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An Algebraic Condition for Finding Exact Solutions to General KdV6

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Abstract: We consider some conditions over the coefficients of the sixth-order KdV equation (KdV6) under which this equation has exact solutions. An algebraic condition for the existence of exact solutions to KdV6 is obtained. A new ansatz is considered to obtain analytic solutions for several forms of it. Additionally, the generalized tanh-coth is used here to obtain periodic and soliton solutions for a special case..

Keywords: nonlinear equation; sixth–order KdV equation; KdV6; integrable equation; generalized tanh-coth method

1 Introduction

It is well know that the general form of the sixth-order KdV equation (KdV6) is given by

$$u_{xxxxxx} + a u_x u_{xxxx} + b u_{xx} u_{xxx} + c u_x^2 u_{xx} + d u_{tt} + e u_{xxxt} + f u_x u_{xt} + g u_t u_{xx} = 0, \tag{1}$$

where a, b, c, d, e, f, g are arbitrary parameters, and $u = u(x,t)$ is a differentiable function. Several forms can be constructed from it by changing the values of the parameters. For instance,

$$\begin{cases} a = 20, b = 40, c = 120, d = 0, e = 1, f = 8, g = 4: \\ u_{xxxxxx} + 20u_x u_{xxxx} + 40u_{xx}u_{xxx} + 120u_x^2 u_{xx} + u_{xxxt} + 8u_x u_{xt} + 4u_t u_{xx} = 0. \end{cases} \tag{2}$$

$$\begin{cases} a = -9, b = -18, c = 18, d = -\frac{1}{2}, e = \frac{1}{2}, f = 0, g = 0: \\ u_{xxxxxx} - 9u_x u_{xxxx} - 18u_{xx}u_{xxx} + 18u_x^2 u_{xx} - \frac{1}{2}u_{tt} + \frac{1}{2}u_{xxxt} = 0. \end{cases} \tag{3}$$

$$\begin{cases} a = -15, b = -15, c = 45, d = -5, e = -5, f = 15, g = 15: \\ u_{xxxxxx} - 15u_x u_{xxxx} - 15u_{xx}u_{xxx} + 45u_x^2 u_{xx} - 5u_{tt} - 5u_{xxxt} + 15u_x u_{xt} + 15u_t u_{xx} = 0. \end{cases} \tag{4}$$

and

$$\begin{cases} a = -15, b = -\frac{75}{2}, c = 45, d = -5, e = -5, f = 15, g = 15: \\ u_{xxxxxx} - 15u_x u_{xxxx} - \frac{75}{2}u_{xx}u_{xxx} + 45u_x^2 u_{xx} - 5u_{tt} - 5u_{xxxt} + 15u_x u_{xt} + 15u_t u_{xx} = 0. \end{cases} \tag{5}$$

respectively. It has been proved that (2), (3), (4) and (5) are particular integrable cases of (1). More exactly, the five authors of [1] have found the Lax Pair, an auto-B$\ddot{a}$cklund transformation, traveling wave solutions and third-order generalized symmetries for (2). More recently, Kupershmidt [2] showed that (2) is integrable in the usual sense. The two authors of [3] found a B$\ddot{a}$cklund self-transformation for (3), and multisoliton solutions for it were studied by the authors of [4]. On the other hand, (4) and (5) have been obtained from equations

$$5\partial_x^{-1} v_{tt} + 5v_{xxt} - 15vv_t - 15v_x \partial_x^{-1} v_t - 45v^2 v_x + 15v_x v_{xx} + 15vv_{xxx} - v_{xxxxx} = 0 \tag{6}$$

and

$$5\partial_x^{-1} v_{tt} + 5v_{xxt} - 15vv_t - 15v_x \partial_x^{-1} v_t - 45v^2 v_x + \frac{45}{2} v_x v_{xx} + 15vv_{xxx} - v_{xxxxx} = 0 \tag{7}$$

respectively, after the use of the potential transformation

$$v(x,t) = u_x(x,t), \tag{8}$$

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equations (6) and (7) are fifth-order nonlinear equations which govern wave propagation in two opposite directions. More exactly, (6) is related to Sawada–Kotera-Caudrey–Dodd–Gibbon equation [5], [6], and (7) may be considered a bidirectional version of the Kaup–Kupershmidt equation [7] (see [1][8]). The two authors of [8] have been constructed Lax pair for (6) and (7).

