An-Najah National University

Faculty of Engineering

Industrial Engineering Department

Course :

Quantitative Methods (65211)

Instructor:

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Chapter 4

Continuous Random Variables and Probability Distributions

4-1 CONTINUOUS RANDOM VARIABLES:

➢ **Example:**

- Measurement of the current in a thin copper (different results due to variation).
- Measuring a dimensional length of a part (Also different results due to variations).

The measurement of interest can be represented by a **RANDOM VARIABLE.**

It is reasonable to model the range of possible values of the random variable by an interval (finite or infinite) of real numbers.

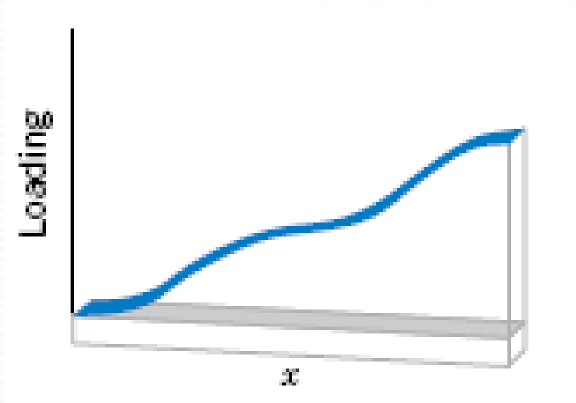
❖ However, because the number of possible values of the random variable X is uncountably infinite, X has a distinctly different distribution from the discrete random variables studied previously. The range of X includes all values in an interval of real numbers; that is, the range of X can be thought of as a continuum.

4-2 PROBABILITY DISTRIBUTION AND PROBABILITY DENSITY FUNCTION:

❖ Density functions are commonly used in engineering to describe physical systems.

➢ **Example:**

Consider the density of a loading on a long, thin beam as shown in following figure.



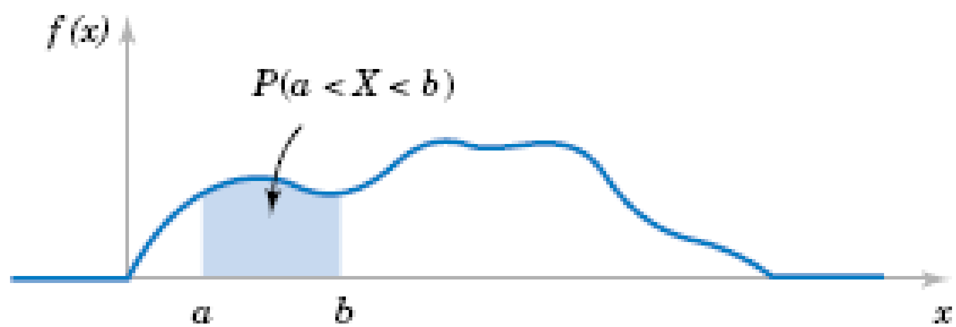
- For any point x along the beam, the density can be described by a function (in grams/cm).

- Intervals with large loadings correspond to large values for the function.

- The total loading between points a and b is determined as the integral of the density function from a to b → **AREA**

✓ Similarly, a **probability density function** f(x) can be used to describe the probability distribution of a **continuous random variable** X.

✓ The Probability that X is between a and b is determined as the integral of f(x) from a to b.



❖ **Probability Density Function:**

For a continuous random variable X, a **probability density function** is a function such that

(1) $f(x) \geq 0$

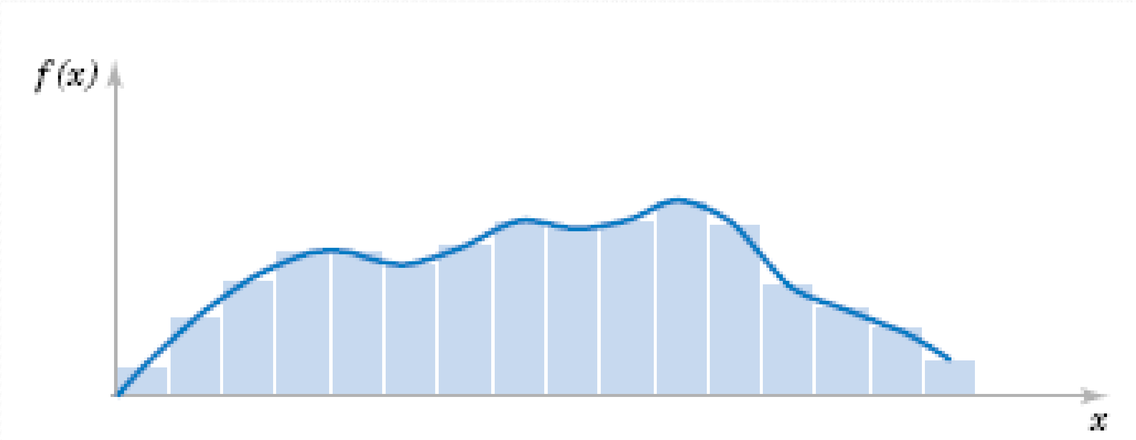
(2) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

(3) $P(a \leq X \leq b) = \int_{a}^{b} f(x)\,dx =$ area under $f(x)$ from a to b

for any a and b

✓ A probability density function provides a simple description of the probabilities associated with a random variable.

✓ A histogram is an approximation to a probability density function.



✓ For each interval of the histogram, the area of the bar equals the relative frequency (proportion) of the measurements in the interval.

✓ The relative frequency is an estimate of the probability that a measurement falls in the interval.

✓ Similarly, the area under *f(x) over any interval equals the* true probability that a measurement falls in the interval.

❖ The important point is that ***f(x) is used to calculate an area that represents the probability*** that X assumes a value in [a, b].

❖ By appropriate choice of the shape of *f(x),* we can represent the probabilities associated with any continuous random variable *X.*

❖For the density function of a loading on a long thin beam, because every point has zero width, the loading at any point is zero. Similarly, for a continuous random variable X and any value x.

$$P(X = x) = 0$$

➢ **Example:**

When a particular current measurement is observed, such as 14.47 milliamperes,

This result can be interpreted as the rounded value of a current measurement that is actually in a range such as

$14.465 \leq x \leq 14.475$ **(NOT ZERO)**

If X is a **continuous random variable**, for any x_1 and x_2,

$$P(x_1 \leq X \leq x_2) = P(x_1 < X \leq x_2) = P(x_1 \leq X < x_2) = P(x_1 < X < x_2)$$

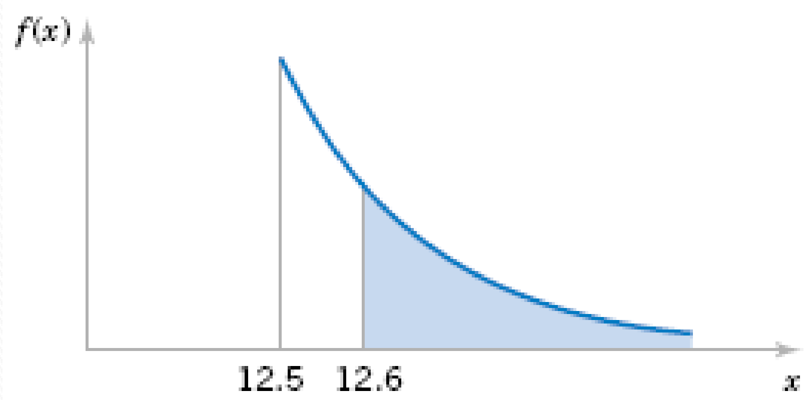
- **Example:**

Let the continuous random variable X denote the diameter of a hole drilled in a sheet metal component. The target diameter is 12.5 millimeters. Most random disturbances to the process result in larger diameters. Historical data show that the distribution of X can be modeled by a probability density function

$$f(x) = 20e^{-20(x-12.5)}, x \geq 12.5$$

$X \geq 12.5$

If a part with a diameter larger than 12.60 millimeters is scrapped, what proportion of parts is scrapped?



$$P(X > 12.60) = \int_{12.6}^{\infty} f(x)\, dx = \int_{12.6}^{\infty} 20e^{-20(x-12.5)}\, dx = -e^{-20(x-12.5)}\Big|_{12.6}^{\infty} = 0.135$$

What proportion of parts is between 12.5 and 12.6 millimeters?

$$P(12.5 < X < 12.6) = \int_{12.5}^{12.6} f(x)\, dx = -e^{-20(x-12.5)}\Big|_{12.5}^{12.6} = 0.865$$

OR

$$P(12.5 < X < 12.6) = 1 - P(X > 12.6) = 1 - 0.135 = 0.865.$$

❖ it is sometimes useful to be able to provide **cumulative probabilities** such as $P(X \leq x)$ and that such probabilities can be used to find the probability mass function of a random variable.

