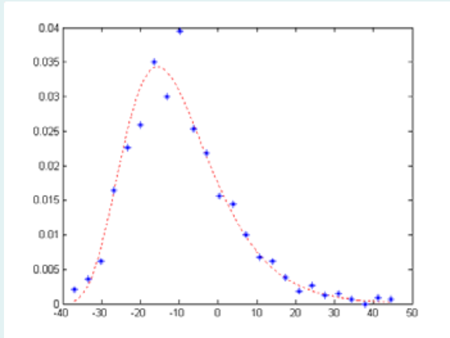
$F_i(t)$:the futures price *i* at time *t*

σ_i :the volatility of asset *i*

$W^{(i)}(t), W^{(j)}(t)$:the Brownian motions driving assets *i* and *j* with correlation $\rho_{i,j}$

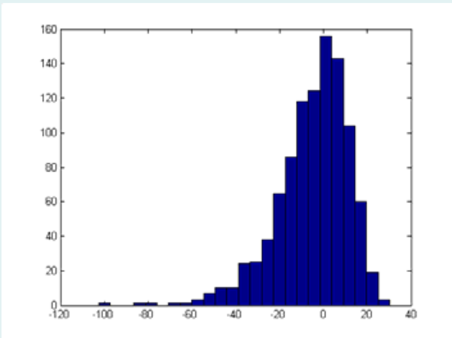
Examples of basket distribution:

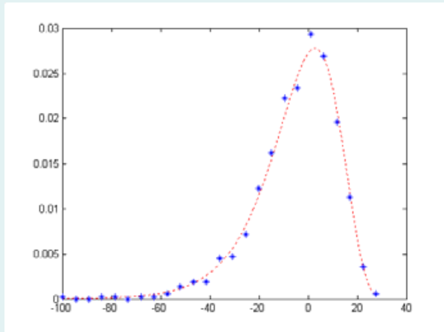




Shifted lognormal

$$Fo = [100;90];\ \sigma = [0.2;0.3];\ a = [-1;1];\ X = -10; r = 3\%;\ T = 1\ year;\ \rho = 0.9$$





Negative shifted lognormal

$$Fo = [105;100];\ \sigma = [0.3;0.2];\ a = [-1;1];\ X = -5; r = 3\%;\ T = 1\ year;\ \rho = 0.9$$

The first three moments and the skewness of basket on maturity date T:

$$E\ B(T) = M_1(T) = \sum_{i=1}^{N} a_i . F_i(0)$$

$$E\ (B(T))^2 = M_2(T) = \sum_{j=1}^{N}\sum_{i=1}^{N} a_i . a_j . F_i(0) . F_j(0) . \exp\left(\rho_{i,j} . \sigma_i . \sigma_j . T\right)$$

$$E\ (B(T))^3 = M_3(T) =$$

$$= \sum_{k=1}^{N}\sum_{j=1}^{N}\sum_{i=1}^{N} a_i . a_j . a_k . F_i(0) . F_j(0) . F_k(0) . \exp\left(\rho_{i.j} . \sigma_i . \sigma_j . T + \rho_{i.k} . \sigma_i . \sigma_k . T + \rho_{j.k} . \sigma_j . \sigma_k . T\right)$$

$$\eta_{B(T)} = \frac{E\left(B(T) - E(B(T))\right)^3}{\sigma_{B(T)}^3}$$

where $\sigma_{B(T)}$: standard deviation of basket at the time T

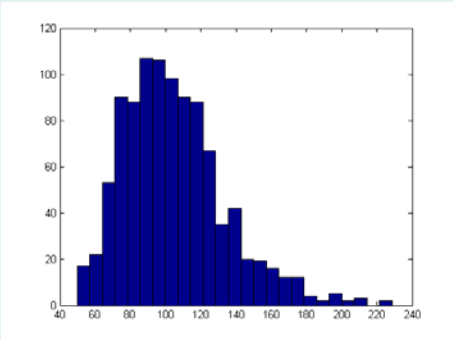
- If we assume the distribution of a basket is shifted lognormal with parameters m, s, τ , the parameters should satisfy non-linear equation system :

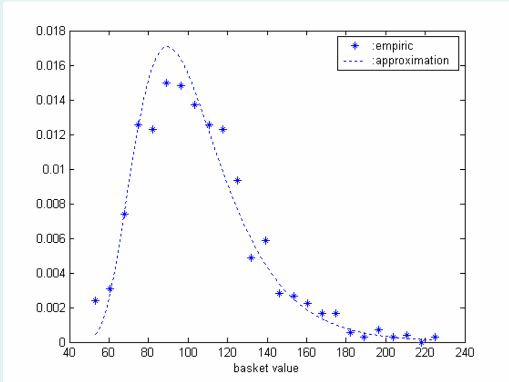
$$M_1(T) = \exp\left((m + \frac{1}{2} s^2 \right)$$

