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0.5 - Exponential and Logarithmic Functions

When logarithms were introduced in the 17th century as a computational tool, they provided scientists of that period computing power that was previously unimaginable. Although computers and calculators have replaced logarithm tables for numerical calculations, the logarithmic functions have wide-ranging applications in mathematics and science. In this section we will review some properties of exponents and logarithms and then use our work on inverse functions to develop results about exponential and logarithmic functions.

Recall from algebra that is *b* is a nonzero real number, then nonzero *integer* powers of *b* are defined by

$$b^n = b \times b \times \ldots \times b \quad \text{and} \quad b^{-n} = \frac{1}{b^n}$$

and if $n = 0$, then $b^0 = 1$. Also, if p/q is a positive *rational* number expressed in lowest terms, then

$$b^{p/q} = \sqrt[q]{b^p} = \left(\sqrt[q]{b}\right)^p \quad \text{and} \quad b^{-p/q} = \frac{1}{b^{p/q}}$$

Here are some other laws of exponents you should be familiar with:

$$b^p b^q = b^{p+q} \qquad \frac{b^p}{b^q} = b^{p-q} \qquad \left(b^p\right)^q = b^{pq}$$

http://www.kutasoftware.com/FreeWorksheets/Alg1Worksheets/Properties%20of%20Exponents.pdf

The Family of Exponential Functions

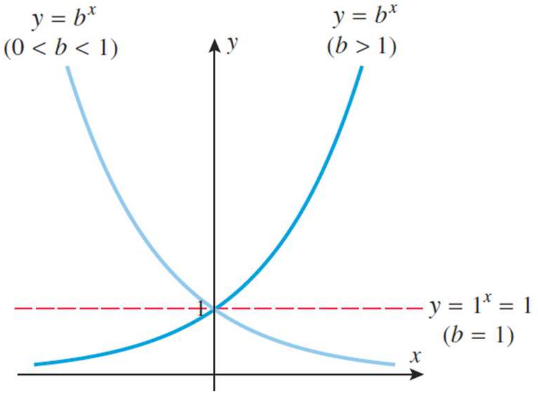
A functions of the form $f(x) = b^x$, where $b > 0$, is called an *exponential functions with base b*. Some example are

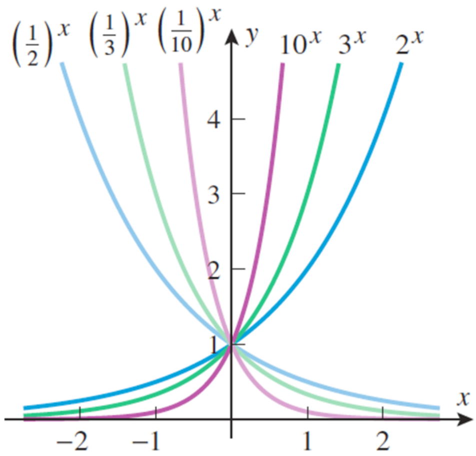
$$f(x) = 3^x, \quad f(x) = (1/2)^x, \quad f(x) = \pi^x$$

Note that an exponential functions has a constant base and variable exponent. Thus, functions such as $f(x) = x^2$ and $f(x) = x^\pi$ would *not* be functions, since they have a variable base and a constant exponent.

The figures below illustrate the graph of $y = b^x$ has one of three general forms, depending on the value of b. The graph of $y = b^x$ has the following properties:

- **The graph passes through (0, 1) because $b^0 = 1$.**
- **If $b > 1$, the value of b^x increases as x increases. The x-axis is a horizontal asymptote of the graph of b^x.**
- **If $0 < b < 1$, the value of b^x decreases as x increases. The x-axis is a horizontal asymptote of the graph of b^x.**

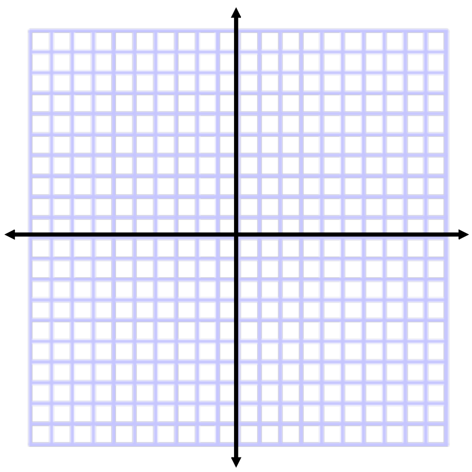




What is the domain and range of an exponential function?