❖ Using cumulative probabilities is an alternate method of describing the probability distribution of a random variable.

In general, for any discrete random variable with possible values $x_1, x_2, \ldots, x_n$, the events $\{X = x_1\}, \{X = x_2\}, \ldots, \{X = x_n\}$ are mutually exclusive. Therefore, $P(X \leq x) = \sum_{x_i \leq x} f(x_i)$.

➢ Question 4-7.

The probability density function of the net weight in pounds of a packaged chemical herbicide is f(x) = 2.0 for 49.75 < x < 50.25 pounds.

a- Determine the probability that a package weighs more than 50 pounds.

$$P(X > 50) = \int_{50}^{50.25} 2.0\,dx = 2x\Big|_{50}^{50.25} = 0.5$$

b- How much chemical is contained in 90% of all packages?

$$P(X > x) = 0.90 = \int_{x}^{50.25} 2.0\,dx = 2x\Big|_{x}^{50.25} = 100.5 - 2x$$

Then, 2x = 99.6 and x = 49.8.

4-3 CUMULATIVE DISTRIBUTION FUNCTION:

❖ An alternative method to describe the distribution of a discrete random variable can also be used for continuous random variables.

❖ **Cumulative Distribution Function**

The **cumulative distribution function** of a continuous random variable X is

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(u)\, du$$

for $-\infty < x < \infty$.

➢ Example:

For the drilling operation in the last Example,

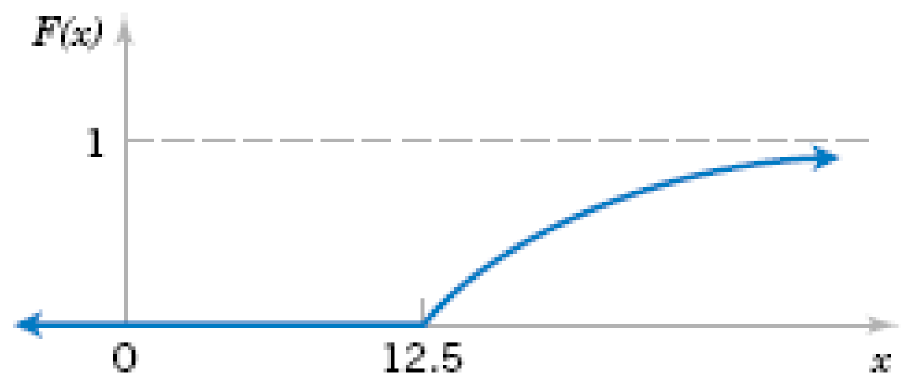
$$F(x) = 0 \quad \text{for} \quad x < 12.5$$

and for $12.5 \le x$

$$F(x) = \int_{12.5}^{x} 20e^{-20(u-12.5)}\,du$$
$$= 1 - e^{-20(x-12.5)}$$

Therefore,

$$F(x) = \begin{cases} 0 & x < 12.5 \\ 1 - e^{-20(x-12.5)} & 12.5 \le x \end{cases}$$



- **Example:**

For the drilling operation in the previous example, where

$$f(x) = 20e^{-20(x-12.5)}, x \geq 12.5$$

F(x) consists of two expressions,

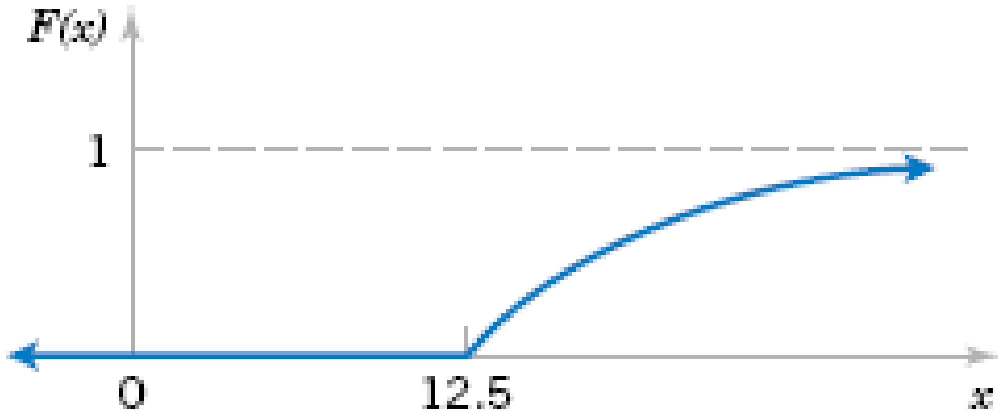
$$F(x) = 0 \quad \text{for} \quad x < 12.5$$

$$F(x) = \int_{12.5}^{x} 20e^{-20(u-12.5)}\, du$$

$$= 1 - e^{-20(x-12.5)} \qquad \text{for } 12.5 \leq x$$

Therefore,

$$F(x) = \begin{cases} 0 & x < 12.5 \\ 1 - e^{-20(x-12.5)} & 12.5 \leq x \end{cases}$$



❖ The probability density function of a continuous random variable can be determined from the cumulative distribution function by differentiating. Recall that the fundamental theorem of calculus states that

$$\frac{d}{dx}\int_{-\infty}^{x} f(u)\,du = f(x)$$

Then, given $F(x)$

$$f(x) = \frac{dF(x)}{dx}$$

As long as the derivative exists.

➢ Example:

The time until a chemical reaction is complete (in milliseconds) is approximated by the cumulative distribution function

$$F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-0.01x} & 0 \le x \end{cases}$$

-Determine the probability density function of X.

$$f(x) = \begin{cases} 0 & x < 0 \\ 0.01e^{-0.01x} & 0 \le x \end{cases}$$

- what proportion of reactions is completed within 200 milliseconds?

$$P(X < 200) = F(200) = 1 - e^{-2} = 0.8647.$$

Question 4-13.

The gap width is an important property of a magnetic recording head. In coded units, if the width is a continuous random variable over the range from 0<x<2 with

$$f(x) = 0.5x,$$

Determine the cumulative distribution function of the gap width.

$$F(x) = \int_0^x 0.5x\,dx = \left.\frac{0.5x^2}{2}\right|_0^x = 0.25x^2 \text{ for } 0 < x < 2. \text{ Then,}$$

$$F(x) = \begin{cases} 0, & x < 0 \\ 0.25x^2, & 0 \le x < 2 \\ 1, & 2 \le x \end{cases}$$

4-4 MEAN AND VARIANCE OF A CONTINUOUS RANDOM VARIABLE:

- Integration replaces summation in the discrete definitions.
- **Mean and Variance**

Suppose X is a continuous random variable with probability density function $f(x)$. The **mean** or **expected value** of X, denoted as μ or $E(X)$, is

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$$

The **variance** of X, denoted as $V(X)$ or σ^2, is

$$\sigma^2 = V(X) = \int_{-\infty}^{\infty} (x - \mu)^2 f(x)\, dx = \int_{-\infty}^{\infty} x^2 f(x)\, dx - \mu^2$$

The **standard deviation** of X is $\sigma = \sqrt{\sigma^2}$.

➢ Example:

For the copper wire example where $f(x) = 0.05$ for $0 \leq x \leq 20$ mA,

The mean of X is,

$$E(X) = \int_0^{20} x f(x)\, dx = 0.05x^2/2 \Big|_0^{20} = 10$$

The variance of X is,

$$V(X) = \int_0^{20} (x - 10)^2 f(x)\, dx = 0.05(x - 10)^3/3 \Big|_0^{20} = 33.33$$

❖ The expected value of a function h(X) of a continuous random variable is defined similarly to a function of a discrete random variable.

❖ **Expected Value of a Function of a Continuous Random Variable:**

If X is a continuous random variable with probability density function $f(x)$,

$$E[h(X)] = \int_{-\infty}^{\infty} h(x) f(x)\, dx$$

➢ **Example:**

For the copper wire example, what is the expected value of the squared current?

$h(x) = X^2$, therefore

$$E[h(X)] = \int_{-\infty}^{\infty} x^2 f(x)\, dx = \int_{0}^{20} 0.05x^2\, dx = 0.05 \frac{x^3}{3} \Big|_0^{20} = 133.33$$

➢ **Question 4-29.**

The thickness of a conductive coating in micrometers has a density function of $600x^{-2}$ for $100\mu m < x < 120\mu m$.

A- Determine the mean and variance of the coating thickness.

$$E(X) = \int_{100}^{120} x \frac{600}{x^2} dx = 600 \ln x \Big|_{100}^{120} = 109.39$$

$$V(X) = \int_{100}^{120} (x - 109.39)^2 \frac{600}{x^2} dx = 600 \int_{100}^{120} 1 - \frac{2(109.39)}{x} + \frac{(109.39)^2}{x^2} dx$$

$$= 600(x - 218.78 \ln x - 109.39^2 x^{-1}) \Big|_{100}^{120} = 33.19$$

B- If the coating costs $0.50 per micrometer of thickness on each part, what is the average cost of the coating per part?