The present work has several objectives: The first, to present some conditions over the coefficients of (1) to obtain exact solutions for it in a special form. This special form of the solutions can be considered as a new ansatz to construct exact solutions for evolution nonlinear partial differential equations. The second objective is to present exact solutions to four integrable KdV6 that appear in the literature and that are related with another important evolution nonlinear partial differential equations. The third objective, is to present a special KdV6 equation with arbitrary coefficients which has exact solutions according with the theory we present and by using the generalized tanh-coth [**?**] to obtain periodic and soliton solutions for this new equation.

This paper is organized as follows: In Sec. 2, we consider a new ansatz and we obtain some conditions over the coefficients of (1) that allow us to get exact solutions. In Sec. 3 we obtain exact solutions for equations (2), (3), (4) and (5) using the results of Sec. 2. In Sec. 4 we review briefly the generalized tanh-coth method. In Sec. 5, we present a new sixth-order KdV equation, which has exact solutions accordingly the conditions given in Sec. 2 and by using the generalized tanh-coth method we obtain exact solutions which include periodic and soliton solutions for it. Finally some conclusions are given.

2 A new ansatz to construct exact solutions of the general KdV6

To construct exact solutions of (1), we consider the ansatz

$$u(x,t) = Bx - \frac{A}{1+e^{x+Ct}}, \tag{9}$$

where A, B and C are some constants to be determined later. To avoid trivial solutions, we will suppose that $A \neq 0$ and $C \neq 0$. Inserting (9) into (1), we obtain a polynomial equation in the variable $\zeta = e^{x+Ct}$. Equating the coefficients of the powers of ζ to zero, we obtain the following algebraic system :

$$\begin{cases} cA^3 - 11aA^2 - 5bA^2 + 2BcA^2 + CfA^2 + CgA^2 - 10aBA + 2B^2cA + \\ 2C^2dA - 10CeA + 2BCfA + 302A = 0, \\ aA^2 + bA^2 + 2BcA^2 + CfA^2 + CgA^2 - 9aBA + 3B^2cA + 3C^2dA - \\ 9CeA + 3BCfA - 57A = 0, \\ AcB^2 + aAB + ACfB + A + AC^2d + ACe = 0. \end{cases} \tag{10}$$

Solving the previous system with with the aid of either *Mathematica 7 or Maple 13* we obtain

$$b = \frac{cA^2 - 12aA + 360}{6A}, \tag{11}$$

$$e = \frac{-cB^2 - aB - CfB - C^2d - 1}{C}, \tag{12}$$

$$g = \frac{-cA^2 + 6aA - 12BcA - 6CfA - 72B^2c - 72C^2d - 72BCf - 72}{6AC}. \tag{13}$$

The equations (11), (12) and (13) give conditions to obtain exact solutions of (1) in the form (9). Furthermore, the system defined by these equations may be reduced to the polynomial equation

