Recap, and outline of Lecture 9

Previously

- Every LP is either
  - infeasible,
  - unbounded,
  - or has (one or more) optimal solutions
- Every LP can be converted into an equivalent LP in standard form
- Feasible LPs in standard form
  - always have at least one basic feasible solutions, and hence
  - (if not unbounded) always have at least one optimal solution which is a BFS

Today

- Better understanding of LPs in standard form

Standard form problems

$$\begin{aligned} \min \quad & \mathbf{c}'\mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

- Converting any LP into standard form (✓)
- Interpretation and visualization
- Full row rank assumption on **A**
- Basic solutions and *bases* in standard form polyhedra
- Degeneracy in standard form polyhedra
- Adjacent solutions and adjacent bases
- Optimality conditions (for general and standard form LPs)
- Developing an algorithm (the Simplex Method) for solving LPs in standard form

Interpretation of standard form LPs

$$\begin{aligned} \min \quad & \mathbf{c}'\mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

- $\mathbf{A} \in \Re^{m \times n}$, $\mathbf{b} \in \Re^m$
- View so far: let $\mathbf{a}_i \in \Re^n$ be the transpose of the ith row of $\mathbf{A}$
  - $\mathbf{A}\mathbf{x} = \mathbf{b} \Leftrightarrow \mathbf{a}_i'\mathbf{x} = b_i, \ i = 1, \ldots, m$
- Let $\mathbf{A}_j \in \Re^m$ be the jth column of $\mathbf{A}$, $j = 1, \ldots, n$
  - $\mathbf{A}\mathbf{x} = \mathbf{b} \Leftrightarrow \sum_{j=1}^{n} \mathbf{A}_j x_j = \mathbf{b}$
- Interpretation: we are trying to "synthesize" the target vector $\mathbf{b}$ as a (nonnegative) linear combination of columns of $\mathbf{A}$, while minimizing the cost.
  - E.g., in the diet problem, suppose we are trying to *exactly* meet the nutritional demands summarized by $\mathbf{b}$ by consuming a combination of n foods whose nutritional contents (per oz.) are given by vectors $\mathbf{A}_j, \ j = 1, \ldots, n$

Rank of matrix $\mathbf{A} \in \Re^{m \times n}$ (Section 1.5)

$$\begin{aligned} \min \quad & \mathbf{c}'\mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

- The *column space* of $\mathbf{A}$ is the subspace spanned by the columns of $\mathbf{A}$: $\text{span}(\mathbf{A}_1, \ldots, \mathbf{A}_n) \subseteq \Re^m$
- The *row space* of $\mathbf{A}$ is the subspace spanned by the rows of $\mathbf{A}$: $\text{span}(\mathbf{a}_1, \ldots, \mathbf{a}_m) \subseteq \Re^n$
- The dimensions of the column space and the row space are the same, and called the *rank of* $\mathbf{A}$
  - Clearly, $\text{rank}(\mathbf{A}) \leq \min\{m, n\}$
  - If $\text{rank}(\mathbf{A}) = \min\{m, n\}$, we say that $\mathbf{A}$ has *full rank*
    - full row rank if $\text{rank}(\mathbf{A}) = m$; full column rank if $\text{rank}(\mathbf{A}) = n$
- When considering LPs in standard form, usually make the assumption that $\mathbf{A}$ has *full row rank*
  - which implies $m \leq n$, and $\text{span}(\mathbf{A}_1, \ldots, \mathbf{A}_n) = \Re^m$

Full row rank assumption on $\mathbf{A}$

Modified Theorem 2.5

Let $S = \{\mathbf{x} \in \Re^n \mid \mathbf{A}\mathbf{x} = \mathbf{b}\}$ be a nonempty set, where $\mathbf{A}$ is a matrix of dimensions $m \times n$, with rows $\mathbf{a}_1', \ldots, \mathbf{a}_m'$. Suppose that $\text{rank}(\mathbf{A}) = k < m$ and that the rows $\mathbf{a}_{i_1}', \ldots, \mathbf{a}_{i_k}'$ are linearly independent. Consider the set

$$T = \{\mathbf{x} \mid \mathbf{a}_{i_1}'\mathbf{x} = b_{i_1}, \ldots, \mathbf{a}_{i_k}'\mathbf{x} = b_{i_k}\}.$$

Then $S = T$.

Proof: To show that $S = T$, show that $S \subseteq T$ and $T \subseteq S$.

- $S \subseteq T$ is easy
- Note that $\mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_k}$ form a basis of the row space...