Average cost per part = $0.50*109.39 = $54.70

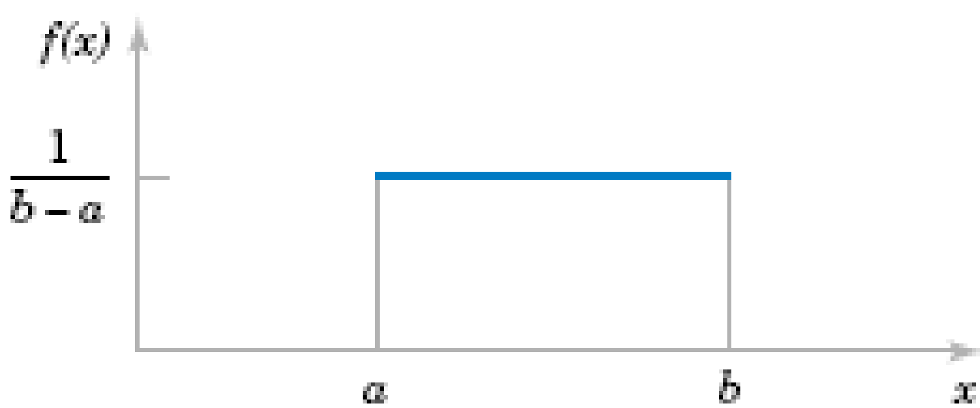
4-5 CONTINUOUS UNIFORM DISTRIBUTION:

❖ The simplest continuous distribution is analogous to its discrete counterpart.

A continuous random variable X with probability density function

$$f(x) = 1/(b - a), \qquad a \le x \le b$$

is a **continuous uniform random variable.**



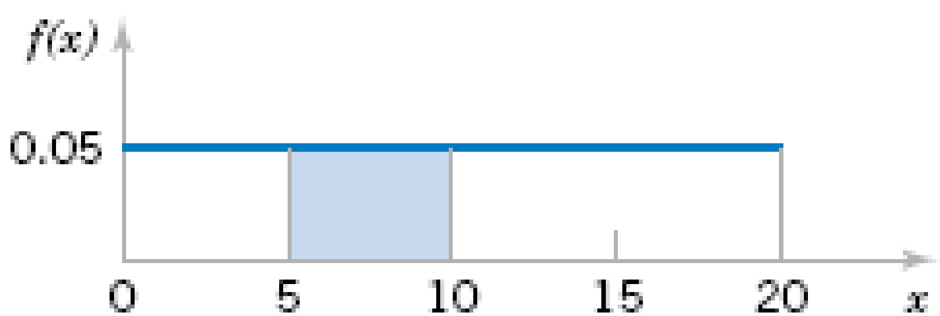
If X is a continuous uniform random variable over $a \le x \le b$,

$$\mu = E(X) = \frac{(a+b)}{2} \quad \text{and} \quad \sigma^2 = V(X) = \frac{(b-a)^2}{12}$$

➢ **Example:**

Let the continuous random variable X denote the current measured in a thin copper wire in milliamperes. Assume that the range of X is [0, 20 mA], and assume that the probability density function of X is f(x) = 0.05, $0 \le x \le 20$.

- What is the probability that a measurement of current is between 5 and 10 milliamperes?



$$P(5 < X < 10) = \int_5^{10} f(x)\,dx$$

$$= 5(0.05) = 0.25$$

The mean and variance formulas can be applied with a=0 and b=20. Therefore,

$$E(X) = 10 \text{ mA} \quad \text{and} \quad V(X) = 20^2/12 = 33.33 \text{ mA}^2$$

S.D. = 5.77 mA.

The Cumulative distribution function is: (If a < x < b)

$$F(x) = \int_a^x 1/(b-a)\,du = x/(b-a) - a/(b-a)$$

Therefore, the complete description of the cumulative distribution function of a continuous uniform random variable is:

$$F(x) = \begin{cases} 0 & x < a \\ (x-a)/(b-a) & a \le x < b \\ 1 & b \le x \end{cases}$$

➢ **Question 4-36.**

Suppose the time it takes a data collection operator to fill out an electronic form for a database is uniformly between 1.5 and 2.2 minutes.

A- What is the mean and variance of the time it takes an operator to fill out the form?

$E(X) = (1.5+2.2)/2 = 1.85$ min

$V(X) = (2.2-1.5)^2/12 = 0.0408\ \text{min}^2$

B- What is the probability that it will take less than two minutes to fill out the form?

$$P(X < 2) = \int_{1.5}^{2} \frac{1}{(2.2-1.5)} dx = \int_{1.5}^{2} 0.7 dx = 0.7x\Big|_{1.5}^{2} = 0.7(.5) = 0.7143$$

C- Determine the cumulative distribution function of the time it takes to fill out the form.

$$F(X) = \int_{1.5}^{x} \frac{1}{(2.2-1.5)} dx = \int_{1.5}^{x} 0.7 dx = 0.7x\Big|_{1.5}^{x} \quad \text{for } 1.5 < x < 2.2. \text{ Therefore,}$$

$$F(x) = \begin{cases} 0, & x < 1.5 \\ 0.7x - 2.14, & 1.5 \le x < 2.2 \\ 1, & 2.2 \le x \end{cases}$$

4-6 NORMAL DISTRIBUTION:

❖ Undoubtedly, the most widely used model for the distribution of a random variable is a ***normal distribution***.

❖ Whenever a random experiment is replicated, the random variable that equals the average (or total) result over the replicates tends to have a normal distribution as the number of replicates becomes large.

CENTRAL LIMIT THEORIM

❖ Normal Distribution is also referred to as a ***Gaussian distribution***.

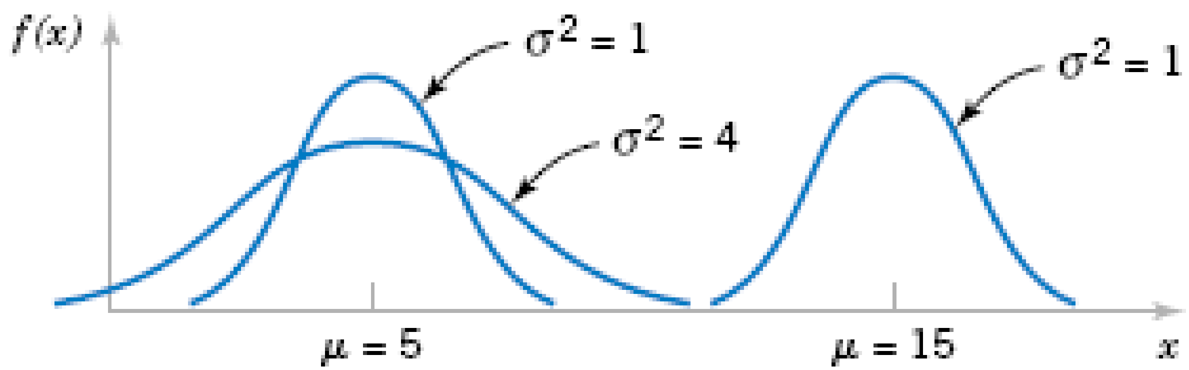
➢ Example:

- Assume that the deviation (or error) in the length of a machined part is the sum of a large number of infinitesimal effects, such as:

- Temperature and humidity drifts, vibrations, cutting angle variations, cutting tool wear, bearing wear, rotational speed variations, mounting and fixture variations, variations in numerous raw material characteristics, and variation in levels of contamination.

- If the component errors are independent and equally likely to be positive or negative, the total error can be shown to have an approximate normal distribution.

❖ Random variables with different means and variances can be modeled by normal probability density functions with appropriate choices of the center and width of the curve.



-The Center of the probability density function $E(X) = \mu$
- The width of the curve $V(X) = \sigma^2$

❖ All Normal functions has the characteristic symmetric bell-shaped curve, but the centers and dispersions differ.

❖ Normal Distribution:

A random variable X with probability density function

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}} \qquad -\infty < x < \infty$$

is a **normal random variable** with parameters μ, where $-\infty < \mu < \infty$, and $\sigma > 0$. Also,

$$E(X) = \mu \quad \text{and} \quad V(X) = \sigma^2$$

and the notation $N(\mu, \sigma^2)$ is used to denote the distribution.