<u>Sketch the graph of the function and find its domain and range.</u>

f(x)* = -2 - 3*^x



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The Natural Exponential Function

Among all possible bases for exponential functions there is one particular base that plays a special role in calculus. That base, denoted by the letter e, is a certain irrational number whose value is

$$e \approx 2.71828\ldots$$

This base is important in calculus because, as we will prove later, $b = e$ is the only base for which the slope of the tangent line to the curve $y = b^x$ at any point P on the curve is equal to the y-coordinate at P. The function $f(x) = e^x$ is called the natural exponential function.

Logarithmic Functions

Recall from algebra that a logarithm is an exponent. More precisely, if $b > 0$ and $b \neq 1$, then for a positive value of x the expression

$$\log_b x$$

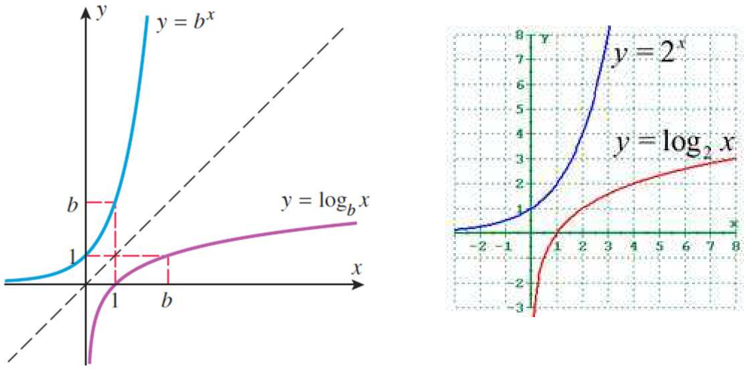
(read "the logarithm to the base b of x") denotes that exponent to which b must be raised to produce x. Thus, for example,

$$\log_{10}100 = 2 \qquad \log_{10}(1/1000) = -3 \qquad \log_2 16 = 4$$

We call the function $f(x) = \log_b x$ the *logarithmic function with base b*. Logarithmic functions can also be viewed as inverses of exponential functions. So if,

$$y = \log_b x \qquad \text{then} \qquad x = b^y$$

If $b > 0$ and $b \neq 1$, then b^x and $\log_b x$ are inverse functions.



The most important logarithms in application are those with base *e*. These are called *natural logarithms* because the function $log_e x$ is the inverse of the natural exponential function e^x. It is standard to denote the natural logarithm of *x* by *ln x* rather than $log_e x$. For example,

$$\ln 1 = 0 \qquad \ln e = 1 \qquad \ln 1/e = -1 \qquad \ln e^2 = 2$$

In general,

$$y = \ln x \qquad \text{IFF} \qquad x = e^y$$

Table 0.5.3

CORRESPONDENCE BETWEEN PROPERTIES OF LOGARITHMIC AND EXPONENTIAL FUNCTIONS

PROPERTY OF b^x	PROPERTY OF $\log_b x$
$b^0 = 1$	$\log_b 1 = 0$
$b^1 = b$	$\log_b b = 1$
Range is $(0, +\infty)$	Domain is $(0, +\infty)$
Domain is $(-\infty, +\infty)$	Range is $(-\infty, +\infty)$
x-axis is a horizontal asymptote	y-axis is a vertical asymptote

$$\log_b(b^x) = x \quad \text{for all real values of } x$$
$$b^{\log_b x} = x \quad \text{for } x > 0$$

$$\ln(e^x) = x \quad \text{for all real values of } x$$
$$e^{\ln x} = x \quad \text{for } x > 0$$

In words, the functions b^x and $log_b x$ cancel out the effect of one another when composed in either order; for example,

$$\log 10^x = x \quad 10^{\log x} = x \quad \ln e^x = x \quad e^{\ln x} = x \quad \ln e^5 = 5$$

Solving Equations Involving Exponentials and Logarithms

0.5.2 THEOREM (*Algebraic Properties of Logarithms*) *If $b > 0$, $b \neq 1$, $a > 0$, $c > 0$, and r is any real number, then:*

(*a*) $\log_b(ac) = \log_b a + \log_b c$ Product property

(*b*) $\log_b(a/c) = \log_b a - \log_b c$ Quotient property

(*c*) $\log_b(a^r) = r \log_b a$ Power property

(*d*) $\log_b(1/c) = -\log_b c$ Reciprocal property

These properties are often used to <u>expand</u> a single logarithm into sums, differences, and multiples of other logarithms and, conversely, to <u>condense</u> sums, differences, and multiples of logarithms into a single logarithm.

