Math 370, Spring 2008
Prof. A.J. Hildebrand

Practice Test 2 Solutions

About this test. This is a practice test made up of a random collection of 15 problems from past Course 1/P actuarial exams. Most of the problems have appeared on the Actuarial Problem sets passed out in class, but I have also included some additional problems.

Topics covered. This test covers the topics of Chapters 1–5 in Hogg/Tanis and Actuarial Problem Sets 1–5.

Ordering of the problems. In order to mimick the conditions of the actual exam as closely as possible, the problems are in no particular order. Easy problems are mixed in with hard ones. In fact, I used a program to select the problems and to put them in random order, with no human intervention. If you find the problems hard, it's the luck of the draw!

Suggestions on taking the test. Try to take this test as if it were the real thing. Take it as a closed books, notes, etc., time yourself, and stop after 2 hours. In the actuarial exam you have 3 hours for 30 problems, so 2 is an appropriate time limit for a 20 problem test.

Answers/solutions. Answers and solutions will be posted on the course webpage, `www.math.uiuc.edu/~hildebr/370`.

1. [4-115]

   Let X and Y denote the values of two stocks at the end of a five-year period. X is uniformly distributed on the interval $(0, 12)$. Given $X = x$, Y is uniformly distributed on the interval $(0, x)$. Determine $\text{Cov}(X, Y)$ according to this model.

   (A) 0 (B) 4 (C) 6 (D) 12 (E) 24

   Answer: C: 6

   Solution: To find $\text{Cov}(X, Y)$, we need to compute the three expectations $E(XY)$, $E(X)$ and $E(Y)$. Now $E(X) = 6$, since X is uniform on the interval $(0, 12)$. To find the remaining two expectations we need to first compute the joint density $f(x, y)$. To this end we use the formula $f(x, y) = h(y|x)f_X(x)$. Since $f_X(x)$ is uniform on the interval $(0, 12)$, we have $f_X(x) = 1/12$ for $0 < x < 12$. Since the conditional density of Y given $X = x$ is uniform on $[0, x]$, we have $h(y|x) = 1/x$ for $0 \le y \le x \le 12$. Hence

$$f(x, y) = h(y|x)f_X(x) = \frac{1}{12x}, \quad 0 \le y \le x \le 12.$$

   The rest is now routine:

$$\begin{aligned} E(XY) &= \int_{x=0}^{12} \int_{y=0}^{x} xy \frac{1}{12x} dy dx = \frac{1}{24} \int_{x=0}^{12} x^2 dx = 24, \\ E(Y) &= \int_{x=0}^{12} \int_{y=0}^{x} y \frac{1}{12x} dy dx = \frac{1}{24} \int_{x=0}^{12} x dx = 3, \\ \text{Cov}(X, Y) &= E(XY) - E(X)E(Y) = 24 - 6 \cdot 3 = 6. \end{aligned}$$

2. [4-17]

   A device contains two circuits. The second circuit is a backup for the first, so the second is used only when the first has failed. The device fails when and only when the second circuit fails. Let X and Y be the times at which the first and second circuits fail, respectively. X and Y have joint probability density function

$$f(x, y) = \begin{cases} 6e^{-x}e^{-2y} & \text{for } 0 < x < y < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

   What is the expected time at which the device fails?

   (A) 0.33 (B) 0.50 (C) 0.67 (D) 0.83 (E) 1.50

   Answer: D: 0.83

   Solution: The time point at which the device fails is given by Y, the failure time of the second (back-up) device, so we need to compute $E(Y)$. Now, $E(Y) = \iint_R y f(x, y) dx dy$, where $f(x, y)$ is the given density and R the region given by $0 \le y \le \infty, 0 \le x \le y$. Therefore,

$$\begin{aligned} E(Y) &= \int_{y=0}^{\infty} \int_{x=0}^{y} y 6e^{-x}e^{-2y} dx dy \\ &= \int_{y=0}^{\infty} 6ye^{-2y}(1 - e^{-y}) dy = 6 \int_{y=0}^{\infty} y \left(e^{-2y} - e^{-3y}\right) dy \\ &= 6 \left[y \left(\frac{e^{-2y}}{-2} - \frac{e^{-3y}}{-3} \right) \right]_{y=0}^{\infty} + 6 \int_{y=0}^{\infty} \left(\frac{e^{-2y}}{2} - \frac{e^{-3y}}{3} \right) dy \\ &= 6 \left(\frac{1}{4} - \frac{1}{9} \right) = \frac{5}{6}. \end{aligned}$$

3. [2-3]

   An actuary has discovered that policyholders are three times as likely to file two claims as to file four claims. If the number of claims filed has a Poisson distribution, what is the variance of the number of claims filed?

   (A) $1/\sqrt{3}$ (B) 1 (C) $\sqrt{2}$ (D) 2 (E) 4

   Answer: D: 2

   Solution: Let X denote the number of claims. We are given that X has Poisson distribution and that $P(X = 2) = 3P(X = 4)$. Substituting the formula for a Poisson p.m.f., $P(X = x) = e^{-\lambda}\lambda^x/x!$ into this equation, we get $e^{-\lambda}\lambda^2/2! = 3e^{-\lambda}\lambda^4/4!$, which implies $\lambda^2 = 4$, and therefore $\lambda = 2$ since λ must be positive. Since the variance of a Poisson distribution is given by $\sigma^2 = \lambda$, it follows that $\sigma^2 = 2$.