$$p_4 + p_3C + p_2C^2 + p_1C^3 + p_0C^4 = 0, \tag{14}$$

where

$$p_4 = \left(a^2 + ba - 10c\right)^2 \left(b^2 - 3ab + 9c\right), \tag{15}$$

$$p_3 = -(6a - 7b)\left(a^2 + ab - 10c\right)^2 g, \tag{16}$$

$$\begin{aligned}p_2 =\,&2a^5bd + 5a^4b^2d - 2a^4bef - 12a^4cd - 6a^4fg + 4a^4g^2 + 4a^3b^3d - 5a^3b^2ef - 64a^3bcd + 2a^3bce^2+\\&8a^3bf^2 + 7a^3bfg + 19a^3bg^2 + 12a^3cef + 12a^3ceg + a^2b^4d - 4a^2b^3ef - 72a^2b^2cd + 5a^2b^2ce^2 + 9a^2b^2f^2+\\&10a^2b^2fg + 11a^2b^2g^2 + 32a^2bcef - 14a^2bceg + 240a^2c^2d - 12a^2c^2e^2 - 30a^2cf^2 + 60a^2cfg - 110a^2cg^2-\\&ab^4ef - 20ab^3cd + ab^3ce^2 + 5ab^3f^2 + 5ab^3fg + 36ab^2cef - 20ab^2ceg + 440abc^2d - 32abc^2e^2 - 110abcf^2\\&-130abcfg - 220abcg^2 - 4120ac^2ef - 120ac^2eg + b^4ce^2 - 10b^3ceg + 100b^2c^2d - 36b^2c^2e^2 - 25b^2cf^2+\\&25b^2cg^2 + 260bc^2eg - 1200c^3d + 120c^3e^2 + 300c^2f^2 + 700c^2g^2,\end{aligned}\tag{17}$$

$$\begin{aligned}p_1 =\,&g(4a^5d + 10a^4bd - 4a^4ef + 8a^3b^2d - 10a^3bef - 80a^3cd + 4a^3ce^2 + 10a^3f^2 + 20a^3fg + 10a^3g^2+\\&2a^2b^3d - 8a^2b^2ef - 120a^2bcd + 10a^2bce^2 + 15a^2bf^2 + 20a^2bfg + 5a^2bg^2 + 40a^2cef - 40a^2ceg-\\&2ab^3ef - 40ab^2cd + 8ab^2ce^2 + 10ab^2f^2 + 10ab^2fg + 60abcef - 40abceg + 400ac^2d - 40ac^2e^2-\\&100acf^2 - 200acfg - 100acg^2 + 2b^3ce^2 - 20b^2ceg + 200bc^2d - 60bc^2e^2 - 50bcf^2 + 50bcg^2 + 400c^2eg),\end{aligned}\tag{18}$$

$$\begin{aligned}p_0 =\,&4a^4cd^2 + 4a^4dfg + 4a^4dg^2 + 8a^3bcd^2 - 2a^3bdf^2 + 2a^3bdfg + 4a^3bdg^2 - 8a^3cdef - 8a^3cdeg - 4a^3ef^2g-\\&4a^3efg^2 + 4a^2b^2cd^2 - a^2b^2df^2 + a^2b^2dg^2 - 8a^2bcdef - 4a^2bcdeg + 2a^2bef^3 - 2a^2bef^2g - 4a^2befg^2-\\&80a^2c^2d^2 + 8a^2c^2de^2 + 20a^2cdf^2 - 40a^2cdfg - 60a^2cdg^2 + 4a^2ce^2f^2 + 12a^2ce^2fg + 4a^2ce^2g^2 + 10a^2f^3g+\\&20a^2f^2g^2 + 10a^2fg^3 - 4ab^2cdef + ab^2ef^3 - ab^2efg^2 - 80abc^2d^2 + 8abc^2de^2 + 40abcdf^2 + 20abcdfg-\\&20abcdg^2 - 2abce^2f^2 + 6abce^2fg + 4abce^2g^2 - 5abf^4 - 5abf^3g + 5abf^2g^2 + 5abfg^3 + 80ac^2def+\\&80ac^2deg - 8ac^2e^3f - 8ac^2e^3g - 20acef^3 - 20acef^2g - 20acefg^2 - 20aceg^3 + 4b^2c^2de^2 - b^2ce^2f^2+\\&b^2ce^2g^2 - 40bc^2deg - 4bc^2e^3g + 10bcef^2g - 10bceg^3 + 400c^3d^2 - 80c^3de^2 + 4c^3e^4 - 200c^2df^2 + 200c^2dg^2+\\&20c^2e^2f^2 + 20c^2e^2g^2 + 25cf^4 - 50cf^2g^2 + 25cg^4.\end{aligned}\tag{19}$$

It is clear that solving (14), we find C and therefore A y B, so that we obtain exact solutions to (1) in the form (9). In the case when $c \neq 0$, from Eqs. (11) and (12) we obtain the following formulas for calculating A and B in terms of C :

i.

