LECTURE / DISCUSSION

Simultaneous Equations

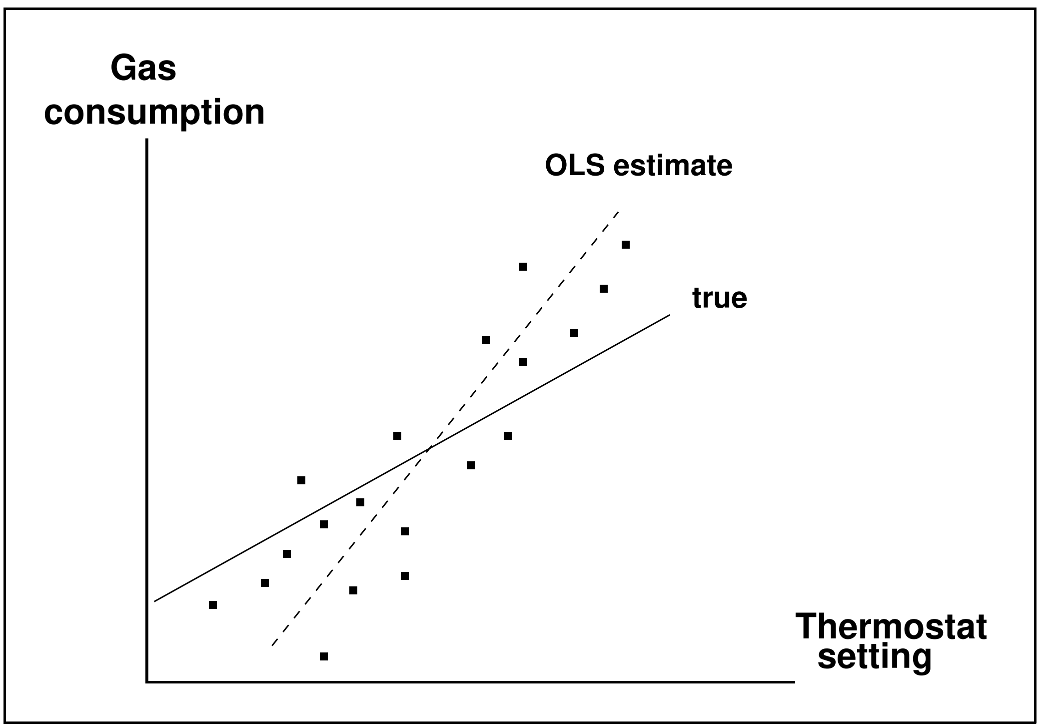
Simultaneous Equations

Example 1

Gas consumption = $\alpha + \beta$(thermostat setting) + θ(sq. ft.) + ε

Thermostat setting = $\eta + \phi$(income) + μ

ε and μ are correlated.



Problem: Because μ is correlated with ε, and μ affects thermostat setting, thermostat setting is correlated with ε.

