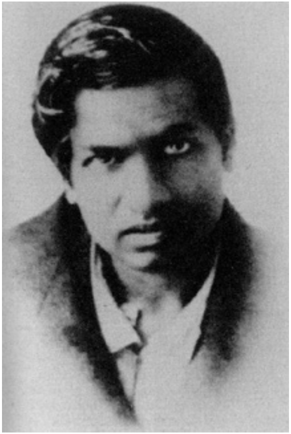
The legacy of Ramanujan's mock theta functions: Harmonic Maass forms in number theory

Ken Ono
Emory University



Srinivasa Ramanujan (1887-1920)

"Death bed letter"

Dear Hardy,

"I am extremely sorry for not writing you a single letter up to now. . . . I discovered very interesting functions recently which I call "Mock" ϑ-functions. . . . they enter into mathematics as beautifully as the ordinary theta functions. I am sending you with this letter some examples."

Ramanujan, January 12, 1920.

Some examples

$$f(q) := 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2 \cdots (1+q^n)^2},$$

$$\omega(q) := \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(1-q)^2(1-q^3)^2 \cdots (1-q^{2n+1})^2},$$

$$\lambda(q) := \sum_{n=0}^{\infty} \frac{(-1)^n(1-q)(1-q^3)\cdots(1-q^{2n-1})q^n}{(1+q)(1+q^2)\cdots(1+q^{n-1})}.$$

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- Works by Atkin, Andrews, Dyson, Selberg, Swinnerton-Dyer, and Watson on these **22 series**.

Aftermath of the letter

Although Ramanujan's secrets died with him, we have:

- Works by Atkin, Andrews, Dyson, Selberg, Swinnerton-Dyer, and Watson on these **22 series**.
- Bolster the view that Ramanujan had found something.

G. N. Watson's 1936 Presidential Address

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"Ramanujan's discovery of the mock theta functions makes it obvious that his skill and ingenuity did not desert him at the oncoming of his untimely end.

As much as any of his earlier work..., the mock theta functions are an achievement sufficient to cause his name to be held in lasting remembrance. ..."

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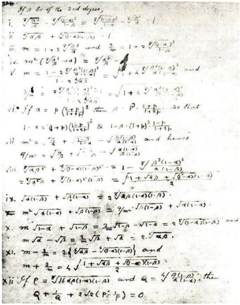
Andrews unearths the "Lost Notebook" (1976)



Forgotten in the Trinity College archives.

"Lost Notebook" identities useful for...

- Hypergeometric functions
- Partitions and Additive Number Theory
- Mordell integrals
- Artin L-functions
- Mathematical Physics
- Probability theory. . .

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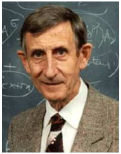
"Mock theta-functions give us tantalizing hints of a grand synthesis still to be discovered... This remains a challenge for the future."

Freeman Dyson, 1987



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In his Ph.D. thesis under Zagier ('02), Zwegers investigated:

- "Lerch-type" series and Mordell integrals.
- Resembling q-series of Andrews and Watson on mock thetas.
- Stitched them together give *non-holomorphic Jacobi forms*.

Important Realizations

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- Previously thought to be difficult to construct.
- ...giving clues of **general theory** which in turn have **applications**.

Some applications

Partition Theory and q-series.

- q-series identities ("mock theta conjectures")
- Congruences (Dyson's ranks)
- Exact formulas

Arithmetic and Modular forms.

- Donaldson invariants
- Eichler-Shimura Theory
- *Moonshine* for affine Lie superalgebras
- Borcherds-type automorphic products
- L-functions and the BSD numbers

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Extend the scope of Serre, Swinnerton-Dyer, Deligne, Ribet....

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Extend and generalize phenomena obtained previously by Rademacher and Zagier*.

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I. (**Maass form congruences**)
Extend the scope of Serre, Swinnerton-Dyer, Deligne, Ribet....

II and III. (**Exact formulas for Maass forms**)
Extend and generalize phenomena obtained previously by Rademacher and Zagier*.