$$A = \frac{6a + 3b - 3\sqrt{(2a+b)^2 - 40c}}{c}, \qquad B = -\frac{a + Cf + \sqrt{(a+Cf)^2 - 4c(C(Cd+e)+1)}}{2c}. \tag{20}$$

ii.

$$A = \frac{3\left(2a + b + \sqrt{(2a+b)^2 - 40c}\right)}{c}, \qquad B = -\frac{a + Cf + \sqrt{(a+Cf)^2 - 4c(C(Cd+e)+1)}}{2c}. \tag{21}$$

iii.

$$A = \frac{6a + 3b - 3\sqrt{(2a+b)^2 - 40c}}{c}, \qquad B = -\frac{a + Cf - \sqrt{(a+Cf)^2 - 4c(C(Cd+e)+1)}}{2c}. \tag{22}$$

iv.

$$A = \frac{3\left(2a + b + \sqrt{(2a+b)^2 - 40c}\right)}{c}, \qquad B = -\frac{a + Cf - \sqrt{(a+Cf)^2 - 4c(C(Cd+e)+1)}}{2c}. \tag{23}$$

The case $c = 0$ is special. We will not consider it here. We only mention the special subcase when $a = -b/2$. If we substitute (9) into (1) then after solving the algebraic system, we get $A = 0$, so the only solution of (1) in the form (9) is the trivial one $u = Bx$.

Remark 1 *It is possible to verify, that if* $b \neq 0$ *and* $g \neq 0$ *then* $C \neq 0$. *The cases when* $g = 0$ *or* $b = 0$ *are special.*

Remark 2 *We may solve equation (14) in an easy way if*

$$c = \frac{a(a+b)}{10} \tag{24}$$

since in this case we obtain $p_4 = p_3 = 0$ *and then (14) converts into quadratic equation*

$$p_2 + p_1 C + p_0 C^2 = 0,$$

where

$$p_2 = -ab^2(3a-5b)^2((a+b)e - 5(f+g))^2,$$
$$p_1 = -ab^2 g((a+b)e - 5(f+g))^2,$$

$$\begin{aligned} p_0 \;\; = \;\; & a((a+b)e-5(f+g))^2(10ab^2d + a^3e^2 + a^2be^2 - 10a^2ef + 25af^2 - \\ & 25bf^2 - 10a^2eg - 10abeg + 50afg + 25ag^2). \end{aligned}$$

Thus, we obtain a wide class of KdV6 equations with exact solutions of the form (9). Observe that equations (2) and (4) satisfy condition (24).

3 Exact solutions to particular cases

In this section we consider the particular cases of (1) obtained by values given in (2), (3), (4) and (5). In all this cases we find A, B, C by solving the system given by equations (11), (12), (13). Solutions in the form (9) for the particular cases considered here are obtained.

3.1 Solutions to (2)

- $A = 1,\ B = \frac{1}{10}\left(-5+\sqrt{15}\right),\ C = 5 - 2\sqrt{15}:$

$$u(x,t) = -\frac{1}{10}\left(5-\sqrt{15}\right)x - \frac{1}{1+e^{x+(5-2\sqrt{15})t}}.$$

- $A = 1,\ B = \frac{1}{10}\left(-5-\sqrt{15}\right),\ C = 5 + 2\sqrt{15}:$

$$u(x,t) = -\frac{1}{10}\left(5+\sqrt{15}\right)x - \frac{1}{1+e^{x+(5+2\sqrt{15})t}}.$$