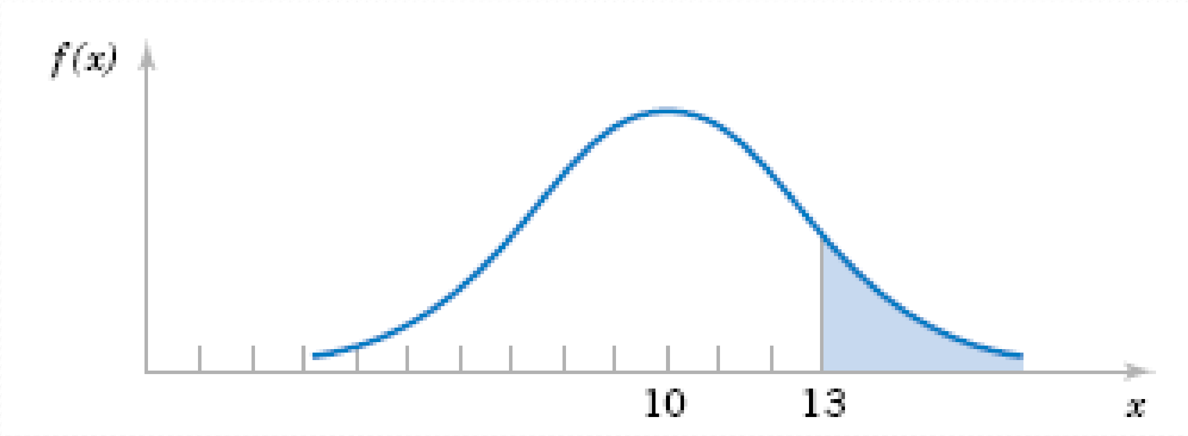
➢ Example:

Assume that the current measurements in a strip of wire follow a normal distribution with a mean of 10 milliamperes and a variance of 4 (milliamperes)^2.

-What is the probability that a measurement exceeds 13 milliamperes?

$P(X > 13) = ?????$

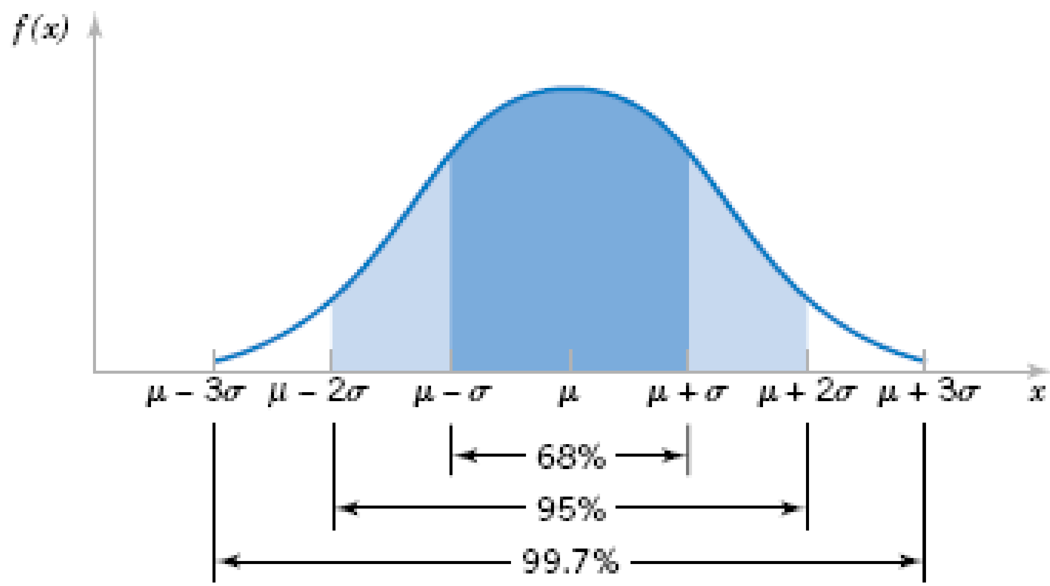
X: the current in mA.



✓ There is no closed-form expression for the integral of a normal probability density function, and probabilities based on the normal distribution are typically found numerically or from a table (that we will later introduce).

❖ Some useful results concerning a normal distribution are as follows:

$$P(\mu - \sigma < X < \mu + \sigma) = 0.6827$$
$$P(\mu - 2\sigma < X < \mu + 2\sigma) = 0.9545$$
$$P(\mu - 3\sigma < X < \mu + 3\sigma) = 0.9973$$



❖ From the symmetry of f(x), $P(X > \mu) = P(X < \mu) = 0.5$

❖ The probability density function decreases as x moves farther from Consequently, the probability that a measurement falls far from μ is small, and at some distance from μ the probability of an interval can be approximated as zero.

❖ Because more than 0.9973 of the probability of a normal distribution is within the interval **($\mu - 3\sigma, \mu + 3\sigma$)** → **$6\sigma$** is often referred as the **width** of a normal distribution

❖ The Integration of normal probability density function between **($-\infty < x < +\infty$)** = 1

❖ Standard Normal Random Variable:

A normal random variable with

$$\mu = 0 \quad \text{and} \quad \sigma^2 = 1$$

is called a **standard normal random variable** and is denoted as Z.

The cumulative distribution function of a standard normal random variable is denoted as

$$\Phi(z) = P(Z \leq z)$$

❖ Appendix Table II provides cumulative probability values for $\phi(z)$, for a standard normal random variable.

$$\Phi(z) = P(Z \le z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2}\, du$$

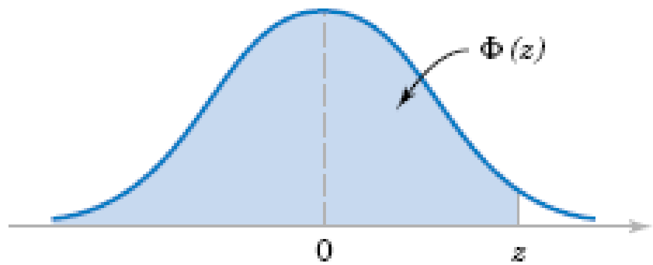


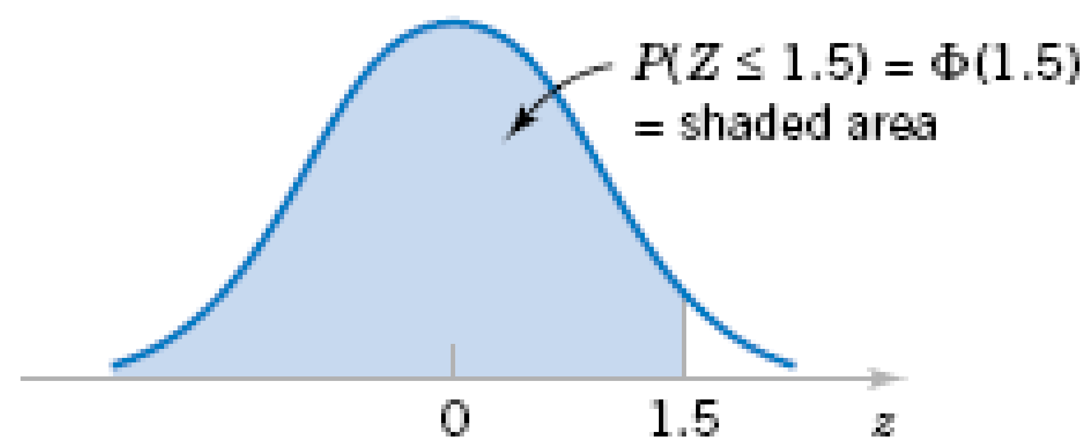
Table II Cumulative Standard Normal Distribution (*continued*)

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.500000	0.503989	0.507978	0.511967	0.515953	0.519939	0.532922	0.527903	0.531881	0.535856
0.1	0.539828	0.543795	0.547758	0.551717	0.555760	0.559618	0.563559	0.567495	0.571424	0.575345
0.2	0.579260	0.583166	0.587064	0.590954	0.594835	0.598706	0.602568	0.606420	0.610261	0.614092
0.3	0.617911	0.621719	0.625516	0.629300	0.633072	0.636831	0.640576	0.644309	0.648027	0.651732
0.4	0.655422	0.659097	0.662757	0.666402	0.670031	0.673645	0.677242	0.680822	0.684386	0.687933
0.5	0.691462	0.694974	0.698468	0.701944	0.705401	0.708840	0.712260	0.715661	0.719043	0.722405
0.6	0.725747	0.729069	0.732371	0.735653	0.738914	0.742154	0.745373	0.748571	0.751748	0.754903
0.7	0.758036	0.761148	0.764238	0.767305	0.770350	0.773373	0.776373	0.779350	0.782305	0.785236
0.8	0.788145	0.791030	0.793892	0.796731	0.799546	0.802338	0.805106	0.807850	0.810570	0.813267
0.9	0.815940	0.818589	0.821214	0.823815	0.826391	0.828944	0.831472	0.833977	0.836457	0.838913
1.0	0.841345	0.843752	0.846136	0.848495	0.850830	0.853141	0.855428	0.857690	0.859929	0.862143

➢ **Example:**

Find P(Z ≤ 1.5) ????