IV. (**Birch and Swinnerton-Dyer Numbers**)
Unify work of Waldspurger and Gross-Zagier on BSD numbers $+\epsilon$.

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- Along the way, I will explain some of the essential features (e.g. definitions) of the theory.

Adding and counting

Definition

A *partition* is any nonincreasing sequence of integers summing to n.

$$p(n) := \#\{\text{partitions of } n\}.$$

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Example

The partitions of 4 are:

$$4, \quad 3+1, \quad 2+2, \quad 2+1+1, \quad 1+1+1+1,$$

and so $p(4) = 5$.

Ramanujan's Congruences

Theorem (Ramanujan)

For every n, we have

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5}, \\ p(7n+5) &\equiv 0 \pmod{7}, \\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned}$$

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Remark

Attempting to explain them, Dyson defined the "rank."

Dyson's Rank

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The *rank* of a partition is its largest part minus its number of parts.

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Example

The ranks of the partitions of 4:

Partition	Largest Part	# Parts	Rank
4	4	1	$3 \equiv 3 \pmod 5$
$3+1$	3	2	$1 \equiv 1 \pmod 5$
$2+2$	2	2	$0 \equiv 0 \pmod 5$
$2+1+1$	2	3	$-1 \equiv 4 \pmod 5$
$1+1+1+1$	1	4	$-3 \equiv 2 \pmod 5$

Dyson's Conjecture

Definition

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Conjecture (Dyson, 1944)

For every n and every r, we have

$$N(r, 5; 5n+4) = p(5n+4)/5,$$
$$N(r, 7; 7n+5) = p(7n+5)/7.$$

A famous theorem

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Remark

The proof depends on the **generating function***:*

$$R(w;q) = \sum_{m,n} N(m,n) w^m q^n := 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(wq;q)_n (w^{-1}q;q)_n},$$

where

$$(a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1}).$$

Revealing specializations

Example

- For $w = 1$ and $q := e^{2\pi iz}$, we have the **modular form**

$$q^{-1}R(1;q^{24}) = \sum_{n=0}^{\infty} p(n)q^{24n-1}.$$

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$$q^{-1}R(1;q^{24}) = \sum_{n=0}^{\infty} p(n)q^{24n-1}.$$

- For $w = -1$, we have Ramanujan's **mock theta**

$$R(-1;q) = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2\cdots(1+q^n)^2}.$$

Modular Forms

"Definition"

A modular form is any meromorphic function $f(z)$ *on* $\mathbb{H}$ *for which*

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \subset \mathrm{SL}_2(\mathbb{Z})$.

Natural Hope

Question

Since the deepest facts about $p(n)$ come from modular form theory, is $R(w;q)$, for roots of unity $w \neq 1$, modular?

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"Theorem" (Bringmann-O)

If $w \neq 1$ is a root of unity, then $R(w;q)$ is the **holomorphic part** *of a* **harmonic Maass form**.

Defining Maass forms

Notation. Throughout, let $z = x + iy \in \mathbb{H}$ with $x, y \in \mathbb{R}$.

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Hyperbolic Laplacian.

$$\Delta_k := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

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$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k \ f(z).$$

2. *We have that $\Delta_k f = 0$.*

A bit more precisely

Definition

If $0 < a < c$ are integers, then let

$$S\left(\frac{a}{c}; z\right) := B(a,c)\int_{-\bar{z}}^{i\infty} \frac{\Theta\left(\frac{a}{c}; \ell_c \tau\right)}{\sqrt{-i(\tau+z)}}\, d\tau,$$

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and define $D\left(\frac{a}{c}; z\right)$ by

$$D\left(\frac{a}{c}; z\right) := -S\left(\frac{a}{c}; z\right) + q^{-\frac{\ell_c}{24}} R(\zeta_c^a; q^{\ell_c})$$

$$\Uparrow \qquad\qquad \Uparrow$$

$$\Theta - \text{integral} \quad \text{Dyson's series}$$

The first theorem

Theorem (Bringmann-O)

If $0 < a < c$, *then* $D\left(\frac{a}{c}; z\right)$ *is a weight* $1/2$ *harmonic Maass form.*

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Remark

By exhibiting Maass forms, this is already interesting.