- $A = 1,\ C = \frac{-120B^2-20B-1}{8B+1}:$

$$u(x,t) = Bx - \frac{1}{1+e^{x-\frac{(120B^2+20B+1)t}{8B+1}}}.$$

3.2 Solutions to (3)

- $A = -10,\ B = \frac{11}{12},\ C = -\frac{7}{2}:$

$$u(x,t) = \frac{11x}{12} + \frac{10}{1+e^{x-\frac{7t}{2}}}.$$

- $A = -10,\ B = \frac{3}{4},\ C = \frac{7}{2}:$

$$u(x,t) = \frac{3x}{4} + \frac{10}{1+e^{x+\frac{7t}{2}}}.$$

- $A = -2,\ B = \frac{1}{4},\ C = \frac{1}{2}$

$$u(x,t) = \frac{x}{4} + \frac{2}{1+e^{x+\frac{t}{2}}}.$$

- $A=-2,\ B=\frac{3}{4},\ C=\frac{7}{2}:$

$$u(x,t)=\frac{3x}{4}+\frac{2}{1+e^{x+\frac{7t}{2}}}.$$

- $A=-2,\ C=6:$

$$u(x,t)=Bx+\frac{2}{1+e^{x+(6B-1)t}}.$$

3.3 Solutions to (4)

- $A=-4,\ B=\frac{1}{10}\left(5-\sqrt{5}\right),\ C=\frac{1}{10}\left(-5+3\sqrt{5}\right):$

$$u(x,t)=\frac{1}{10}\left(5-\sqrt{5}\right)x+\frac{4}{1+e^{x-\frac{1}{10}\left(5-3\sqrt{5}\right)t}}.$$

- $A=-4,\ B=\frac{1}{10}\left(5+\sqrt{5}\right),\ C=\frac{1}{10}\left(-5-3\sqrt{5}\right):$

$$u(x,t)=\frac{1}{10}\left(5+\sqrt{5}\right)x+\frac{4}{1+e^{x-\frac{1}{10}\left(5+3\sqrt{5}\right)t}}.$$

- $A=-2,\ B=\frac{1}{10}\left(5-\sqrt{5}\right),\ C=\frac{1}{10}\left(-5+3\sqrt{5}\right):$

$$u(x,t)=\frac{1}{10}\left(5-\sqrt{5}\right)x+\frac{2}{1+e^{x-\frac{1}{10}\left(5-3\sqrt{5}\right)t}}.$$

- $A=-2,\ B=\frac{1}{10}\left(5+\sqrt{5}\right),\ C=\frac{1}{10}\left(-5-3\sqrt{5}\right):$

$$u(x,t)=\frac{1}{10}\left(5+\sqrt{5}\right)x+\frac{2}{1+e^{x-\frac{1}{10}\left(5+3\sqrt{5}\right)t}}.$$

- $A=\frac{2\left(5-\sqrt{5}\right)}{-5+\sqrt{5}},\ B=0,\ C=\frac{1}{10}\left(-5+3\sqrt{5}\right):$

$$u(x,t)=\frac{2}{1+e^{x-\frac{1}{10}\left(5-3\sqrt{5}\right)t}}.$$

- $A=\frac{2\left(-5-\sqrt{5}\right)}{5+\sqrt{5}},\ B=0,\ C=\frac{1}{10}\left(-5-3\sqrt{5}\right):$

$$u(x,t)=\frac{2}{1+e^{x-\frac{1}{10}\left(5+3\sqrt{5}\right)t}}.$$

- $A=-2,\ B=\frac{1}{30}\left(-5C-\sqrt{5}(5C+1)+5\right):$

$$u(x,t)=\frac{1}{30}\left(\left(-5C-\sqrt{5}(5C+1)+5\right)x+\frac{60}{1+e^{x+Ct}}\right).$$

- $A=-2,\ B=\frac{1}{30}\left(-5C+\sqrt{5}(5C+1)+5\right):$

$$u(x,t)=\frac{1}{30}\left(\left(-5C+\sqrt{5}(5C+1)+5\right)x+\frac{60}{1+e^{x+Ct}}\right).$$

3.4 Solutions to (5)

- $A = -8,\ B = \frac{1}{15}\left(15 - 4\sqrt{5}\right),\ C = \frac{1}{5}\left(-5 + 4\sqrt{5}\right)$:

$$u(x,t) = \frac{4x}{3\sqrt{5}} + x + \frac{8}{1 + e^{x + (\frac{4}{\sqrt{5}} - 1)t}}.$$

- $A = -8,\ B = \frac{1}{15}\left(15 + 4\sqrt{5}\right),\ C = \frac{1}{5}\left(-5 - 4\sqrt{5}\right)$:

$$u(x,t) = \frac{4x}{3\sqrt{5}} + x + \frac{8}{1 + e^{x - (\frac{4}{\sqrt{5}} + 1)t}}.$$

