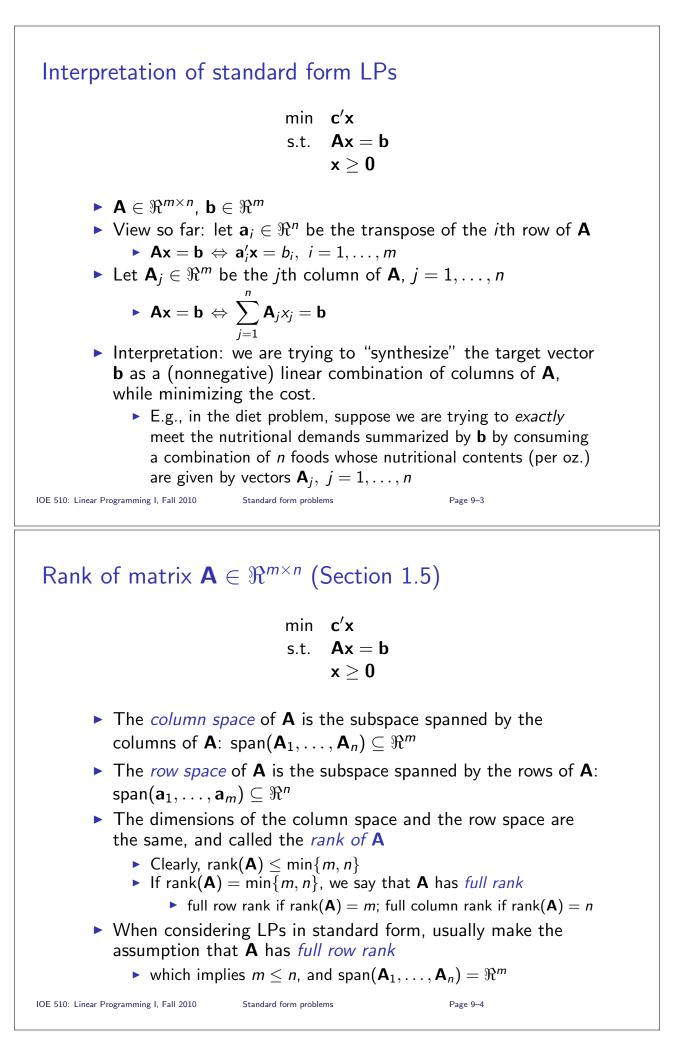
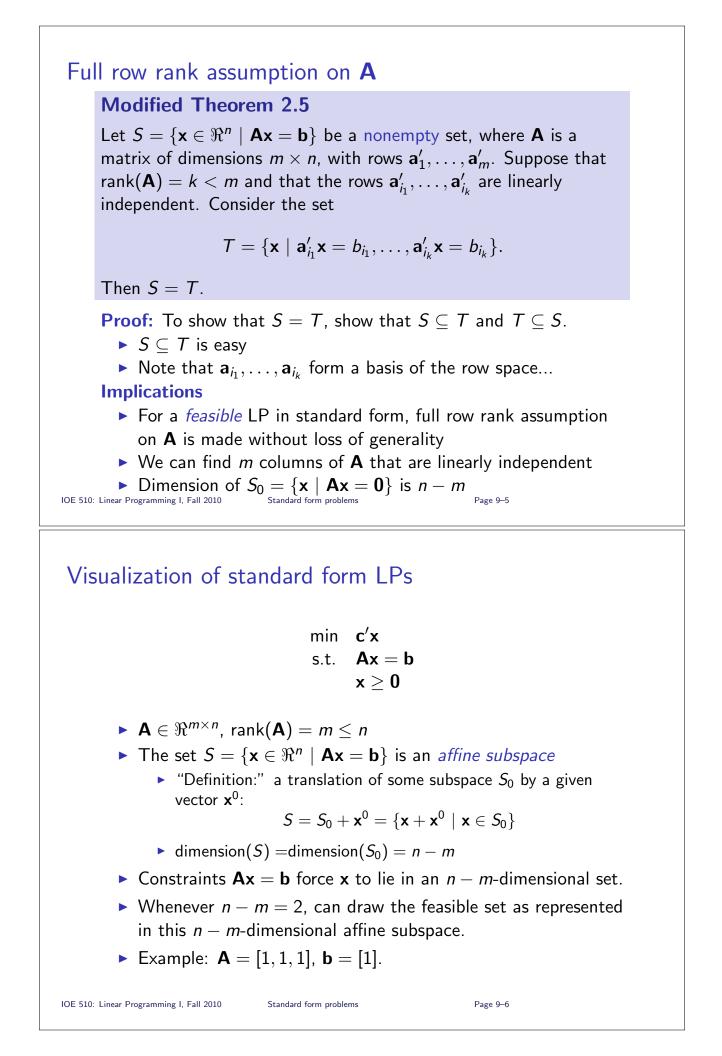
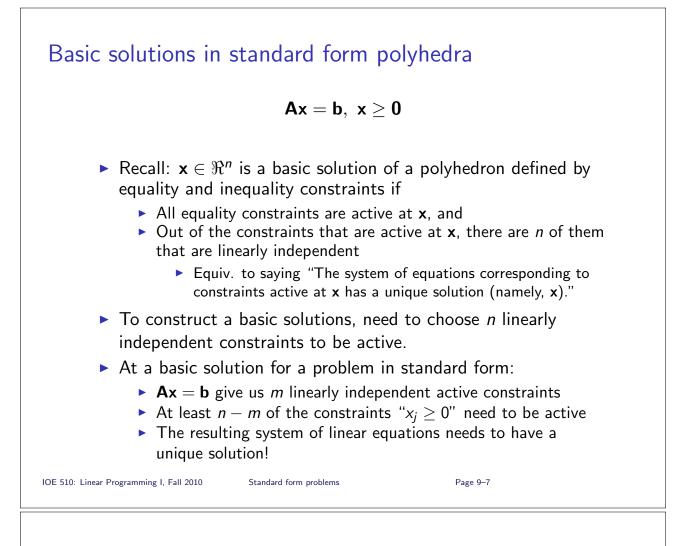