Partition congruences revisited

Theorem (O, 2000)

For primes $Q \geq 5$, there are infinitely many non-nested progressions $An + B$ for which

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For primes $\mathcal{Q} \geq 5$, there are infinitely many non-nested progressions $An + B$ for which

$$p(An + B) \equiv 0 \pmod{\mathcal{Q}}.$$

Examples. Simplest ones for $17 \leq \mathcal{Q} \leq 31$:

$$p(48037937n + 1122838) \equiv 0 \pmod{17},$$
$$p(1977147619n + 815655) \equiv 0 \pmod{19},$$
$$p(14375n + 3474) \equiv 0 \pmod{23},$$
$$p(348104768909n + 43819835) \equiv 0 \pmod{29},$$
$$p(4063467631n + 30064597) \equiv 0 \pmod{31}.$$

I. Maass form congruences

Theorem (Bringmann-O)

There are progressions $An+B$ s.t. for all $0 \leq r < t$

$$N(r,t;An+B) \equiv 0 \pmod{\mathcal{Q}}.$$

I. Maass form congruences

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There are progressions $An + B$ s.t. for all $0 \leq r < t$

$$N(r, t; An + B) \equiv 0 \pmod{\mathcal{Q}}.$$

Remark

This is a Dyson-style proof of

$$p(An + B) \equiv 0 \pmod{\mathcal{Q}}.$$

"Proof" of the second theorem

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- The Fourier expansions in $q := e^{2\pi i z}$ are **special**:

$$D\left(\frac{a}{c}; z\right) = q^{-\frac{\ell_c}{24}} R(\zeta_c^a; q^{\ell_c}) + \sum_{n\in\mathbb{Z}} B(a,c,n)\gamma(c,y;n) q^{-\widetilde{\ell}_c n^2}.$$

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- The "bad" coefficients are so **sparse** that the proof becomes

$$\text{Shimura correspondence} + \ell\text{-adic Galois repns} + \epsilon.$$

II. Exact formulas of Rademacher-type

Problem (Rademacher)

Define $\alpha(n)$ *by*

$$f(q) = R(-1;q) = \sum_{n=0}^{\infty} \alpha(n) q^n = 1 + q - 2q^2 + \cdots + 487q^{47} + 9473q^{89} - \cdots.$$

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Find an **exact formula** *for*

$$\alpha(n) = \underset{\substack{\Uparrow \\ \text{even rank}}}{N(0,2;n)} - \underset{\substack{\Uparrow \\ \text{odd rank,}}}{N(1,2;n)}$$

Rademacher-type exact formula

Conjecture (Andrews-Dragonette, 1966)

If n is a positive integer, then

$$\alpha(n) = \pi(24n-1)^{-\frac{1}{4}} \times \sum_{k=1}^{\infty} \frac{(-1)^{\lfloor \frac{k+1}{2} \rfloor} A_{2k}\left(n - \frac{k(1+(-1)^k)}{4}\right)}{k} \cdot I_{\frac{1}{2}}\left(\frac{\pi\sqrt{24n-1}}{12k}\right),$$

where $A_k(n)$ is a "Kloosterman-type sum".

II. Exact formulas for Maass forms

Theorem (Bringmann-O, 2006)

The Andrews-Dragonette Conjecture is true.

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- By the first theorem, the holomorphic part of the Maass form $D\left(\frac{1}{2}; z\right)$ is $q^{-1}R(-1; q^{24})$.

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Idea of the Proof.

- By the first theorem, the holomorphic part of the Maass form $D\left(\frac{1}{2}; z\right)$ is $q^{-1}R(-1; q^{24})$.
- Construct the "right" Poincaré series

$$P(z) = \frac{2}{\sqrt{\pi}} \sum_{M \in \Gamma_\infty \backslash \Gamma_0(2)} \chi(M)^{-1}(cz+d)^{-\frac{1}{2}}\phi(Mz),$$

where ϕ is a Whittaker function.

Idea of the proof

- The "right" one has a Fourier expansion

$$P(24z) = \text{Nonholomorphic function } + \sum_{n=1}^{\infty} \beta(n) q^{24n-1},$$

where the $\beta(n)$'s **equal** the expressions in the conjecture.