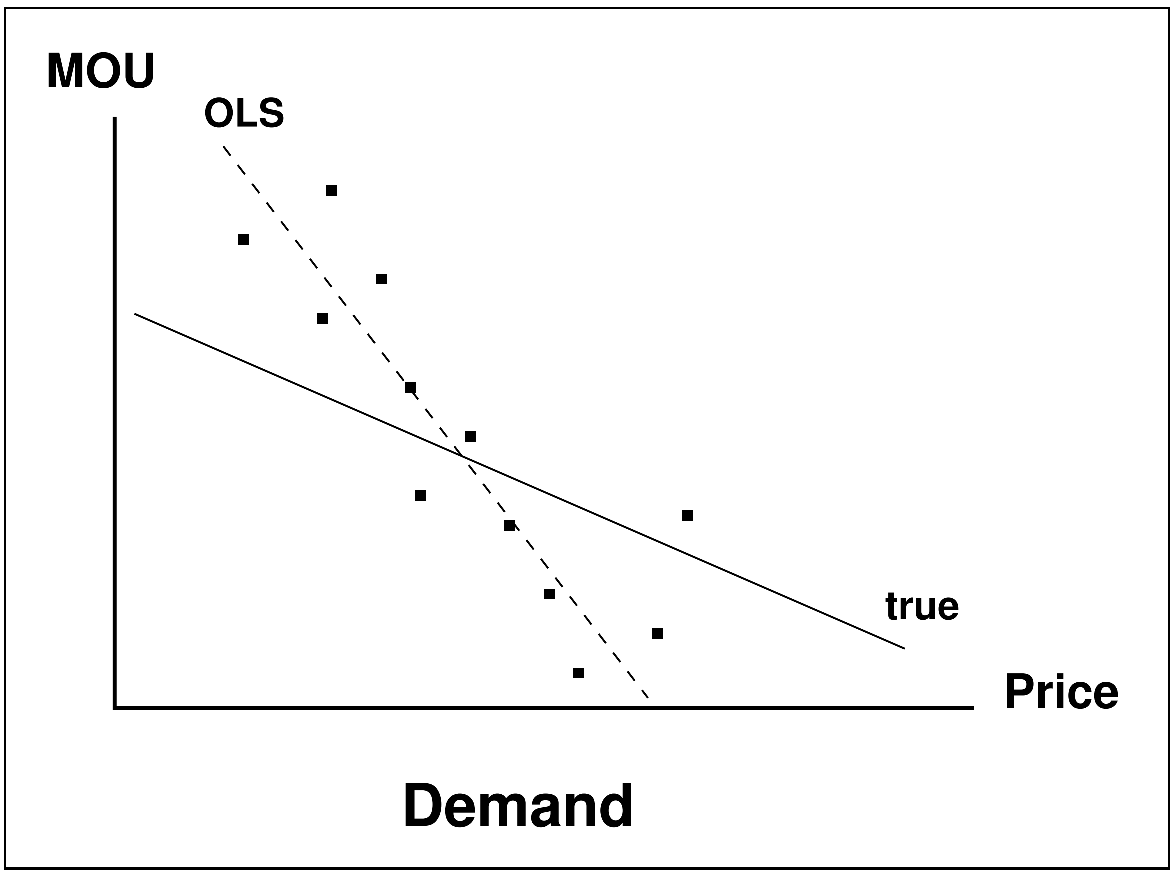
Example 2

Price elasticity of telecommunications demand:

$$\text{MOU} = \alpha + \beta(\text{price}) + \theta(\text{Nemply}) + \varepsilon$$

$$\text{Price} = \lambda + \varphi(\text{MOU}) + \eta(\text{region dum.}) + \mu$$

ε and μ are correlated.



Problem: Because μ is correlated with ε, and μ affects price, price is correlated with ε.

Example 3

Aggregate demand and supply:

Demand: $Q = \alpha + \beta P + \varepsilon$

Supply: $P = \lambda + \theta Q + \mu$

Solution: Apply IV/2SLS

$$M = \alpha + \beta P + \theta N + \varepsilon$$

$$P = \lambda + \varphi M + \eta R + \mu$$

Endogenous: M, P
Exogenous: R, N

Step 1: Regress endogenous variables against all the exogenous variables.

$$M = a + bR + cN + e$$
$$P = d + fR + gN + u$$
$$\text{Get } \hat{M}, \hat{P}$$

Step 2: Regress original equations, replacing endogenous explanatory variables M and P with predicted values $\hat{M}$ and $\hat{P}$.

$$M = \alpha + \beta \hat{P} + \theta N + \varepsilon^*$$

$$P = \lambda + \varphi \hat{M} + \eta R + \mu^*$$

Or apply IV directly.

Regress $M = \alpha + \beta P + \theta N + \varepsilon$
with instruments N and R
and $P = \lambda + \phi M + \eta R + \mu$
with instruments N and R .

TSP commands:

```
2sls(inst = (c,n,r)) m c,p,n ;
2sls(inst = (c,n,r)) p c,m,r ;
```

or

```
inst m c,p,n invr c,n,r ;
inst p c,m,r invr c,n,r ;
```

Reduced-Form Equation

Structural model: causal relationships

$$M = \alpha + \beta P + \theta N + \varepsilon$$
$$P = \lambda + \phi M + \eta R + \mu$$

Structural parameters: $\alpha, \beta, \theta, \lambda, \phi, \eta$.