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
1.3	0.903199	0.904902	0.906582	0.908241	0.909877	0.911492	0.913085	0.914657	0.916207	0.917736
1.4	0.919243	0.920730	0.922196	0.923641	0.925066	0.926471	0.927855	0.929219	0.930563	0.931888
1.5	0.933193	0.934478	0.935744	0.936992	0.938220	0.939429	0.940620	0.941792	0.942947	0.944083
1.6	0.945201	0.946301	0.947384	0.948449	0.949497	0.950529	0.951543	0.952540	0.953521	0.954486

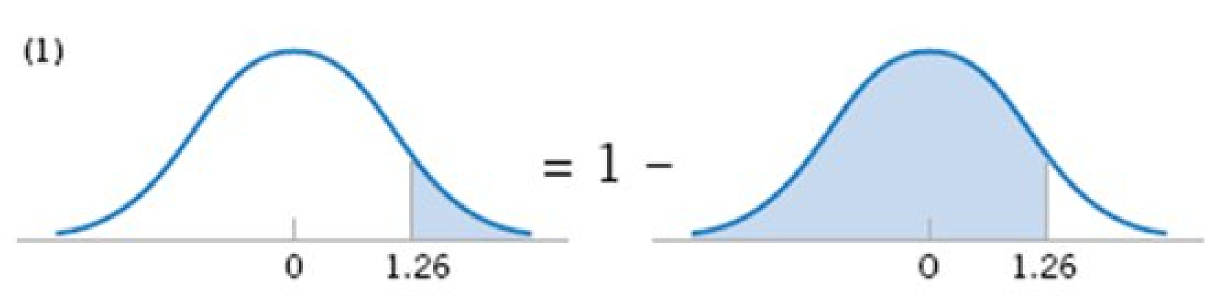


P(Z ≤ 1.5) = 0.933193

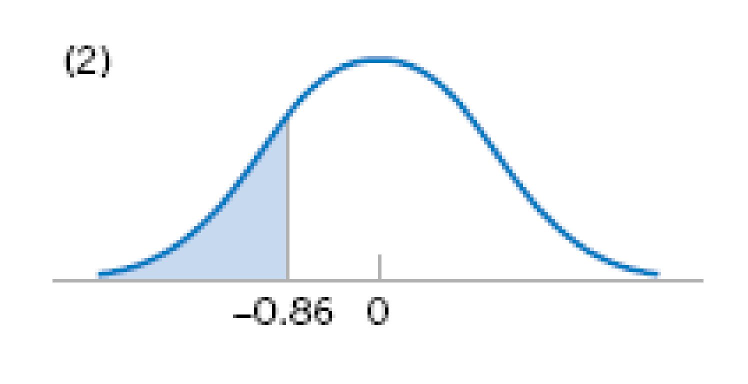
P(Z ≤ 1.36) = 0.913085

- **Example:**

1- $P(Z > 1.26) = 1 - P(Z \le 1.26) = 1 - 0.89616 = 0.10384$

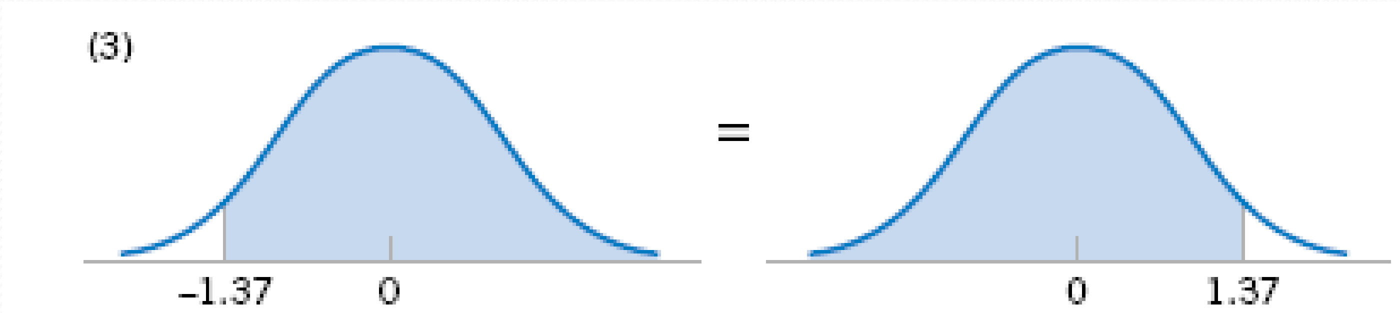


2- $P(Z < -0.86) = 0.19490$

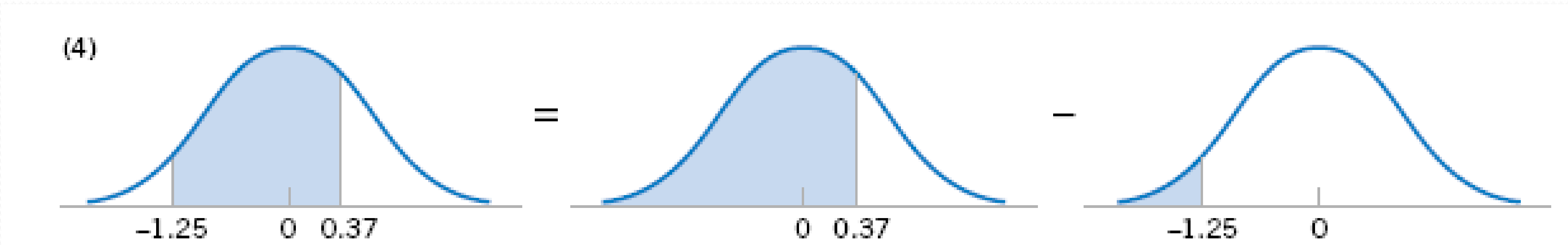


z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
−0.8	0.186733	0.189430	0.192150	0.194894	0.197662	0.200454	0.203269	0.206108	0.208970	0.211855
−0.7	0.214764	0.217695	0.220650	0.223627	0.226627	0.229650	0.232695	0.235762	0.238852	0.241964

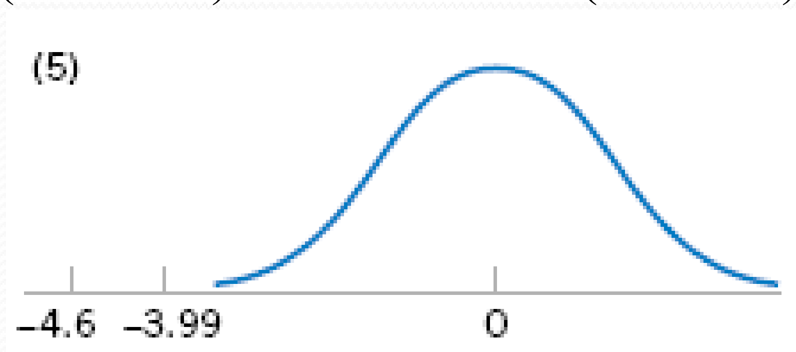
3- $P(Z > -1.37) = P(Z < 1.37) = 0.91465$



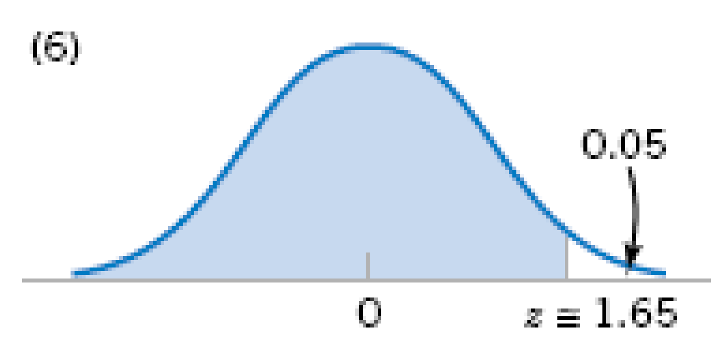
4- $P(-1.25 < Z < 0.37) = P(Z < 0.37) – P(Z < -1.25) = 0.64431 – 0.10565 = 0.53866$



5- $P(Z \leq - 4.6)$ Cannot be found exactly from the standard table.
From Table we can find $P(Z \leq -3.99)$ 0.00003 so that $(Z \leq - 4.6)$ nearly **ZERO**



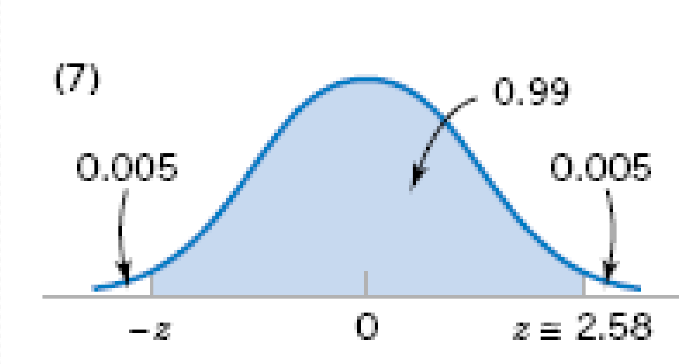
6- Find the value of z such that P(Z > z) = 0.05



Find P(Z < z) = 0.95 then from table….. Z = 1.65

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
1.5	0.933193	0.934478	0.935744	0.936992	0.938220	0.939429	0.940620	0.941792	0.942947	0.944083
1.6	0.945201	0.946301	0.947384	0.948449	0.949497	0.950529	0.951543	0.952540	0.953521	0.954486
1.7	0.955435	0.956367	0.957284	0.958185	0.959071	0.959941	0.960796	0.961636	0.962462	0.963273

7- Find the value of z such that P(-z < Z < z) = 0.99



Z = 2.58

❖ The standard table can be used to find the probabilities associated with an arbitrary normal variable by first using a simple transformation.