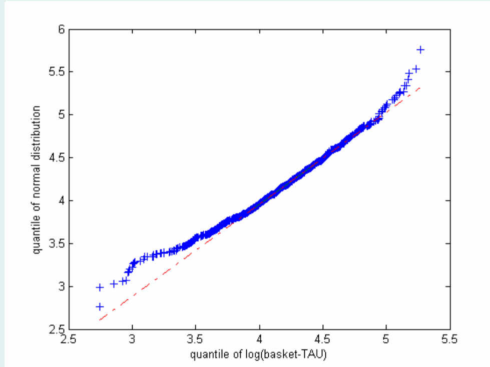
$$M_2(T) = \tau^2 + 2.\tau.\exp\left(m + \frac{1}{2} s^2 \right) + \exp\left(2m + 2s^2\right)$$

$$M_3(T) = \tau^3 + 3.\tau^2.\exp\left(m + \frac{1}{2} s^2 \right) + 3.\tau.\exp\left(2m + 2s^2\right) + \exp\left(3m + \frac{9}{2} s^2 \right)$$

- If we assume the distribution of a basket is negative shifted lognormal, the parameters should satisfy non-linear equation system above by changing $M_1(T)$ to $-M_1(T)$ and $M_3(T)$ to $-M_3(T)$.





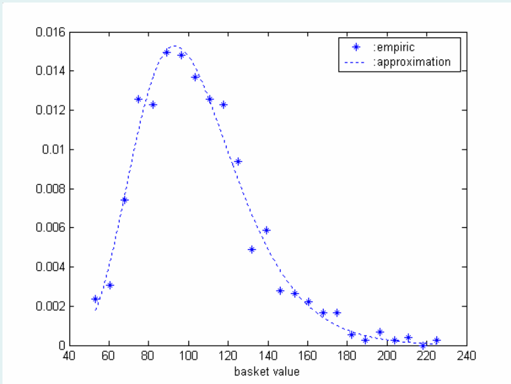


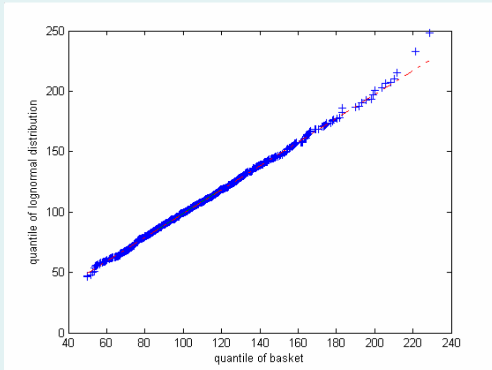
Shifted lognormal

$Fo = [50;175]; \sigma = [0.2;0.3];$

$a = [-1;1]; X = 150;$

$r = 3\%; T = 1\ year; \rho = 0.8$





Regular lognormal

Approximating distribution:

Skewness	$\eta > 0$	$\eta > 0$	$\eta < 0$	$\eta < 0$
Location parameter	$\tau \geq 0$	$\tau < 0$	$\tau \geq 0$	$\tau < 0$
Approximating distribution	***regular***	***shifted***	***negative***	***negative shifted***

Valuation of a call option (shifted lognormal):

- Suppose that the distribution of basket 1 is lognormal.
  Then the option on such basket can be valued by applying the Black-Scholes formula.

- Suppose that the relationship between basket 2 and basket 1 is

$$B^{(2)}(t) = B^{(1)}(t) + \tau$$

- The payoff of a call option on basket 2 with the strike price X is:

$$\left(B^{(2)}(T) - X\right)^{+} = \left(\left(B^{(1)}(T) + \tau\right) - X\right) = \left(B^{(1)}(T) - (X - \tau)\right)^{+}$$

It is the payoff of a call option on basket 1 with the strike price $(X - \tau)$

Valuation of call option (negative lognormal):

- Suppose again that the distribution of basket 1 is lognormal.
  Then the option on such basket can be valued by applying the Black-Scholes formula.

- Suppose that the relationship between basket 2 and basket 1 is

$$B^{(2)}(t) = -B^{(1)}(t)$$

- The payoff of a call option on basket 2 with the strike price X is:

$$\left(B^{(2)}(T) - X\right)^{+} = \left(-B^{(1)}(T) - X\right) = \left(\left(-X\right) - B^{(1)}(T)\right)^{+}$$

It is the payoff of a put option on basket 1 with the strike price $-X$

Closed form formulae of a basket call option:

- For e.g. shifted lognormal :

$$c = \exp(-rT)\left[(M_1(T) - \tau).N(d_1) - (X - \tau).N(d_2)\right]$$

where $$d_1 = \frac{\log(M_1(T) - \tau) - \log(X - \tau) + \frac{1}{2}V^2}{V}$$

$$d_2 = \frac{\log(M_1(T) - \tau) - \log(X - \tau) - \frac{1}{2}V^2}{V}$$

$$V = \sqrt{\log\left(\frac{M_2(T) - 2.\tau.M_1(T) + \tau^2}{(M_1(T) - \tau)^2}\right)}$$

It is the call option price with strike price $(X - \tau)$.