Expanding a Logarithmic Function

$$\log \frac{xy^5}{\sqrt{z}}$$

Condensing a Logarithmic Function

$$5\log 2 + \log 3 - \log 8$$

http://www.kutasoftware.com/FreeWorksheets/Alg2Worksheets/Properties%20of%20Logarithms.pdf

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An equation of the form $\log_b x = y$ can be solved for x by rewriting it in the exponential form $x = b^y$, and an equation of the form $b^x = y$ can be solved by rewriting it in the logarithm form $x = \log_b y$.

http://www.kutasoftware.com/FreeWorksheets/Alg2Worksheets/Logarithmic%20Equations.pdf

Change of Base Formula for Logarithms

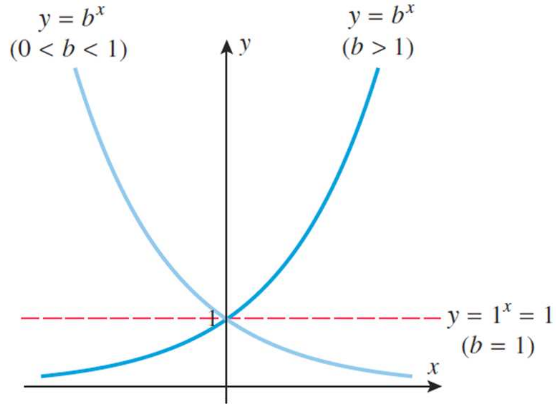
Scientific calculators generally have no keys for evaluating logarithms with bases other than 10 or *e*. However, this is not a serious deficiency because it is possible to express a logarithm with any base in terms of logarithms with any other base by using the following formula and *ln*.

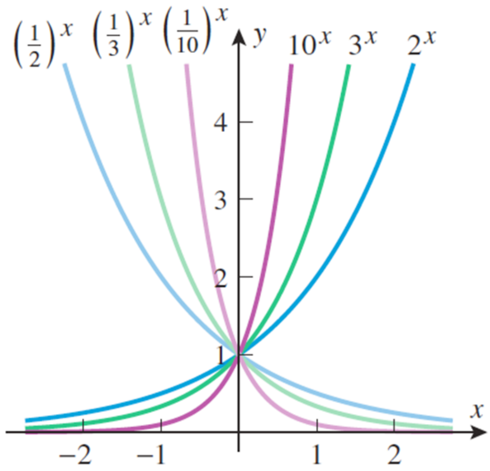
$$\log_b x = \frac{\ln x}{\ln b}$$

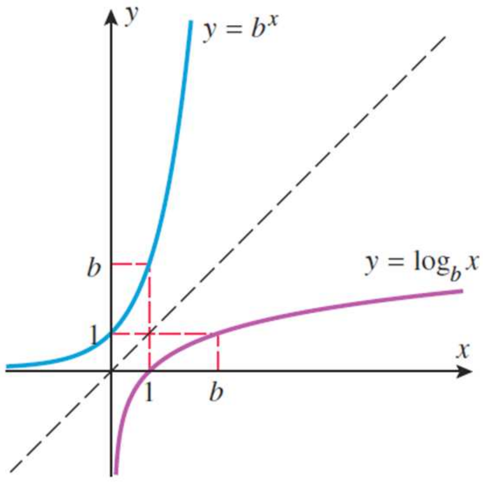
<u>EXAMPLE</u>

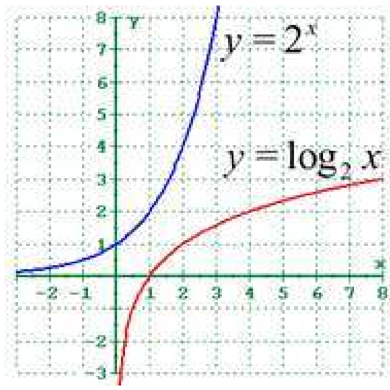
Evaluate $\log_3 6$

<u>HW</u>: PAGES 61 - 62, PROBLEMS 1, 3b, 4a, 5bc, 6d, 11b, 13, 15, 17, 19, 22, 25, 28, 32, 40









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Attachments

exp and log func2.jpg

e and ln func.jpg

exp and log func.jpg