❖ Standardizing a Normal Random Variable:

If X is a normal random variable with $E(X) = \mu$ and $V(X) = \sigma^2$, the random variable

$$Z = \frac{X - \mu}{\sigma}$$

is a normal random variable with $E(Z) = 0$ and $V(Z) = 1$. That is, Z is a standard normal random variable.

❖ The random variable Z represents the distance of X from its mean in terms of standard deviations. It is the key step to calculate a probability for an arbitrary normal random variable.

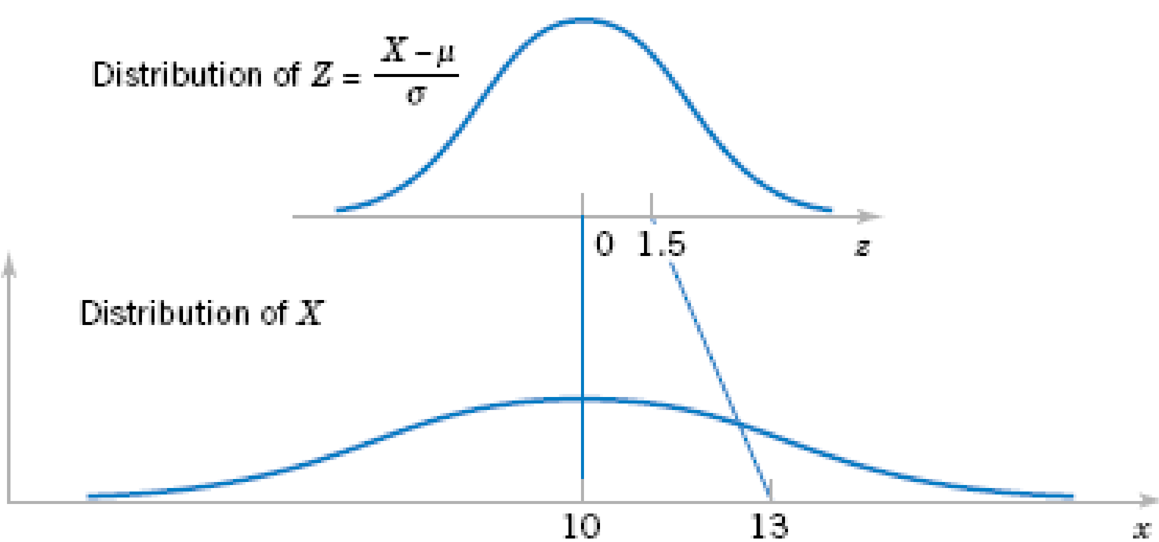
- **Example:**

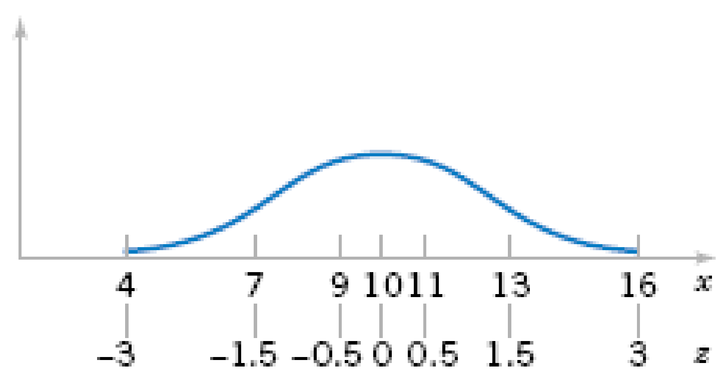
Assume that the current measurements in a strip of wire follow a normal distribution with a mean of 10 milliamperes and a variance of 4 (milliamperes)^2.

-What is the probability that a measurement exceeds 13 milliamperes?

$$P(X > 13) = P\left(\frac{(X - 10)}{2} > \frac{(13 - 10)}{2}\right) = P(Z > 1.5) = 0.06681$$

$$P(X > 13) = P(Z > 1.5) = 1 - P(Z \leq 1.5) = 1 - 0.93319 = 0.06681$$





❖ Standardizing to Calculate a Probability:

Suppose X is a normal random variable with mean μ and variance σ^2. Then,

$$P(X \leq x) = P\left(\frac{X-\mu}{\sigma} \leq \frac{x-\mu}{\sigma}\right) = P(Z \leq z)$$

where Z is a **standard normal random variable,** and $z = \frac{(x-\mu)}{\sigma}$ is the **z-value** obtained by **standardizing** X.

The probability is obtained by entering Appendix Table II with $z = (x-\mu)/\sigma$.

➢ **Example:**

For the previous example, what is the probability that a current measurement is between 9 and 11 mA?

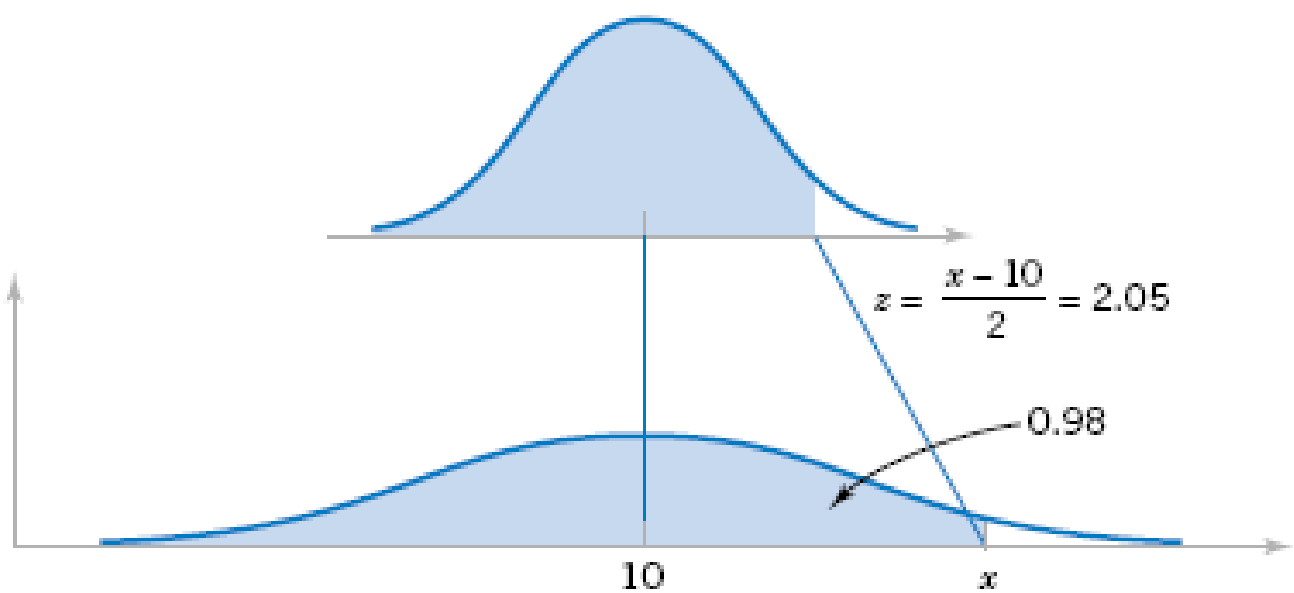
$$\begin{aligned} P(9 < X < 11) &= P((9-10)/2 < (X-10)/2 < (11-10)/2) \\ &= P(-0.5 < Z < 0.5) = P(Z < 0.5) - P(Z < -0.5) \\ &= 0.69146 - 0.30854 = 0.38292 \end{aligned}$$

- Determine the value for which the probability that a current measurement is below this value is 0.98.

$$\begin{aligned} P(X < x) &= P((X - 10)/2 < (x - 10)/2) \\ &= P(Z < (x - 10)/2) \\ &= 0.98 \end{aligned}$$