Algorithm for pricing general basket option:

- Compute the first three moments of the terminal basket value and the skewness of basket.
- If the basket skewness η is positive, the approximating distribution is regular or shifted lognormal. If the basket skewness η is negative, the approximating distribution is negative or negative shifted lognormal.
- By moments matching of the appropriate distribution, estimate parameters m, s, τ .
- Choose the approximating distribution on the basis of skewness and the shift parameter τ.

Simulation results:

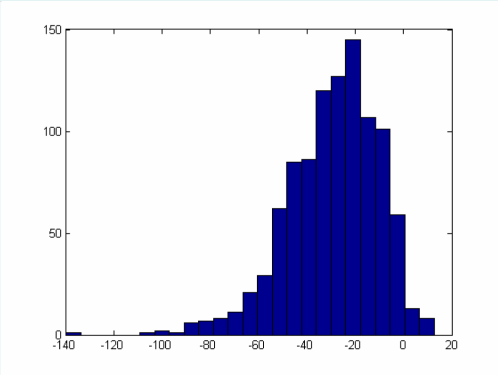
$Basket\ 5:$

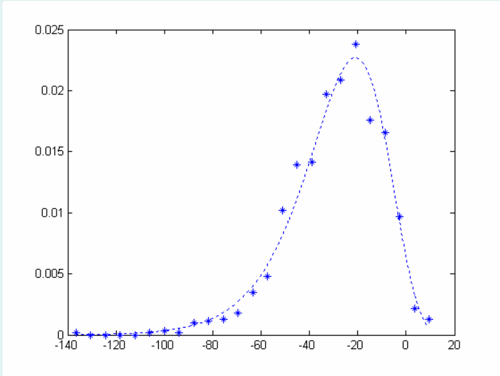
$Fo = [95;90;105]; \sigma = [0.2;0.3;0.25];$

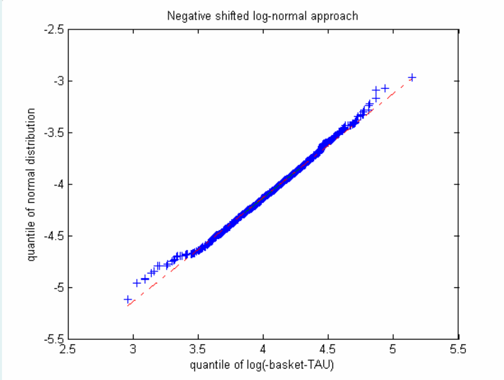
$a = [1;-0.8;-0.5]; X = -30;$

$\rho_{1,2} = \rho_{2,3} = 0.9; \rho_{1,3} = 0.8;$

$T = 1\ year; r = 3\%$





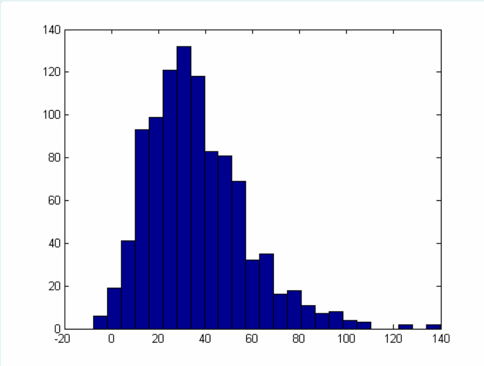


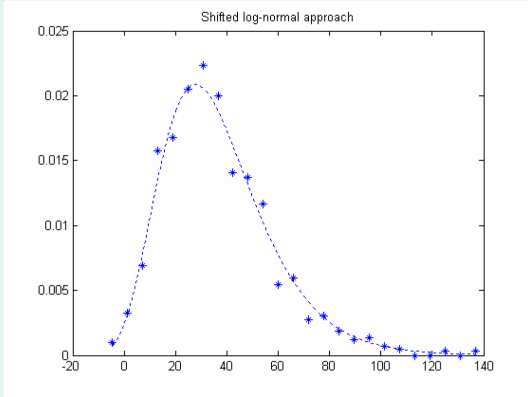
$\eta < 0\ \ ; \tau < 0$ **(neg. shifted)**

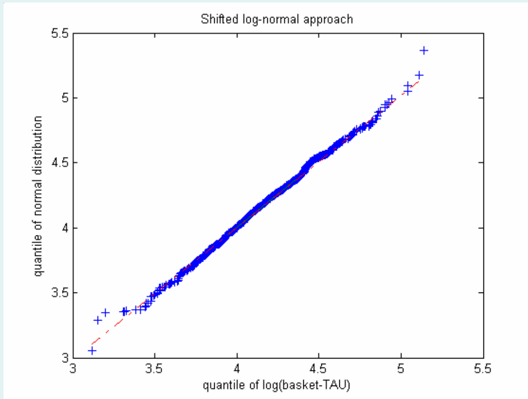
Call price: 7.7587

(*7.7299*)

$$Basket\ 6:$$
$$Fo = [100;90;95]; \sigma = [0.25;0.3;0.2];$$
$$a = [0.6;0.8;-1]; X = 35;$$
$$\rho_{1,2} = \rho_{2,3} = 0.9; \rho_{1,3} = 0.8;$$
$$T = 1\ year; r = 3\%$$