- Adjacent solutions and adjacent bases
- Optimality conditions (for general and standard form LPs)
- Developing an algorithm (the Simplex Method) for solving LPs in standard form







Example

Arbitrarily picking n - m sign constraints to be active might not result in a basic solution!

A =	1	1	2	1	0	0	0	, b =	[8]
	2	1	6	0	1	0	0		12
	1	0	4	0	0	1	0		4
	0	1	0	0	0	0	1		6

n = 7, m = 4; need (at least) 3 sign constraints active for a BS

- 1. Try $x_1 = x_2 = x_3 = 0$ (a BFS)
- 2. Try $x_1 = x_2 = x_4 = 0$ (a BS, but not a BFS)
- 3. Try $x_1 = x_3 = x_4 = 0$ (not a BS: no solutions)
- 4. Try $x_4 = x_5 = x_6 = 0$ (not a BS: multiple solutions, e.g., $x_1 = 2, x_2 = 5, x_3 = 0.5, x_7 = 1$, or $x_1 = 1, x_2 = 5.5, x_3 = 0.75, x_7 = 0.5$)

Basic solutions in standard form polyhedra

Theorem 2.4

Consider the polyhedron represented by constraints $\mathbf{A}\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \ge \mathbf{0}$ and assume that the $m \times n$ matrix \mathbf{A} has linearly independent rows. A vector $\mathbf{x} \in \Re^n$ is a basic solution if and only if we have $\mathbf{A}\mathbf{x} = \mathbf{b}$ and there exist indices $B(1), \ldots, B(m)$ such that: (a) The columns $\mathbf{A}_{B(1)}, \ldots, \mathbf{A}_{B(m)}$ are linearly independent; (b) if $j \ne B(1), \ldots, B(m)$, then $x_i = 0$.

Proof of the "if" part:

Suppose x satisfies (a) and (b). Then x satisfies

$$\sum_{i=1}^{m} \mathbf{A}_{B(i)} x_{B(i)} = \mathbf{b}, \ x_j = 0, \, j \neq B(1), \dots, B(m)$$

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- Above system has a unique solution (since A_{B(1)},..., A_{B(m)} are linearly independent)
- ► Therefore, **x** is a BFS

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Basic solutions in standard form polyhedra

Theorem 2.4

Consider the polyhedron represented by constraints $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \ge \mathbf{0}$ and assume that the $m \times n$ matrix \mathbf{A} has linearly independent rows. A vector $\mathbf{x} \in \mathbb{R}^n$ is a basic solution if and only if we have $A\mathbf{x} = \mathbf{b}$ and there exist indices $B(1), \ldots, B(m)$ such that: (a) The columns $\mathbf{A}_{B(1)}, \ldots, \mathbf{A}_{B(m)}$ are linearly independent;

(b) if $j \neq B(1), ..., B(m)$, then $x_j = 0$.

Proof of the "only if" part:

- ► Suppose **x** is a BS.
- Let $x_{B(1)}, \ldots, x_{B(k)}$ be the non-zero components of **x** ($k \le m$)
- ► The following system has a unique solution (since **x** is a BS):

$$\sum_{i=1}^{k} \mathbf{A}_{B(i)} x_{B(i)} = \mathbf{b}, \ x_{j} = 0, \, j \neq B(1), \dots, B(k)$$

- Hence, $\mathbf{A}_{B(1)}, \ldots, \mathbf{A}_{B(k)}$ are linearly independent
- If k < m, can find additional columns A_{B(k+1)},..., A_{B(m)} so that columns A_{B(1)},..., A_{B(m)} are linearly independent

With this selection of $B(1), \ldots, B(m)$, x satisfies (a) and (b) IOE 510: Linear Programming I, Fall 2010 Standard form problems

Procedure for constructing basic solutions of problems in standard form

- 1. Choose *m* linearly independent columns $\mathbf{A}_{B(1)}, \ldots, \mathbf{A}_{B(m)}$
- 2. Let $x_i = 0$ for all $i \neq B(1), \ldots, B(m)$
- 3. Solve the system of *m* equations $\mathbf{A}\mathbf{x} = \mathbf{b}$ for the unknowns $x_{B(1)}, \dots, x_{B(m)}$