- $A = -1,\ B = \frac{1}{40}\left(5 - \sqrt{5}\right),\ C = \frac{1}{40}\left(-5 + 3\sqrt{5}\right)$:

$$u(x,t) = \frac{1}{40}\left(5 - \sqrt{5}\right)x + \frac{1}{1 + e^{x - \frac{1}{40}\left(5 - 3\sqrt{5}\right)t}}.$$

- $A = -1,\ B = \frac{1}{40}\left(5 + \sqrt{5}\right),\ C = \frac{1}{40}\left(-5 - 3\sqrt{5}\right)$:

$$u(x,t) = \frac{1}{40}\left(5 + \sqrt{5}\right)x + \frac{1}{1 + e^{x - \frac{1}{40}\left(5 + 3\sqrt{5}\right)t}}.$$

4 The generalized tanh-coth method

The wave transformation

$$\xi = x + \lambda t + \xi_0, \tag{25}$$

convert a PDE that does not explicitly involve independent variables to an ODE

$$P(v, v', v'', \ldots) = 0. \tag{26}$$

Using the idea of tanh-coth method introduced by Wazwaz [9], the generalized tanh-coth method admits the use of a finite expansion

$$\sum_{i=0}^{M} a_i \phi(\xi)^i + \sum_{M+1}^{2M} a_i \phi(\xi)^{M-i}, \tag{27}$$

where M is a positive inter that will be determined and $\phi(\xi)$ satisfies the Riccati equation

$$\phi' = k + \phi^2, \tag{28}$$

where k is a constant. The following are particular solutions of (28) :

$$\phi(\xi) = \begin{cases} -\xi^{-1}, & k = 0, \\ \sqrt{k}\tan(\sqrt{k}\,\xi), & k > 0, \\ -\sqrt{k}\cot(\sqrt{k}\,\xi), & k > 0, \\ -\sqrt{-k}\tanh(\sqrt{-k}\,\xi), & k < 0, \\ -\sqrt{-k}\coth(\sqrt{-k}\,\xi), & k < 0. \end{cases} \tag{29}$$

Substituting (27) into (26) and using (28) and (29) results in an algebraic equation in powers of $\phi(\xi)$. Balancing the linear terms of highest order in the resulting equation with the highest order nonlinear terms leads to the determinations of the parameter M. This gives us a set of algebraic equations for k, λ, μ, a_1, $\ldots$, a_{2M} because all coefficients of $\phi(\xi)^i$ $(i = 1, 2, \ldots, 2M)$ have to vanish.

5 A new KdV6 with exact solutions

If we take the values

$$a \neq 0, \qquad b = \frac{3a}{5}, \qquad c = \frac{4a^2}{25}, \qquad f = \frac{8ae}{25} \qquad g = 0,$$

then Eq. (1) converts to

$$u_{xxxxxx} + au_x u_{xxxx} + \frac{3a}{5} u_{xx} u_{xxx} + \frac{4a^2}{25} u_x^2 u_{xx} + du_{tt} + eu_{xxxt} + \frac{8ae}{25} u_x u_{xt} = 0. \tag{30}$$

In this case, we have $p_4 = p_3 = p_2 = p_1 = p_0 = 0$ in (14), so that according with Sec. 2, C may be any number ($C \neq 0$). Therefore, solving the system given by (11),(12),(13) we obtain

$$A = \frac{75}{2a}, \qquad B = \frac{-8Ce - 25 \pm \sqrt{64C^2e^2 - 400C^2d + 225}}{8a}.$$

Using (9), an exact solution of (30) is given by

$$u(x,t) = \frac{\left(-8Ce - 25 \pm \sqrt{64C^2e^2 - 400C^2d + 225}\right) x}{8a} - \frac{75}{2a\left(1 + e^{x+Ct}\right)}.$$