Idea of the proof

- The "right" one has a Fourier expansion

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where the $\beta(n)$'s **equal** the expressions in the conjecture.

- **Somehow** prove that $D\left(\frac{1}{2}; z\right) - P(24z)$ is 0.

$\square$

II. Exact formulas for Maass forms

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We have formulas for **all** *harmonic Maass forms with weight* $\leq 1/2$.

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Remark

Gives the theorem of Rademacher-Zuckerman for **non-positive weight** *modular forms as a special case.*

Relation to classical modular forms

$$S_k(\Gamma) := \text{ weight } k \text{ cusp forms on } \Gamma,$$

$$H_{2-k}(\Gamma) := \text{ weight } 2-k \text{ harmonic Maass forms on } \Gamma.$$

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Lemma

If $w \in \frac{1}{2}\mathbb{Z}$ and $\xi_w := 2iy^w \overline{\frac{\partial}{\partial \overline{z}}}$, then

$$\xi_{2-k} : H_{2-k}(\Gamma) \longrightarrow S_k(\Gamma).$$

Moreover, this map is ***surjective****.*

Harmonic Maass forms have two parts ($q := e^{2\pi iz}$)

Fundamental Lemma

If $f \in H_{2-k}$ and $\Gamma(a,x)$ is the incomplete Γ-function, then

$$f(z) = \sum_{n \gg -\infty} c_f^+(n) q^n + \sum_{n<0} c_f^-(n)\Gamma(k-1, 4\pi|n|y) q^n.$$

$$\updownarrow \qquad\qquad\qquad\qquad \updownarrow$$

Holomorphic part f^+ **Nonholomorphic part** f^-

Relation with classical modular forms

Fundamental Lemma

If $\xi_w := 2iy^w \overline{\frac{\partial}{\partial \overline{z}}}$*, then*

$$\xi : H_{2-k} \longrightarrow S_k$$

satisfies

$$\xi(f) = \xi(f^- + f^+) = \xi(f^-).$$

Relation with classical modular forms

Fundamental Lemma

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satisfies

$$\xi(f) = \xi(f^- + f^+) = \xi(f^-).$$

Question

What does the holomorphic part f^+ **unearth***?*

Source of rich information

Question

Can one use **holomorphic** *parts of Maass forms to* **unearth** *hidden information related to* $S_k(\Gamma)$*?*

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Answer. Yes, and we discuss two applications:

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Answer. Yes, and we discuss two applications:

- III. Exact formulas for Maass forms of Zagier-type.
- BSD numbers

Rademacher's "exact formula"

Theorem (Rademacher (1943))

If n is a positive integer, then

$$p(n) = \text{CRAZY convergent } \textbf{infinite} \text{ sum.}$$

III. Exact formulas of Zagier-type

Theorem (Bruinier-O)

Let F be the Maass form for which $\xi_{-2}(F)$ is in $S_4(6)$.

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$$\mathbb{P}(z) := -\left(\frac{1}{2\pi i} \cdot \frac{d}{dz} + \frac{1}{2\pi y}\right) F(z)$$

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has the property that its sum over disc $-24n+1$ CM points is

$$p(n) = \frac{1}{24n-1}\cdot\left(\mathbb{P}(\alpha_{n,1})+\mathbb{P}(\alpha_{n,2})+\cdots+\mathbb{P}(\alpha_{n,h_n})\right).$$

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$$p(n)=\frac{1}{24n-1}\cdot\left(\mathbb{P}(\alpha_{n,1})+\mathbb{P}(\alpha_{n,2})+\cdots+\mathbb{P}(\alpha_{n,h_n})\right).$$

Moreover, each $(24n-1)\mathbb{P}(\alpha_{n,m})$ is an **algebraic integer**.

Hard proof that $p(1) = 1$.