Implications

- For a *feasible* LP in standard form, full row rank assumption on $\mathbf{A}$ is made without loss of generality
- We can find m columns of $\mathbf{A}$ that are linearly independent
- Dimension of $S_0 = \{\mathbf{x} \mid \mathbf{A}\mathbf{x} = \mathbf{0}\}$ is $n - m$

Visualization of standard form LPs

$$\begin{array}{ll} \min & \mathbf{c}'\mathbf{x} \\ \text{s.t.} & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{array}$$

- $\mathbf{A} \in \Re^{m \times n}$, $\text{rank}(\mathbf{A}) = m \leq n$
- The set $S = \{\mathbf{x} \in \Re^n \mid \mathbf{A}\mathbf{x} = \mathbf{b}\}$ is an *affine subspace*
  - "Definition:" a translation of some subspace S_0 by a given vector $\mathbf{x}^0$:
  $$S = S_0 + \mathbf{x}^0 = \{\mathbf{x} + \mathbf{x}^0 \mid \mathbf{x} \in S_0\}$$
  - $\text{dimension}(S) = \text{dimension}(S_0) = n - m$
- Constraints $\mathbf{A}\mathbf{x} = \mathbf{b}$ force $\mathbf{x}$ to lie in an $n - m$-dimensional set.
- Whenever $n - m = 2$, can draw the feasible set as represented in this $n - m$-dimensional affine subspace.
- Example: $\mathbf{A} = [1, 1, 1]$, $\mathbf{b} = [1]$.

Basic solutions in standard form polyhedra

$$\mathbf{Ax} = \mathbf{b}, \ \mathbf{x} \geq \mathbf{0}$$

- Recall: $\mathbf{x} \in \Re^n$ is a basic solution of a polyhedron defined by equality and inequality constraints if
  - All equality constraints are active at $\mathbf{x}$, and
  - Out of the constraints that are active at $\mathbf{x}$, there are n of them that are linearly independent
    - Equiv. to saying "The system of equations corresponding to constraints active at $\mathbf{x}$ has a unique solution (namely, $\mathbf{x}$)."
- To construct a basic solutions, need to choose n linearly independent constraints to be active.
- At a basic solution for a problem in standard form:
  - $\mathbf{Ax} = \mathbf{b}$ give us m linearly independent active constraints
  - At least $n - m$ of the constraints "$x_j \geq 0$" need to be active
  - The resulting system of linear equations needs to have a unique solution!

Example

Arbitrarily picking $n - m$ sign constraints to be active might not result in a basic solution!

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 & 0 \\ 2 & 1 & 6 & 0 & 1 & 0 & 0 \\ 1 & 0 & 4 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} 8 \\ 12 \\ 4 \\ 6 \end{bmatrix}$$

$n = 7, \ m = 4$; need (at least) 3 sign constraints active for a BS

1. Try $x_1 = x_2 = x_3 = 0$ (a BFS)
2. Try $x_1 = x_2 = x_4 = 0$ (a BS, but not a BFS)
3. Try $x_1 = x_3 = x_4 = 0$ (not a BS: no solutions)
4. Try $x_4 = x_5 = x_6 = 0$ (not a BS: multiple solutions, e.g., $x_1 = 2, \ x_2 = 5, \ x_3 = 0.5, \ x_7 = 1$, or $x_1 = 1, \ x_2 = 5.5, \ x_3 = 0.75, \ x_7 = 0.5$)

Basic solutions in standard form polyhedra

Theorem 2.4

Consider the polyhedron represented by constraints $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$ and assume that the $m \times n$ matrix $\mathbf{A}$ has linearly independent rows. A vector $\mathbf{x} \in \Re^n$ is a basic solution if and only if we have $\mathbf{Ax} = \mathbf{b}$ and there exist indices $B(1), \ldots, B(m)$ such that:
(a) The columns $\mathbf{A}_{B(1)}, \ldots, \mathbf{A}_{B(m)}$ are linearly independent;
(b) if $j \neq B(1), \ldots, B(m)$, then $x_j = 0$.

Proof of the "if" part:

- Suppose $\mathbf{x}$ satisfies **(a)** and **(b)**. Then $\mathbf{x}$ satisfies

$$\sum_{i=1}^{m} \mathbf{A}_{B(i)} x_{B(i)} = \mathbf{b}, \quad x_j = 0, \, j \neq B(1), \ldots, B(m)$$

- Above system has a unique solution (since $\mathbf{A}_{B(1)}, \ldots, \mathbf{A}_{B(m)}$ are linearly independent)
- Therefore, $\mathbf{x}$ is a BFS

Basic solutions in standard form polyhedra

Theorem 2.4

Consider the polyhedron represented by constraints $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$ and assume that the $m \times n$ matrix $\mathbf{A}$ has linearly independent rows. A vector $\mathbf{x} \in \Re^n$ is a basic solution if and only if we have $\mathbf{Ax} = \mathbf{b}$ and there exist indices $B(1), \ldots, B(m)$ such that:
(a) The columns $\mathbf{A}_{B(1)}, \ldots, \mathbf{A}_{B(m)}$ are linearly independent;
(b) if $j \neq B(1), \ldots, B(m)$, then $x_j = 0$.