$\eta > 0 \ ; \tau < 0$ **(shifted)**

Call price : 9.0264

(*9.0222*)

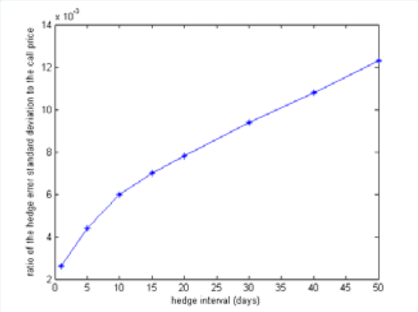
	Basket 1	*Basket 2*	*Basket 3*	*Basket 4*	*Basket 5*	*Basket 6*
Futures price (Fo)	[100;120]	[150;100]	[50;175]	[200;50]	[95;90;105]	[100;90;95]
Volatility (σ)	[0.2;0.3]	[0.3;0.2]	[0.2;0.3]	[0.1;0.15]	[0.2;0.3;0.25]	[0.25;0.3;0.2]
Weights (a)	[-1;1]	[-1;1]	[0.7;0.3]	[-1;1]	[1; -0.8; -0.5]	[0.6;0.8; -1]
Correlation (ρ)	0.9	0.3	0.8	0.8	$\rho_{1,2} = \rho_{2,3} =$ 0.9 $\rho_{1,3} =$ 0.8	$\rho_{1,2} = \rho_{2,3} =$ 0.9 $\rho_{1,3} =$ 0.8
Strike price (X)	20	-50	104	-140	-30	35
skewness (η)	$\eta > 0$	$\eta < 0$	$\eta > 0$	$\eta < 0$	$\eta < 0$	$\eta > 0$
Location parameter (τ)	$\tau < 0$	$\tau < 0$	$\tau > 0$	$\tau > 0$	$\tau < 0$	$\tau < 0$

T= 1 year; r = 3 %

Method	*Basket 1*	*Basket 2*	*Basket3*	*Basket 4*	*Basket 5*	*Basket 6*
Our approach	7.7514 *shifted*	16.9099 *neg. shifted*	10.8439 *regular*	1.9576 neg. regular	7.7587 neg. shifted	9.0264 shifted
Bachelier	8.0523	17.2365	-	2.1214	-	-
Kirk	7.7341	16.6777	-	1.5065	-	-
Monte carlo	7.7441 *(0.0143)*	16.7569 *(0.0224)*	10.8211 *(0.0183)*	1.9663 *(0.0044)*	7.7299 *(0.0095)*	9.0222 (0.0151)

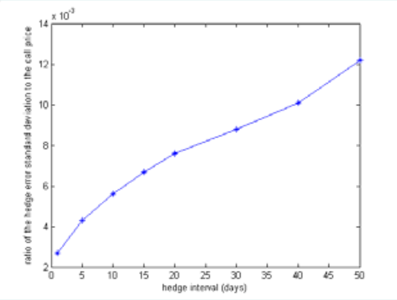
Performance of Delta-hedging:

• Hedge error: the difference between the option price and the discounted hedge cost (the cost of maintaining the deta-hedged portfolio); computed on the basis of simulations.



$Basket\ 1: Fo = [100;110]; \sigma = [0.1;0.15];$
$a = [-1;1]; \rho = 0.9; X = 10; T = 1\ year; r = 3\%$

• Plot the ratio between the hedge error standard deviation to call price vs. hedge interval.