If $\beta := 161529092 + 18648492\sqrt{69}$, then

$$\frac{1}{23}\cdot\mathbb{P}\left(\frac{-1+\sqrt{-23}}{12}\right) = \frac{1}{3} + \frac{\beta^{2/3}+127972}{6\beta^{1/3}},$$

$$\frac{1}{23}\cdot\mathbb{P}\left(\frac{-13+\sqrt{-23}}{24}\right) = \frac{1}{3} - \frac{\beta^{2/3}+127972}{12\beta^{1/3}} + \frac{\beta^{2/3}-127972}{4\sqrt{-3}\beta^{1/3}},$$

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and we see directly that

$$p(1) = 1 = \frac{1}{23}\left(\mathbb{P}(\alpha_1) + \mathbb{P}(\alpha_2) + \mathbb{P}(\alpha_3)\right).$$

First few minimal polynomials

n	$x^{h_n} - (24n-1)p(n)x^{h_n-1} + \dots$
1	$x^3 - 23\cdot 1x^2 + \frac{3592}{23}x - 419$
2	$x^5 - 47\cdot 2x^4 + \frac{169659}{47}x^3 - 65838x^2 + \frac{1092873176}{47^2}x + \frac{1454023}{47}$
3	$x^7 - 71\cdot 3x^6 + \frac{1312544}{71}x^5 - 723721x^4 + \frac{44648582886}{71^2}x^3 + \frac{9188934683}{71}x^2 + \frac{166629520876208}{71^3}x + \frac{2791651635293}{71^2}$
4	$x^8 - 95\cdot 5x^7 + \frac{9032603}{95}x^6 - 9455070x^5 + \frac{3949512899743}{95^2}x^4 - \frac{97215753021}{19}x^3 + \frac{9776785708507683}{95^3}x^2 - \frac{53144327916296}{19^2}x - \frac{134884469547631}{5^4\cdot 19}.$

Zagier's famous Berkeley lectures

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Theorem (Bruinier-O)

The generating function for the CM Galois traces of a **good** *$\mathbb{Q}$-rational Maass form with Laplacian eigenvalue $\lambda = -2$ is the* **holomorphic part** f^+ *of a weight $-1/2$ harmonic Maass form.*

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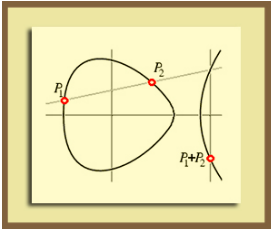
Remark (Bruinier-O-Sutherland)

The coeffs are computable using CM and the CRT.

IV: BSD Numbers



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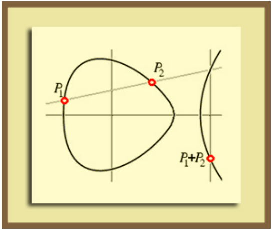

Group Law

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Theorem (Mordell-Weil)

The rational points of an elliptic curve over a number field form a finitely generated abelian group.

The Congruent Number Problem

Problem (Open)

Determine the integers which are areas of rational right triangles.

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2. The number 157 is congruent, since it is the area of

$$\left(\frac{411340519227716149383203}{21666555693714761309610}, \frac{680\cdots540}{411\cdots203}, \frac{224\cdots041}{891\cdots830}\right).$$

A Classical Diophantine Criterion

Theorem (Easy)

An integer D is congruent if and only if the elliptic curve

$$E_D : \quad Dy^2 = x^3 - x$$

has positive rank.

Quadratic twists

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If Δ is a fund. disc., then the Δ-quadratic twist of E is

$$E(\Delta): \quad \Delta y^2 = x^3 + Ax + B.$$

Birch and Swinnerton-Dyer Conjecture

Conjecture

If $E/\mathbb{Q}$ is an elliptic curve and $L(E,s)$ is its L-function, then

$$\mathrm{ord}_{s=1}(L(E,s)) \;=\; \text{rank of } E(\mathbb{Q}).$$

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A good question. How does one compute $\mathrm{ord}_{s=1}(L(E,s))$?