Proof of the "only if" part:

- Suppose $\mathbf{x}$ is a BS.
- Let $x_{B(1)}, \ldots, x_{B(k)}$ be the non-zero components of $\mathbf{x}$ ($k \leq m$)
- The following system has a unique solution (since $\mathbf{x}$ is a BS):

$$\sum_{i=1}^{k} \mathbf{A}_{B(i)} x_{B(i)} = \mathbf{b}, \quad x_j = 0, \, j \neq B(1), \ldots, B(k)$$

- Hence, $\mathbf{A}_{B(1)}, \ldots, \mathbf{A}_{B(k)}$ are linearly independent
- If $k < m$, can find additional columns $\mathbf{A}_{B(k+1)}, \ldots, \mathbf{A}_{B(m)}$ so that columns $\mathbf{A}_{B(1)}, \ldots, \mathbf{A}_{B(m)}$ are linearly independent
- With this selection of $B(1), \ldots, B(m)$, $\mathbf{x}$ satisfies **(a)** and **(b)**

Procedure for constructing basic solutions of problems in standard form

1. Choose m linearly independent columns $\mathbf{A}_{B(1)}, \ldots, \mathbf{A}_{B(m)}$
2. Let $x_i = 0$ for all $i \neq B(1), \ldots, B(m)$
3. Solve the system of m equations $\mathbf{Ax} = \mathbf{b}$ for the unknowns $x_{B(1)}, \ldots, x_{B(m)}$

$$\text{Example: } \mathbf{A} = \begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 & 0 \\ 2 & 1 & 6 & 0 & 1 & 0 & 0 \\ 1 & 0 & 4 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \; \mathbf{b} = \begin{bmatrix} 8 \\ 12 \\ 4 \\ 6 \end{bmatrix}$$

1. Let $B(1) = 4,\ B(2) = 1,\ B(3) = 6,\ B(4) = 2.$
2. $x_3 = x_5 = x_7 = 0$
3. Solve

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_{B(1)} + \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} x_{B(2)} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} x_{B(3)} + \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} x_{B(4)} = \begin{bmatrix} 8 \\ 12 \\ 4 \\ 6 \end{bmatrix}$$

$$x_4 = x_{B(1)} = -1,\ x_1 = x_{B(2)} = 3,\ x_6 = x_{B(3)} = 1,\ x_2 = x_{B(4)} = 6$$

Terminology of BSs for standard form systems

If $\mathbf{x}$ is a basic solution, and $x_{B(1)}, \ldots, x_{B(m)}$ are as above,

- Columns $\mathbf{A}_{B(1)}, \ldots, \mathbf{A}_{B(m)}$ — *basic columns*; they form a basis of $\Re^m$.
- The matrix $\mathbf{B} = \begin{bmatrix} | & | & & | \\ \mathbf{A}_{B(1)} & \mathbf{A}_{B(2)} & \cdots & \mathbf{A}_{B(m)} \\ | & | & & | \end{bmatrix}$ is the *basis matrix*
- Variables $\mathbf{x}_B = (x_{B(1)}, \ldots, x_{B(m)})'$ — *basic variables*; the remaining variables are *nonbasic*
- Unique solution of $\mathbf{Bx}_B = b$ is $\mathbf{x}_B = \mathbf{B}^{-1}b$
- A BFS "synthesizes" the target vector $\mathbf{b}$ as a (nonnegative) linear combination of basic columns of $\mathbf{A}$

Correspondence between bases and basic solutions

- Two bases are *distinct* or *different* if they involve different set of indices $\{B(1), \ldots, B(m)\}$.
- Two different bases may correspond to the same basic solution.
  - In the second part of Theorem 2.4, if the number of non-zero components in a BS is $< m$, we might have a choice of which columns of $\mathbf{A}$ to use to complete the basis
  - In this case, the basic solution is *degenerate*
- However, different basic solutions correspond to different bases
  - because a basis uniquely determines the corresponding basic solution

Adjacent basic solutions and adjacent bases

Definition: Adjacent basic solutions

Two distinct basic solution to a set of linear constraints in $\Re^n$ are adjacent if we can find $n-1$ linearly independent constraints that are active at both of them.

Definition: Adjacent bases

In a standard form problem, two bases are adjacent if they share all but one basic column.

- Adjacent basic solutions can always be obtained from adjacent bases
- If two adjacent bases lead to two different basic solutions, then these solutions are adjacent.

Degeneracy in standard form polyhedra

Definition 2.10

A basic solution $\mathbf{x} \in \Re^n$ is said to be **degenerate** if more than n of the constraints are active at $\mathbf{x}$.

Definition 2.11

Consider the standard from polyhedron $P = \{\mathbf{x} \in \Re^n \mid \mathbf{A}\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0}\}$ and let $\mathbf{x}$ be a basic solution. Let $\mathbf{A} \in \Re^{m \times n}$ have full row rank. The vector $\mathbf{x}$ is a **degenerate** basic solution if more than $n - m$ of the components of $\mathbf{x}$ are zero.

Degeneracy is not a purely geometric property! It depends on problem representation.

$$P = \{(x_1, x_2, x_3) : x_1 - x_2 = 0,\ x_1 + x_2 + 2x_3 = 2,\ x_1, x_2, x_3 \geq 0\}$$

vs.

$$P = \{(x_1, x_2, x_3) : x_1 - x_2 = 0,\ x_1 + x_2 + 2x_3 = 2,\ x_1, x_3 \geq 0\}$$

$(0, 0, 1)$ is degenerate in the 1st representation, but not the 2nd.