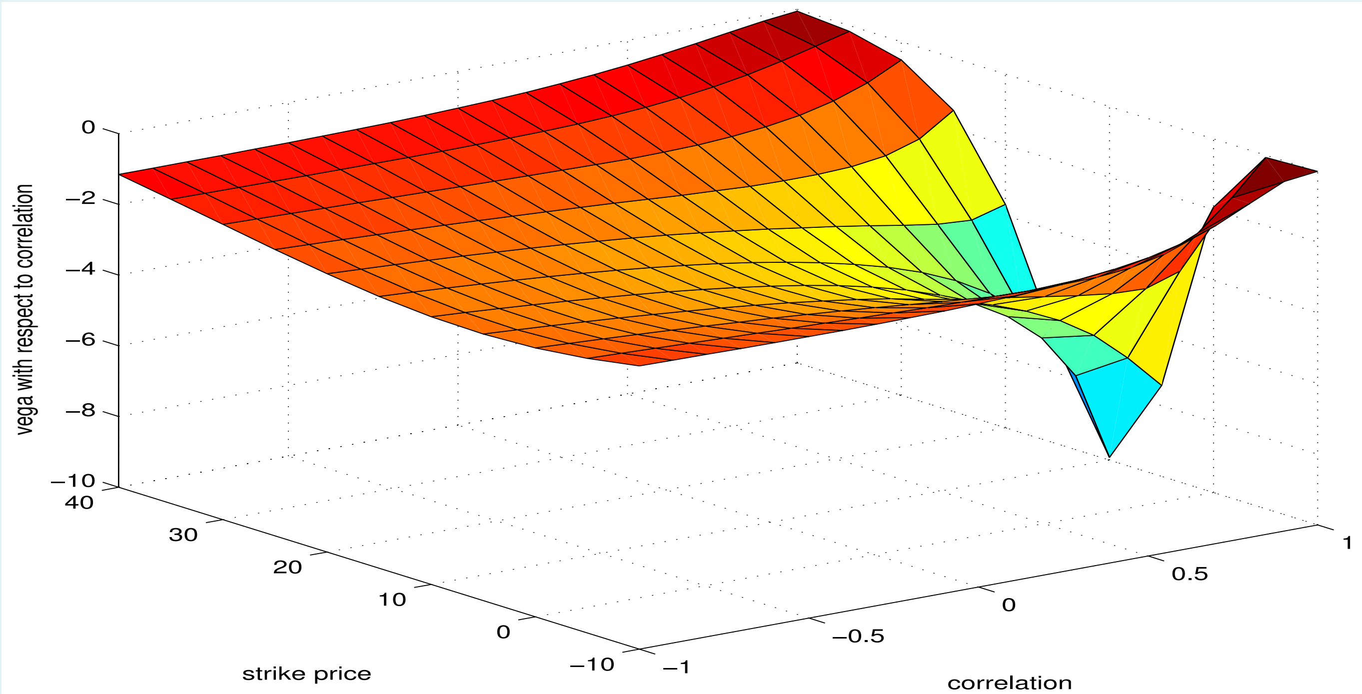


$Basket\ 2: Fo = [95;90;105]; \sigma = [0.2;0.3;0.25]$
$a = [1;-0.8;-0.5]; \rho_{1,2} = \rho_{2,3} = 0.9; \rho_{1,3} = 0.8;$
$X = -30; T = 1\ year; r = 3\%$

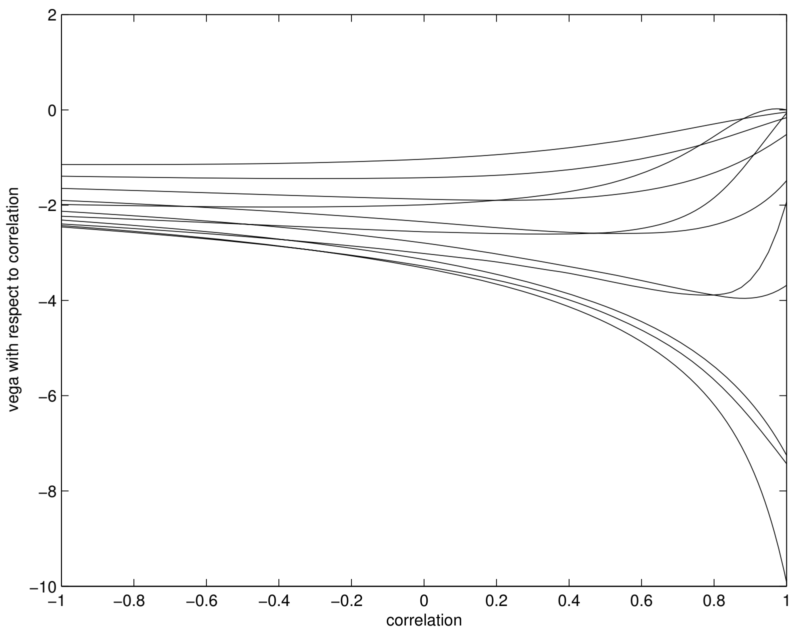
• Mean of hedge error is 4 % for basket 1 and 7 % for basket 2.

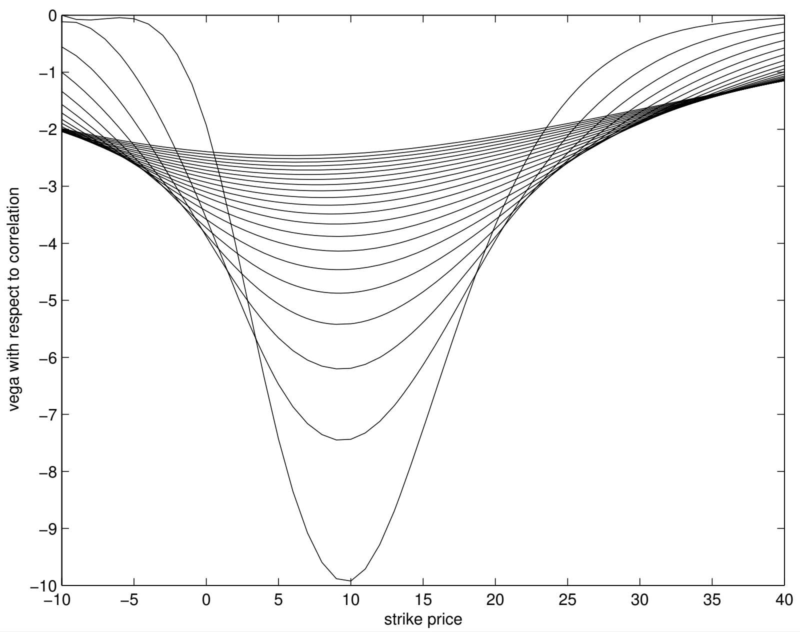
Greeks: correlation vega

Spread [110,10], vols= [0.15,0.1]