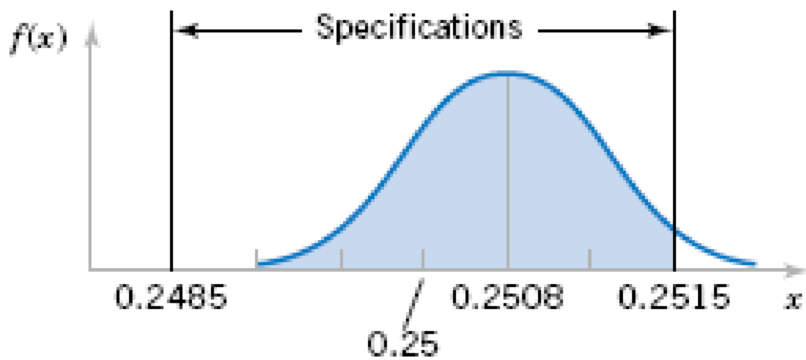
$$P(Z < 2.05) = 0.97982$$

$$x = 2(2.05) + 10 = 14.1 \text{ milliamperes}$$



➢ **Example:**

The diameter of a shaft in an optical storage drive is normally distributed with mean 0.2508 inch and standard deviation 0.0005 inch. The specifications on the shaft are 0.2500 + or - 0.0015 inch. What proportion of shafts conforms to specifications?



Let X denote the shaft diameter in inches.

$$P(0.2485 < X < 0.2515) = P\left(\frac{0.2485 - 0.2508}{0.0005} < Z < \frac{0.2515 - 0.2508}{0.0005}\right)$$
$$= P(-4.6 < Z < 1.4) = P(Z < 1.4) - P(Z < -4.6)$$
$$= 0.91924 - 0.0000 = 0.91924$$

✓ Find the probability when the process is centered………………..

Question 4-49.

The compressive strength of samples of cement can be modeled by a normal distribution with a mean of 6000 kilograms per square centimeter and a standard deviation of 100 kilograms per square centimeter.

a) What is the probability that a sample's strength is less that 6250 Kg/cm^2?

$$P(X < 6250) = P\left(Z < \frac{6250 - 6000}{100}\right)$$
$$= P\,(Z < 2.5)$$
$$= 0.99379$$

b) What is the probability that a sample's strength is between 5800 and 5900 Kg/cm^2?

$$P\,(5800 < X < 5900) = P\left(\frac{5800 - 6000}{100} < Z < \frac{5900 - 6000}{100}\right)$$
$$= P\,(-2 < Z < -1)$$
$$= P\,(Z < -1) - P(Z < -2)$$
$$= 0.13591$$

c) What strength is exceeded by 95% of the samples?

$$P(X > x) = P\left(Z > \frac{x-6000}{100}\right) = 0.95.$$

$$\text{Therefore, } \frac{x-6000}{100} = -1.65 \text{ and } x = 5835.$$

➢ **Question 4-63.**

The weight of a sophisticated running shoe is normally distributed with a mean of 12 ounces and a standard deviation of 0.5 ounce.

a) What is the probability that a shoe weights more that 13 ounces?

$$P(X > 13) = P\left(Z > \frac{13-12}{0.5}\right)$$

$$= P(Z > 2)$$

$$= 0.02275$$

b) What must the standard deviation of weight be in order for the company to state that 99.9% of its shoes are less that 13 ounces?

If $P(X < 13) = 0.999$, then $P\left(Z < \frac{13-12}{\sigma}\right) = 0.999$.

Therefore, $1/\sigma = 3.09$ and $\sigma = 1/3.09 = 0.324$.

c) If the standard deviation remains at 0.5 ounce, what must be the mean weight in order for the company to state that 99.9% of its shoes are less that 13 ounces?

If $P(X < 13) = 0.999$, then $P\left(Z < \frac{13-\mu}{0.5}\right) = 0.999$.

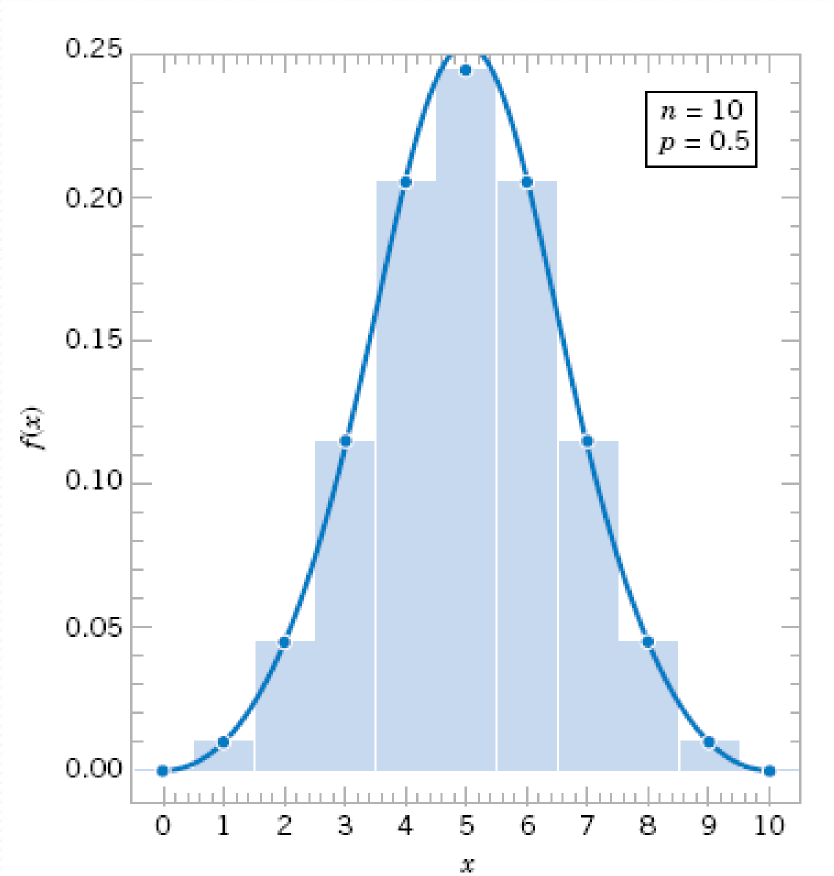
Therefore, $\frac{13-\mu}{0.5} = 3.09$ and $\mu = 11.455$

4-7 NORMAL APPROXIMATION TO THE BINOMIAL AND POISSONDISTRIBUTION:

❖ The normal distribution can be used to approximate binomial probabilities for cases in which n is large.

❖ In some systems, it is difficult to calculate probabilities by using the binomial distribution.

➢ **Example:**



❖ The area of each bar equals the binomial probability of x.

❖ Notice that the area of bars can be approximated by areas under the normal density function.

❖ From the last figure, it can be seen that a probability such as $P(3 \leq X \leq 7)$ is better approximated by the area under the normal curve from 2.5 to 7.5.

❖ This observation provides a method to improve the approximation of binomial probabilities.

The modification is referred to a ***CONTINUITY CORRECTION***

➢ Example:

In a digital communication channel, assume that the number of bits received in error can be modeled by a binomial random variable, and assume that the probability that a bit is received in error is $1X10^{-5}$. If 16 million bits are transmitted, what is the probability that 150 or fewer errors occur?

$$P(x \leq 150) = \sum_{x=0}^{150} \binom{16,000,000}{x} (10^{-5})^x (1 - 10^{-5})^{16,000,000-x}$$

❖ Normal Approximation to the Binomial Distribution:

If X is a binomial random variable,

$$Z = \frac{X - np}{\sqrt{np(1-p)}}$$

is approximately a standard normal random variable. The approximation is good for

$$np > 5 \quad \text{and} \quad n(1-p) > 5$$

To approximate a binomial probability with a normal distribution a continuity correction is applied as follows:

$$P(X \le x) = P(X \le x+0.5) \approx P(Z \le (x + 0.5 - np)/[\sqrt{np(1-p)}])$$

and

$$P(x \le X) = P(x - 0.5 \le X) \approx P((x - 0.5 - np)/[\sqrt{np(1-p)}] \le Z)$$

- **Example:**

The digital communication problem in the previous example is solved as follows:

$$P(X \le 150) = P(X \le 150.5) = P\left(\frac{X - 160}{\sqrt{160(1 - 10^{-5})}} \le \frac{150.5 - 160}{\sqrt{160(1 - 10^{-5})}}\right)$$

$$= P(Z \le -0.75) = 0.227$$

➤ Example:

Again consider the transmission of bits. To judge how well the normal approximation works, assume only n = 50 bits are to be transmitted and that the probability of an error is p = 0.1. The exact probability that 2 or less errors occur is

$$P(X \le 2) = \binom{50}{0} 0.9^{50} + \binom{50}{1} 0.1(0.9^{49}) + \binom{50}{2} 0.1^2(0.9^{48}) = 0.112$$

$$P(X \le 2) = P(X \le 2.5) \approx P\left(Z \le \frac{2 + 0.5 - 5}{2.12}\right) = P(Z \le -1.18) = 0.119$$

Find P(X ≥ 9) = ??????

$$P(9 \le X) = P(8.5 \le X) \approx P\left(\frac{9 - 0.5 - 5}{2.12} \le Z\right) = P(1.65 \le Z) = 0.05$$

❖Conditions for approximating hypergeometric and binomial probabilities:

hypergometric distribution	$\approx$	binomial distribution	$\approx$	normal distribution
	$\frac{n}{N} < 0.1$		$np > 5$ $n(1-p) > 5$	

❖Recall that the Poisson distribution was developed as the limit of a binomial distribution as the number of trials increased to infinity. So that, the normal distribution can also be used to approximate probabilities of a Poisson random variable.