Reduced-form equations: endogenous variables expressed as a function of exogenous variables.

$$M = \alpha + \beta(\lambda + \varphi M + \eta R + \mu) + \theta N + \varepsilon$$

$$M = \alpha + \beta\lambda + \beta\varphi M + \beta\eta R + \theta N + \beta\mu + \varepsilon$$

$$(1 - \beta\varphi)M = \alpha + \beta\lambda + \beta\eta R + \theta N + \beta\mu + \varepsilon$$

$$M = \left(\frac{\alpha + \beta\lambda}{1 - \beta\varphi}\right) + \left(\frac{\beta\eta}{1 - \beta\varphi}\right) R + \left(\frac{\theta}{1 - \beta\varphi}\right) N + \left(\frac{\beta\mu + \varepsilon}{1 - \beta\varphi}\right)$$

$$M = a + b\,R + c\,N + e$$

Similarly,

$$P = \left(\frac{\lambda + \varphi\alpha}{1 - \beta\varphi}\right) + \left(\frac{\eta}{1 - \beta\varphi}\right) R + \left(\frac{\varphi\theta}{1 - \beta\varphi}\right) N + \left(\frac{\varphi\varepsilon + \mu}{1 - \beta\varphi}\right)$$

$$P = d + f\,R + g\,N + u$$

Reduced form parameters: a, b, c, d, f, g .

1. Reduced-form equations are estimated in first step of 2SLS. Structural equations are estimated in second step of 2SLS.

2. Reduced-form equations can be estimated by OLS because exogenous variables are uncorrelated with errors.

3. There is a relation between the reduced-form parameters and the structural parameters.

   For example:

$$c = \frac{\theta}{1 - \beta\varphi}$$

Reduced form parameters give the full effect of a change in an exogenous variable.

$$M = \alpha + \beta P + \theta N + \varepsilon$$
$$P = \lambda + \phi M + \eta R + \mu$$

N rises by 1 unit:

M increases by θ
P increases by $\phi\theta$
M increases by $\beta[\phi\theta]$
P increases by $\phi[\beta\phi\theta] = \beta\phi^2\theta$
M increases by $\beta[\beta\phi^2\theta] = \beta^2\phi^2\theta$
P increases by $\phi[\beta^2\phi^2\theta] = \beta^2\phi^3\theta$
M increases by $\beta[\beta^2\phi^3\theta] = \beta^3\phi^3\theta$

.
.
.

Total effect on M

$$\theta + \beta\varphi\theta + (\beta\varphi)^2\theta + (\beta\varphi)^3\theta + \ldots = \sum_{t=0}^{\infty} (\beta\varphi)^t\theta$$
$$= \frac{\theta}{1 - \beta\varphi}$$

Total effect on P

$$\varphi\theta + \beta\varphi^2\theta + \beta^2\varphi^3\theta = \varphi\theta + (\beta\varphi)\varphi\theta + (\beta\varphi)^2\varphi\theta$$

$$= \sum_{t=0}^{\infty} (\beta\varphi)^t\varphi\theta = \frac{\varphi\theta}{1 - \beta\varphi}$$

Indirect Least Squares

Sometimes, structural parameters can be calculated from reduced-form parameters.

$$a = \frac{\alpha + \beta\lambda}{1 - \beta\varphi} \qquad d = \frac{\lambda + \varphi\alpha}{1 - \beta\varphi}$$

$$b = \frac{\beta\eta}{1 - \beta\varphi} \qquad f = \frac{\eta}{1 - \beta\varphi}$$

$$c = \frac{\theta}{1 - \beta\varphi} \qquad g = \frac{\varphi\theta}{1 - \beta\varphi}$$

Six equations, six unknowns.

$$\frac{b}{f} = \beta \qquad \frac{g}{c} = \varphi \qquad \text{etc.}$$

Example:

if	$\hat{a} = 4$	$\hat{d} = 8$
	$\hat{b} = 3$	$\hat{f} = 2$
	$\hat{c} = 7$	$\hat{g} = 3.5$
then	$\hat{\alpha} = -8.$	$\hat{\lambda} = 6$
	$\hat{\beta} = 1.5$	$\hat{\varphi} = 0.50$
	$\hat{\theta} = 1.75$	$\hat{\eta} = 0.50$

Note: ILS gives same estimates as 2SLS.