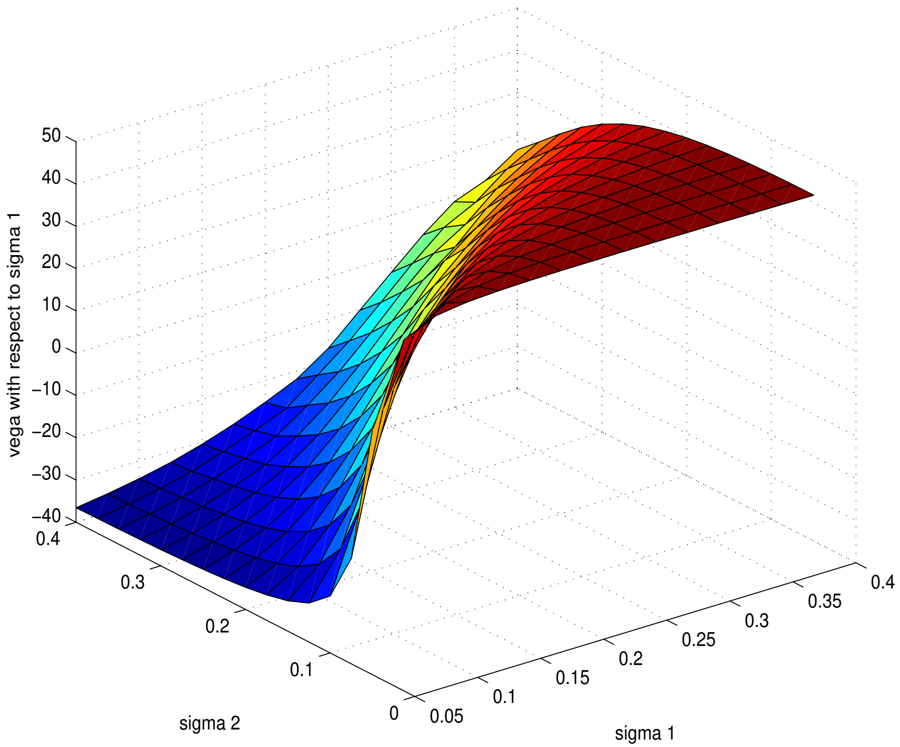
Correlation vega
vs. correlation and vs. strike