❖ **Normal Approximation to the Poisson Distribution:**

If X is a Poisson random variable with $E(X) = \lambda$ and $V(X) = \lambda$,

$$Z = \frac{X - \lambda}{\sqrt{\lambda}}$$

is approximately a standard normal random variable. The approximation is good for

$$\lambda > 5$$

➢ Example:

Assume that the number of asbestos particles in a squared meter of dust on a surface follows a Poisson distribution with a mean of 1000. If a squared meter of dust is analyzed, what is the probability that less than 950 particles are found?

$$P(X \leq 950) = \sum_{x=0}^{950} \frac{e^{-1000} x^{1000}}{x!}$$

$$P(X \leq x) = P\left(Z \leq \frac{950 - 1000}{\sqrt{1000}}\right) = P(Z \leq -1.58) = 0.057$$

4-8 EXPONENTIAL DISTRIBUTION:

❖ The discussion of the Poisson distribution defined a random variable to be the number of flaws along a length of copper wire.

❖ The distance between flaws is another random variable that is often of interest.

❖ Let the random variable X denote the length from any starting point on the wire until a flaw is detected.

❖ The distribution of X can be obtained from knowledge of the distribution of the number of flaws. For example:

The distance to the first flaw exceeds 3 millimeters if and only if there are no flaws within a length of 3 millimeters

❖ In general, let the random variable N denote the number of flaws in x millimeters of wire. If the mean number of flaws is λ per millimeter, N has a Poisson distribution with mean λx.

❖ We assume that the wire is longer than the value of *x. Now,*

$$P(X > x) = P(N = 0) = \frac{e^{-\lambda x}(\lambda x)^0}{0!} = e^{-\lambda x}$$

$$F(x) = P(X \le x) = 1 - e^{-\lambda x}, \quad x \ge 0$$

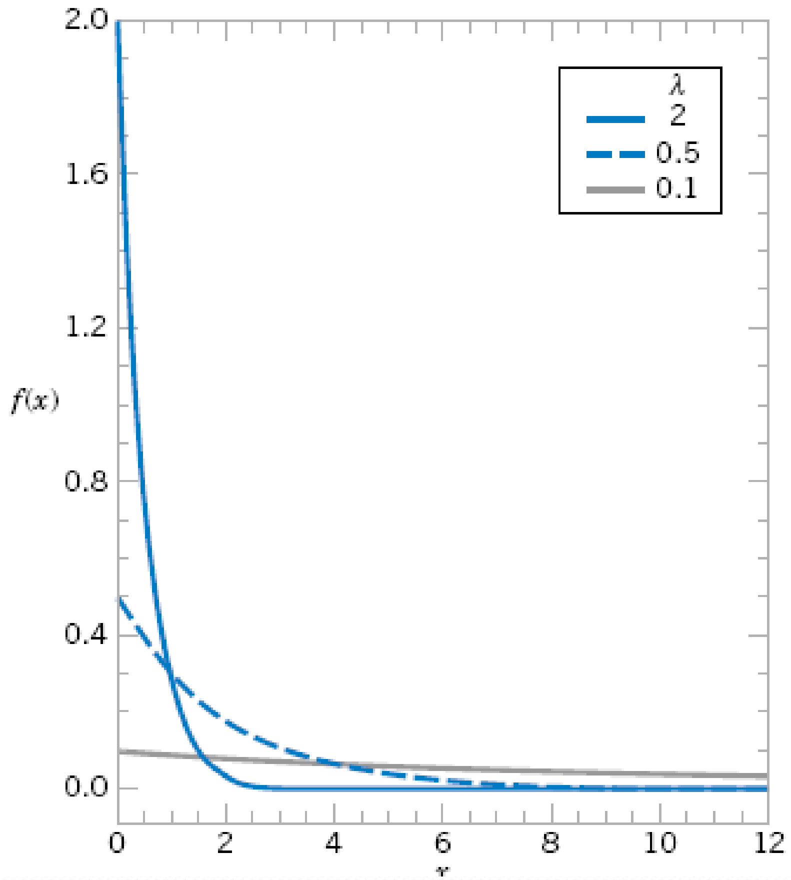
$$f(x) = \lambda e^{-\lambda x}, \quad x \ge 0$$

❖ The derivation of the distribution of X depends only on the assumption that the flaws in the wire follow a **Poisson process**.

❖Exponential Distribution:

The random variable X that equals the distance between successive counts of a Poisson process with mean $\lambda > 0$ is an **exponential random variable** with parameter λ. The probability density function of X is

$$f(x) = \lambda e^{-\lambda x} \quad \text{for} \quad 0 \le x < \infty$$



❖Exponential Distribution Mean and Variance:

If the random variable X has an exponential distribution with parameter λ,

$$\mu = E(X) = \frac{1}{\lambda} \quad \text{and} \quad \sigma^2 = V(X) = \frac{1}{\lambda^2}$$

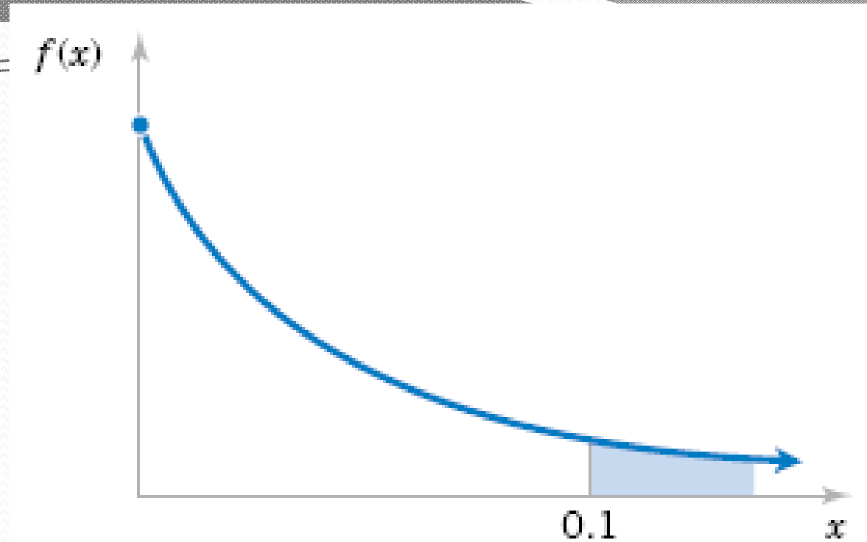
➢ **Example:**

In a large corporate computer network, user log-ons to the system can be modeled as a Poisson process with a mean of 25 log-ons per hour. What is the probability that there are no logons in an interval of 6 minutes?

Let X denote the time in hours from the start of the interval until the first log-on.

X has an exponential distribution with $\lambda = 25$ log-ones/hr

Find the probability of X exceeds 6 min or 0.1 hour



$$P(X > 0.1) = \int_{0.1}^{\infty} 25e^{-25x}\,dx = e^{-25(0.1)} = 0.082$$

$$P(X > 0.1) = 1 - F(0.1) = e^{-25(0.1)}$$

What is the probability that the time until the next log-on is between 2 and 3 minutes?

$$P(0.033 < X < 0.05) = \int_{0.033}^{0.05} 25e^{-25x}\,dx = -e^{-25x}\Big|_{0.033}^{0.05} = 0.152$$

$$P(0.033 < X < 0.05) = F(0.05) - F(0.033) = 0.152$$

Determine the interval of time such that the probability that no log-on occurs in the interval is 0.90.

$$P(X > x) = e^{-25x} = 0.90$$

$$x = 0.00421 \text{ hour} = 0.25 \text{ minute}$$

Furthermore, the mean time until the next log-on is

$$\mu = 1/25 = 0.04 \text{ hour} = 2.4 \text{ minutes}$$

The standard deviation of the time until the next log-on is

$$\sigma = 1/25 \text{ hours} = 2.4 \text{ minutes}$$