Cannot Always Apply ILS

I. Add an exogenous variable:

$$M = \alpha + \beta P + \theta N + \tau I + \varepsilon$$

$$P = \lambda + \varphi M + \eta R + \mu$$

7 structural parameters

Reduced-form equations:

$$M = a + bN + cI + dR + e$$

$$P = f + gN + hI + \ell R + u$$

8 reduced-form parameters

Eight equations for seven unknowns. No solution.

Can still apply 2SLS.

2SLS finds structural parameters that best fit the reduced form parameters.

II. Omit an exogenous variable:

$$M = \alpha + \beta P + \varepsilon$$

$$P = \lambda + \varphi M + \eta R + \mu$$

5 structural parameters

Reduced-form equations

$$M = a + bR + e$$

$$P = c + dR + u$$

4 reduced-form parameters

Four equations, five unknowns. Infinite number of solutions.

Also: Cannot apply 2SLS in Case II

Step 1:
$$\hat{M} = \hat{a} + \hat{b}R$$
$$\hat{P} = \hat{c} + \hat{d}R$$

Step 2:
$$M = \alpha + \beta\hat{P} + \varepsilon$$

$$\begin{aligned} P &= \lambda + \varphi\hat{M} + \eta R + \mu \\ &= \lambda + \varphi\hat{a} + \varphi\hat{b}R + \eta R + \mu \\ &= (\lambda + \varphi\hat{a}) + (\varphi\hat{b} + \eta)R + \mu \end{aligned}$$

Cannot estimate ϕ and η separately.

Identification

Name		Estimation method
Just identified	NRFP = NSP	2SLS = ILS
Over identified	NRFP > NSP	2SLS
Under identified	NRFP < NSP	Cannot estimate

NRFP: Number of reduced form parameters.
NSP: Number of structural parameters.

Efficiency

2SLS gives unbiased estimate.

But 2SLS ignores information contained in correlation between errors. So, 2SLS is not efficient.

$$M = \alpha + \beta P + \theta N + \varepsilon$$

$$P = \lambda + \varphi M + \eta R + \mu$$

$$\varepsilon, \mu \text{ correlated}$$

If we knew μ was high, we would know that ε is probably also high and hence M is higher than predicted from P and N only.

Similarly for ε and P .

3SLS: An Efficient Estimator

Step 1: Estimate reduced form equations. Get $\hat{M}$, $\hat{P}$.

Step 2: Estimate structural equations with $\hat{M}$ and $\hat{P}$.

Save residuals from these regressions, labeled $\hat{\varepsilon}$ and $\hat{\mu}$.

Step 3: Re-estimate structural equations with $\hat{\varepsilon}$ and $\hat{\mu}$ included as explanatory variables.

$$M = \alpha + \beta\hat{P} + \theta N + \rho\hat{\mu} + \varepsilon^{**}$$

$$P = \lambda + \varphi\hat{M} + \eta R + \tau\hat{\varepsilon} + \mu^{**}$$

Because ε and μ are correlated, $\hat{\mu}$ provides information for explaining M and $\hat{\varepsilon}$ provides information for explaining P . Including this information makes the estimates better.

Special Case:
Seemingly Unrelated Equations

$$Y_n = \alpha + \beta X_n + \varepsilon_n$$

$$W_n = \lambda + \theta S_n + \mu_n$$

No endogenous explanatory variables.
But: ε_n and μ_n are correlated.

Example: Y_n = temperature in San Francisco.
W_n = temperature in Monterey.

3SLS can be done in two steps, because reduced-form equations are the same as the structural equations.

Step 1: Estimate equations by OLS.
Get residuals $\hat{\varepsilon}_n$ and $\hat{\mu}_n$.

Step 2: Re-estimate equations including residuals as explanatory variables:

$$Y_n = \alpha + \beta X_n + \rho \hat{\mu}_n + \varepsilon_n^*$$

$$W_n = \lambda + \theta S_n + \tau \hat{\varepsilon}_n + \mu_n^*$$