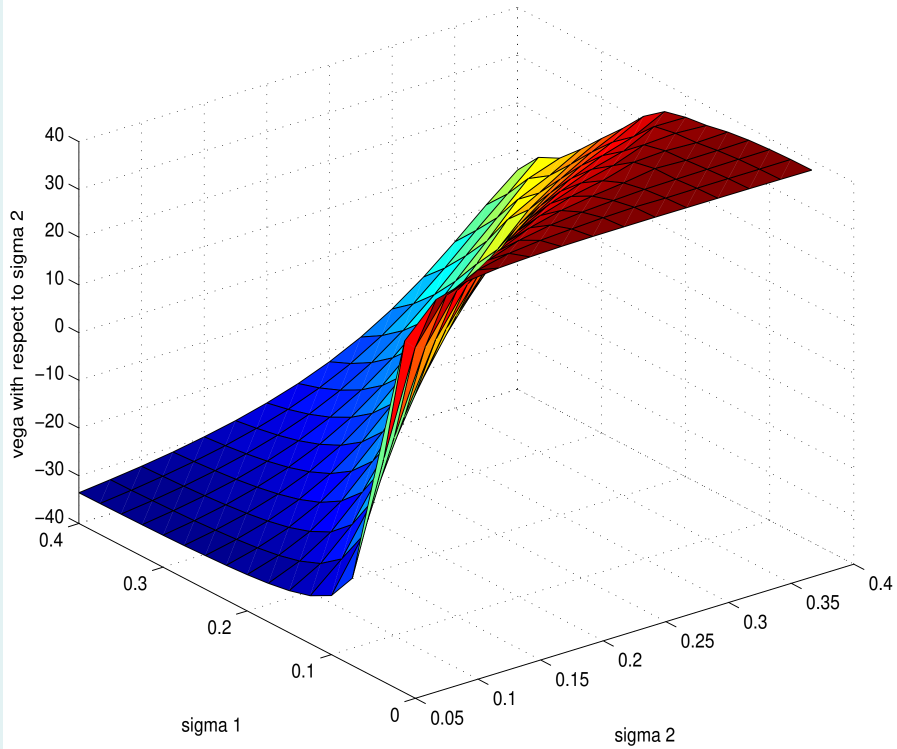




Volatility vegas vs. volatilities

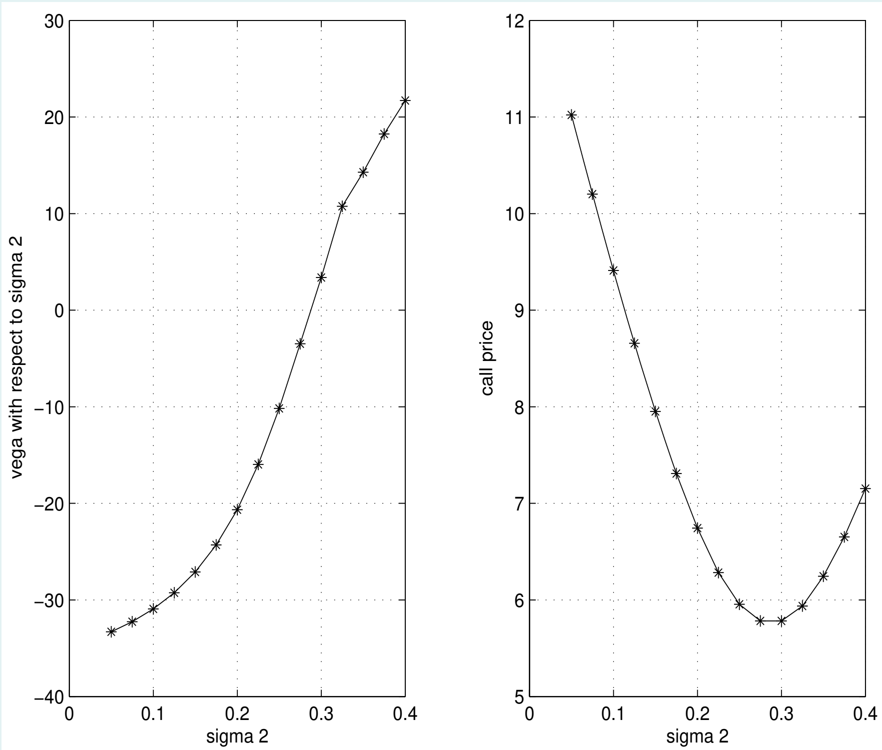
same spread, X= 10, correlation= 0.8

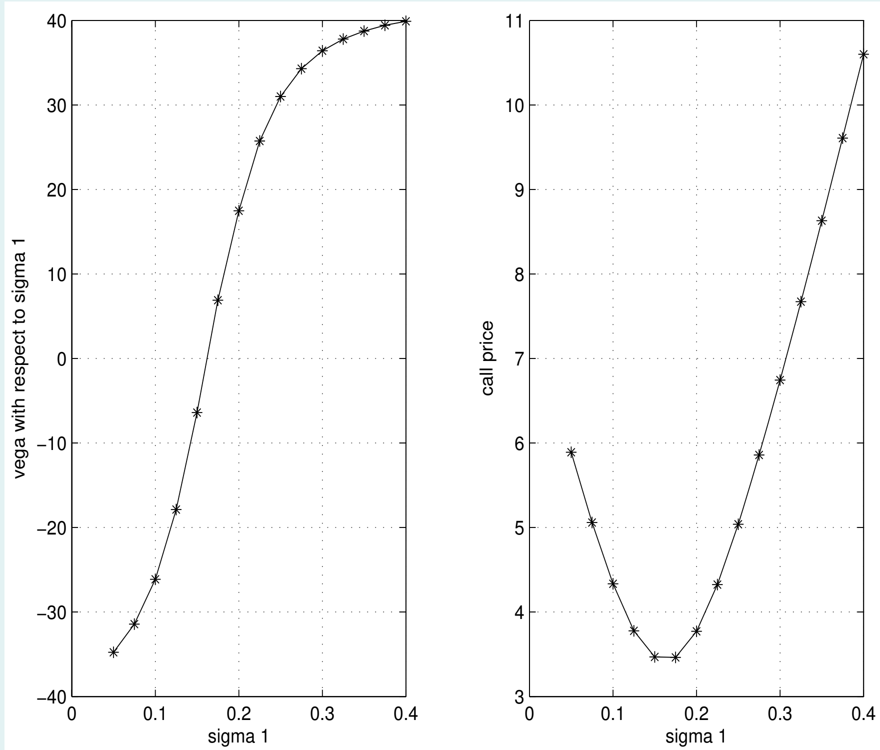




Volatility vegas and call price

sigma1=0.3 sigma2=0.2





Asian baskets

•Underlying value: (arithmetic) discrete average basket value over a certain interval

•The same approach as above applies, because

$$A_B(T) = \frac{1}{n}\sum_{k=t_1}^{t_n} B(t_k) = \frac{1}{n}\sum_{k=t_1}^{t_n}\sum_{i=1}^{N} a_i F_i(t_k) = \sum_{i=1}^{N} a_i A_i(T)$$

So the average basket value is simply the basket of individual assets' averages, with the same weights, so

→ the above approach applies directly, only with different moments!