Example:
$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 & 0 \\ 2 & 1 & 6 & 0 & 1 & 0 & 0 \\ 1 & 0 & 4 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} 8 \\ 12 \\ 4 \\ 6 \end{bmatrix}$$

1. Let $B(1) = 4$, $B(2) = 1$, $B(3) = 6$, $B(4) = 2$.
2. $x_3 = x_5 = x_7 = 0$
3. Solve

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_{B(1)} + \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} x_{B(2)} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} x_{B(3)} + \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} x_{B(4)} = \begin{bmatrix} 8 \\ 12 \\ 4 \\ 6 \end{bmatrix}$$

 $x_4 = x_{B(1)} = -1, \ x_1 = x_{B(2)} = 3, \ x_6 = x_{B(3)} = 1, \ x_2 = x_{B(4)} = 6$
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Terminology of BSs for standard form systems

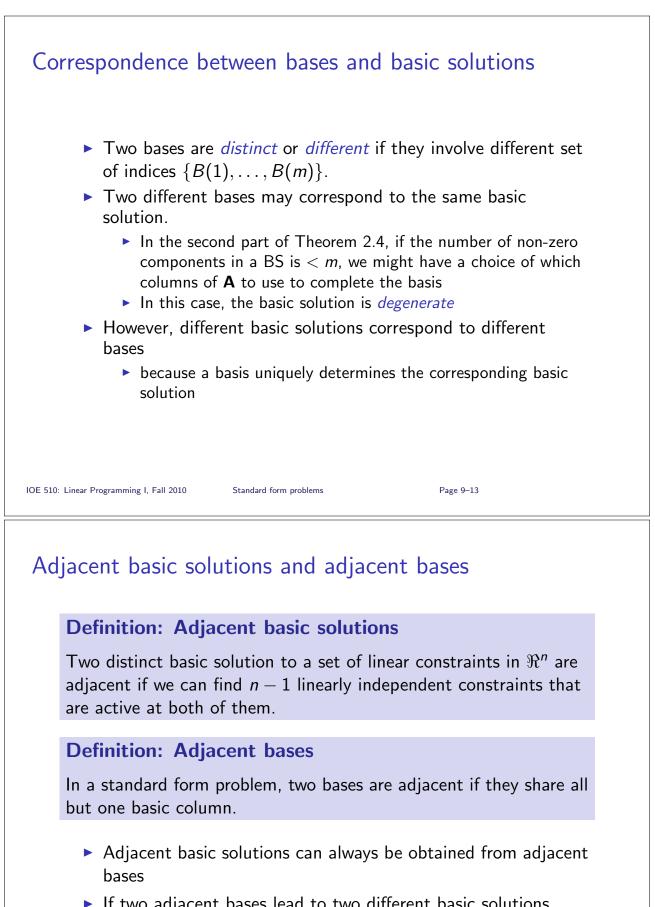
If **x** is a basic solution, and $x_{B(1)}, \ldots, x_{B(m)}$ are as above,

Columns A_{B(1)},..., A_{B(m)} — basic columns; they form a basis of R^m.

• The matrix
$$\mathbf{B} = \begin{bmatrix} | & | & | & | \\ \mathbf{A}_{B(1)} & \mathbf{A}_{B(2)} & \cdots & \mathbf{A}_{B(m)} \\ | & | & | \end{bmatrix}$$
 is the *basis*

matrix

- Variables x_B = (x_{B(1)},...,x_{B(m)})' basic variables; the remaining variables are nonbasic
- Unique solution of $\mathbf{B}\mathbf{x}_B = b$ is $\mathbf{x}_B = \mathbf{B}^{-1}b$
- A BFS "synthesizes" the target vector b as a (nonnegative) linear combination of basic columns of A



If two adjacent bases lead to two different basic solutions, then these solutions are adjacent.

Degeneracy in standard form polyhedra

Definition 2.10

A basic solution $\mathbf{x} \in \Re^n$ is said to be **degenerate** if more than *n* of the constraints are active at \mathbf{x} .

Definition 2.11

Consider the standard from polyhedron

 $P = {\mathbf{x} \in \Re^n \mid \mathbf{A}\mathbf{x} = \mathbf{b}, \ \mathbf{x} \ge \mathbf{0}}$ and let \mathbf{x} be a basic solution. Let $\mathbf{A} \in \Re^{m \times n}$ have full row rank. The vector \mathbf{x} is a **degenerate** basic solution if more than n - m of the components of \mathbf{x} are zero.

Degeneracy is not a purely geometric property! It depends on problem representation.

$$P = \{(x_1, x_2, x_3) : x_1 - x_2 = 0, x_1 + x_2 + 2x_3 = 2, x_1, x_2, x_3 \ge 0\}$$

VS.

$$P = \{(x_1, x_2, x_3) : x_1 - x_2 = 0, \ x_1 + x_2 + 2x_3 = 2, \ x_1, x_3 \ge 0\}$$

(0, 0, 1) is degenerate in the 1st representation, but not the 2nd. IOE 510: Linear Programming I, Fall 2010 Standard form problems Page 9–15