We make use of the generalized tanh-coth method to obtain more solutions to (30), which include periodic and soliton solutions. Let

$$u(x,t) = u(\xi), \tag{31}$$

where ξ is given by (25), (30) reduces to ordinary differential equation

$$u^{(vi)} + au'u^{(iv)} + \frac{3a}{5} u''u''' + \frac{4a^2}{25} (u')^2 u'' + d\lambda^2 u'' + \lambda e u^{(iv)} + \frac{8ae}{25} \lambda u'u'' = 0. \tag{32}$$

Inserting (27) into (32), using (28) and balancing $u^{(vi)}$ with $(u')^2 u''$we obtain

$$M + 6 = 3M + 4,$$

so that $M = 1$. According with the method, we seek solutions to (32) using the expansion

$$u(\xi) = a_0 + a_1 \phi(\xi) + a_2 \phi(\xi)^{-1}.$$

Solving the algebraic system that we obtain in this case, using (25), (28) and (29) and introducing the notations

$$C = \frac{k}{d}(e - \sqrt{e^2 - 4d}) \text{ and } D = \frac{k}{d}(e + \sqrt{e^2 - 4d})$$

we get the following set of periodic and soliton solutions to (30):

$$u_1(x,t) = a_0 - \frac{75\sqrt{k}}{2a} \tan\left(\sqrt{k}\,(x + 2Ct + \xi_0)\right)$$

$$u_2(x,t) = a_0 - \frac{75\sqrt{k}}{2a} \tan\left(\sqrt{k}\,(x + 2Dt + \xi_0)\right)$$

$$u_3(x,t) = a_0 + \frac{75\sqrt{-k}}{2a} \tanh\left(\sqrt{-k}\,(x + 2Ct + \xi_0)\right)$$

$$u_4(x,t) = a_0 + \frac{75\sqrt{-k}}{2a} \tanh\left(\sqrt{-k}\,(x + 2Dt + \xi_0)\right)$$

$$u_5(x,t) = a_0 + \frac{75\sqrt{k}}{2a} \cot\left(\sqrt{k}\,(x + 2Ct + \xi_0)\right)$$

$$u_6(x,t) = a_0 + \frac{75\sqrt{k}}{2a} \cot\left(\sqrt{k}\,(x + 2Dt + \xi_0)\right)$$

$$u_7(x,t) = a_0 + \frac{75\sqrt{-k}}{2a}\coth\left(\sqrt{-k}\,(x+2Ct+\xi_0)\right)$$
$$u_8(x,t) = a_0 + \frac{75\sqrt{-k}}{2a}\coth\left(\sqrt{-k}\,(x+2Dt+\xi_0)\right)$$
$$u_9(x,t) = a_0 + \frac{75\sqrt{k}}{2a}\left(\cot\left(\sqrt{k}\,(x+8Ct+\xi_0)\right) - \tan\left(\sqrt{k}\,(x+8Ct+\xi_0)\right)\right)$$
$$u_{10}(x,t) = a_0 + \frac{75\sqrt{k}}{2a}\left(\cot\left(\sqrt{k}\,(x+8Dt+\xi_0)\right) - \tan\left(\sqrt{k}\,(x+8Dt+\xi_0)\right)\right)$$
$$u_{11}(x,t) = a_0 + \frac{75\sqrt{-k}}{2a}\left(\coth\left(\sqrt{-k}\,(x+8Ct+\xi_0)\right) + \tanh\left(\sqrt{-k}\,(x+8Ct+\xi_0)\right)\right)$$
$$u_{12}(x,t) = a_0 + \frac{75\sqrt{-k}}{2a}\left(\coth\left(\sqrt{-k}\,(x+8Dt+\xi_0)\right) + \tanh\left(\sqrt{-k}\,(x+8Dt+\xi_0)\right)\right)$$

6 Conclusions

In this paper we have derived solutions to several integrable forms of the nonlinear evolution wave equation of sixth order, by using a new ansatz. Conditions over the parameters of the generalized KdV6 equation to obtain exact solutions using this new ansatz have been derived. A new KdV6 equation has been studied and some its exact solutions have beeen derived by using this new ansatz and using the generalized tanh-coth method. The methods used here can be considered as a powerful technique to analyze several forms of nonlinear partial differential equations. Other methods to find exact solutions to NLPDE's may be found in [10]-[37].

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