Kolyvagin's Theorem

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Question

How does one compute $L(E,1)$ *and* $L'(E,1)$*?*

Formulas for $L(E(\Delta), 1)$

Theorem (Shimura-Kohnen/Zagier-Waldspurger)

There is a modular form

$$g(z) = \sum_{n=1}^{\infty} b_E(n) q^n$$

such that if $\Delta < 0$ *and* $\left(\frac{\Delta}{p}\right) = 1$, *then*

$$L(E(\Delta), 1) = \alpha_E(\Delta) \cdot b_E(|\Delta|)^2.$$

The Gross-Zagier Theorem

Question

What about derivatives?

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Theorem (Gross and Zagier)

If $\Delta > 0$ and $\left(\frac{\Delta}{p}\right) = 1$, then for suitable $d < 0$ the global Neron-Tate height on $J_0(p)(H)$ of $y_{\Delta,r}(-n,h)$ is

$$\beta_E(\Delta, d) \cdot L(E(d), 1) \cdot L'(E(\Delta), 1).$$

Natural Question

Question

Find an extension of the Kohnen-Waldspurger theorem giving both

$$L(E(\Delta),1) \quad \textit{and} \quad L'(E(\Delta),1).$$

Theorem (Bruinier-Ono)

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$$
\begin{array}{ccccc}
 & & & & E/\mathbb{Q} \\
 & & & & \updownarrow \\
f_g = f_g^+ + f_g^- & \longrightarrow & g & \longrightarrow & G \\
\cap & & \cap & & \cap \\
H_{\frac{1}{2}}^+(4p) & \longrightarrow & S_{3/2}^+(4p) & \longrightarrow & S_2(p) \\
 & \Uparrow & \Uparrow & & \Uparrow \\
 & \xi & \textit{Kohnen-Shimura} & & \textit{Modularity}
\end{array}
$$

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2. *If* $\Delta > 0$ *and* $\left(\frac{\Delta}{p}\right) = 1$, *then*

$$L'(E(\Delta), 1) = 0 \iff c_g^+(\Delta) \text{ is algebraic.}$$

Example for $E: \quad y^2 = x^3 + 10x^2 - 20x + 8.$

Δ	$c_g^+(-\Delta)$	$L'(E(\Delta), 1)$
-3	$1.0267149116\ldots$	$1.4792994920\ldots$
-4	$1.2205364009\ldots$	$1.8129978972\ldots$
$\vdots$	$\vdots$	$\vdots$
-136	$-4.8392675993\ldots$	$5.7382407649\ldots$
-139	-6	0
-151	$-0.8313568817\ldots$	$6.6975085515\ldots$
$\vdots$	$\vdots$	$\vdots$
-815	$121.1944103120\ldots$	$4.7492583693\ldots$
-823	312	0

Overview of the proof

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- The first formula follows from ξ.
- The equivalence of $L'(E(\Delta), 1) = 0$ and the algebraicity of $c_g^+(\Delta)$ involves a **detailed study** of Heegner divisors.
- Algebraicity is dictated by the vanishing of Heegner divisors, and Gross-Zagier gives the connection to

$$L'(E(\Delta), 1).$$

"Detailed study" of Heegner divisors

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Theorem (Bruinier-O)

We have that $\eta_\Delta(z, f_g) := -\frac{1}{2}\partial\Phi_\Delta(z, f_g)$ *is a differential on* $X_0(p)$ *with Heegner divisor. Moreover, we have*

$$\eta_\Delta(z, f_g) =$$
$$\left(\rho_{f_g,\ell} - \operatorname{sgn}(\Delta)\sqrt{\Delta}\sum_{n\geq 1}\sum_{d|n}\frac{n}{d}\left(\frac{\Delta}{d}\right)c_g^+\left(\frac{|\Delta|n^2}{4Nd^2}\right)e(nz)\right)\cdot 2\pi i\, dz.$$

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Building general theory has applications, such as:

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Building general theory has applications, such as:

I. (**Maass form congruences**)
Extend the scope of Serre, Swinnerton-Dyer, Deligne, Ribet....

II and III. (**Exact formulas for Maass forms**)
Extend and generalize phenomena obtained previously by Rademacher and Zagier*.

IV. (**Birch and Swinnerton-Dyer Numbers**)
Unify work of Waldspurger and Gross-Zagier on BSD numbers $+\epsilon$.