→ option prices and greeks again calculated analytically.

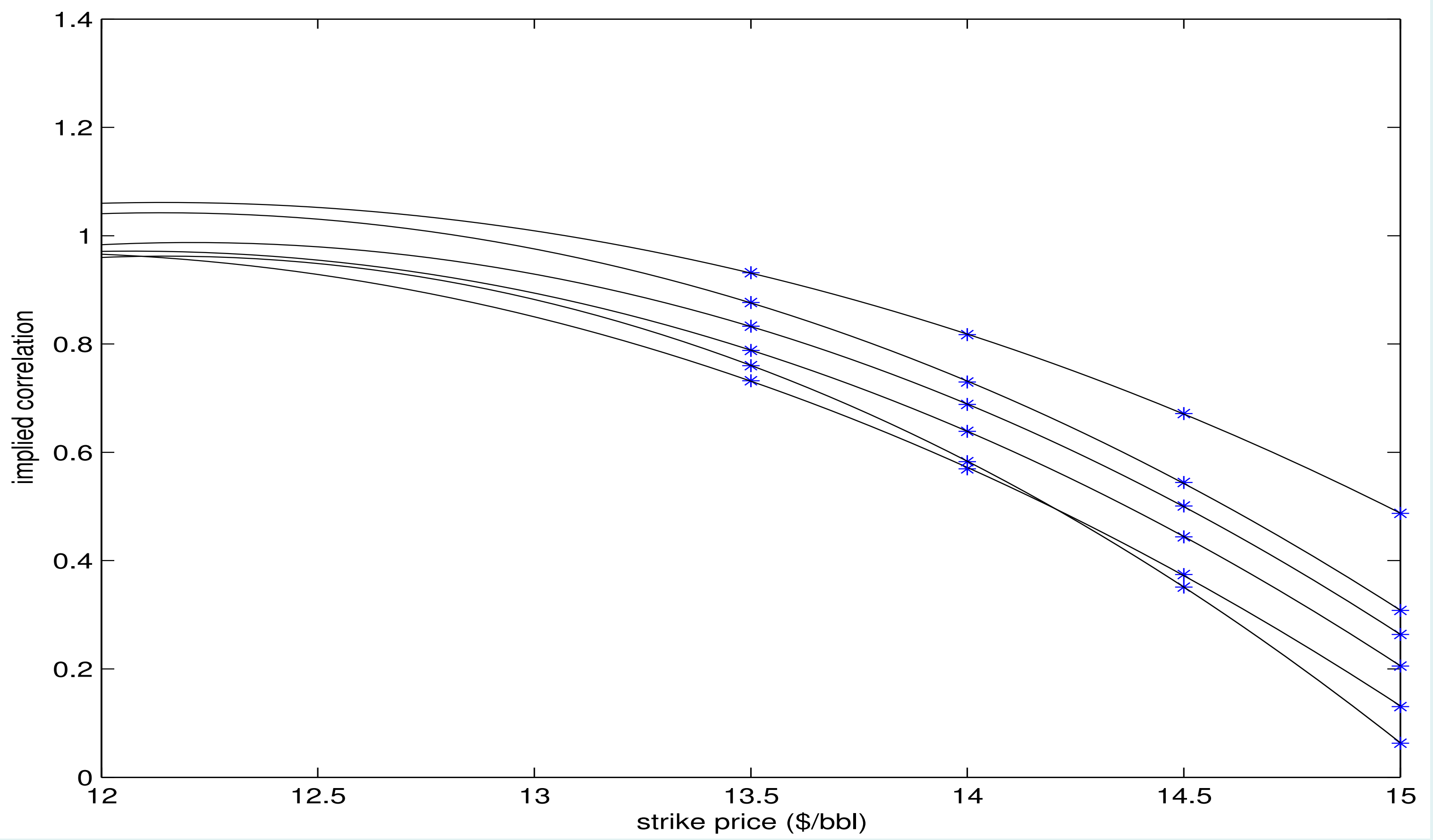
Implied correlations from spreads

For spreads, when liquid option prices are observed, the option price formula can be now inverted to obtain implied correlation (volatilities implied from individual assets' options).

Correlations implied from NYMEX Brent crude oil/heating oil Asian spread options:

Strike	Oct. 12	Oct. 13	Oct. 16	Oct. 17	Oct. 18	Oct. 19
15	0.15	0.21	0.48	0.26	0.31	0.10
14.5	0.37	0.44	0.67	0.51	0.54	0.35
14	0.57	0.64	0.82	0.69	0.73	0.59
13.5	*0.73*	*0.79*	*0.93*	*0.83*	*0.88*	*0.76*
12	0.96	0.97	***	0.98	***	0.96

Implied correlations vs strikes



Conclusions

Our proposed method:

- Has advantages of lognormal approximation
- Applicable to several assets, negative weights and Asian basket options
- Provides good approximation of option prices
- Gives closed-form expressions for the greeks
- Performs well on the basis of delta-hedging
- Allows to imply correlations from liquid spread options