4. [1-103]

   An insurance company examines its pool of auto insurance customers and gathers the following information:

   (i) All customers insure at least one car.
   (ii) 70% of the customers insure more than one car.
   (iii) 20% of the customers insure a sports car.
   (iv) Of those customers who insure more than one car, 15% insure a sports car.

   Calculate the probability that a randomly selected customer insures exactly one car and that car is not a sports car.

   (A) 0.13 (B) 0.21 (C) 0.24 (D) 0.25 (E) 0.30

   Answer: B: 0.21

   Solution: The tricky part here is the proper interpretation of the given data and the question asked. Letting A denote the event "insures more than one car" and B the event "insures a sports car", we need to compute $P(A' \cap B')$. (Note that the complement to A, "at most one car", is equivalent to "exactly one car", by the assumption that all customers insure at least one car.)

   In terms of this notation, the given data translates into $P(A) = 0.7$, $P(B) = 0.2$, $P(B|A) = 0.15$, and from this we deduce $P(A \cap B) = P(B|A)P(A) = 0.105$. We now have everything at hand to compute the requested probability:

$$
\begin{aligned}
P(A' \cap B') &= 1 - P(A \cup B) = 1 - P(A) - P(B) + P(A \cap B) \\
&= 1 - 0.7 - 0.2 + 0.105 = 0.205.
\end{aligned}
$$

5. [3-20]

   A device that continuously measures and records seismic activity is placed in a remote region. The time, T, to failure of this device is exponentially distributed with mean 3 years. Since the device will not be monitored during its first two years of service, the time to discovery of its failure is $X = \max(T, 2)$. Determine $E(X)$.

   (A) $2 + \frac{1}{3}e^{-6}$
   (B) $2 - 2e^{-2/3} + 5e^{-4/3}$
   (C) 3

(D) $2 + 3e^{-2/3}$

(E) 5

Answer: D

Solution: Since T is exponentially distributed with mean 3, the density of T is $f(t) = (1/3)e^{-t/3}$ for $t > 0$. Since $X = \max(T, 2)$, we have $X = 2$ if $0 \le T \le 2$ and $X = T$ if $2 < T < \infty$.

Thus,

$$\begin{aligned} E(X) &= \int_0^2 2\frac{1}{3}e^{-t/3}dt + \int_2^\infty t \cdot \frac{1}{3}e^{-t/3}dt \\ &= 2(1 - e^{-2/3}) - te^{-t/3}\Big|_2^\infty + \int_2^\infty e^{-t/3}dt \\ &= 2(1 - e^{-2/3}) + 2e^{-t/3} + 3e^{-2/3} = 2 + 3e^{-2/3} \end{aligned}$$

6. [4-2]

A car dealership sells 0, 1, or 2 luxury cars on any day. When selling a car, the dealer also tries to persuade the customer to buy an extended warranty for the car. Let X denote the number of luxury cars sold in a given day, and let Y denote the number of extended warranties sold, and suppose that

$$P(X = x, Y = y) = \begin{cases} 1/6 & \text{for } (x, y) = (0, 0), \\ 1/12 & \text{for } (x, y) = (1, 0), \\ 1/6 & \text{for } (x, y) = (1, 1), \\ 1/12 & \text{for } (x, y) = (2, 0), \\ 1/3 & \text{for } (x, y) = (2, 1), \\ 1/6 & \text{for } (x, y) = (2, 2). \end{cases}$$

What is the variance of X?

(A) 0.47 (B) 0.58 (C) 0.83 (D) 1.42 (E) 2.58

Answer: B: 0.58

Solution: We first compute the marginal distribution of X by adding appropriate probabilities: $P(X = 0) = 1/6$, $P(X = 1) = 1/12 + 1/6 = 1/4$, $P(X = 2) = 1/12 + 1/3 + 1/6 = 7/12$. Therefore

$$\begin{aligned} E(X) &= 0 \cdot \frac{1}{6} + 1 \cdot \frac{1}{4} + 2 \cdot \frac{7}{12} = \frac{17}{12}, \\ E(X^2) &= 0^2 \cdot \frac{1}{6} + 1^2 \cdot \frac{1}{4} + 2^2 \cdot \frac{7}{12} = \frac{14}{12}, \\ \mathrm{Var}(X) &= E(X^2) - E(X)^2 = 0.58. \end{aligned}$$

7. [2-53]

A company prices its hurricane insurance using the following assumptions:

(i) In any calendar year, there can be at most one hurricane.

(ii) In any calendar year, the probability of a hurricane is 0.05.

(iii) The number of hurricanes in any calendar year is independent of the number of hurricanes in any other calendar year.

Using the company's assumptions, calculate the probability that there are fewer than 3 hurricanes in a 20-year period.

(A) 0.06 (B) 0.19 (C) 0.38 (D) 0.62 (E) 0.92

Answer: E: 0.92

Solution: By the given assumptions the occurrence of hurricanes can be modeled as success/failure trials, with success meaning that a hurricane occurs in a given year and occurring with probability $p = 0.05$. Thus, the probability that there are fewer than 3 hurricanes in a 20-year period is equal to the probability of having less than 3 successes in 20 success/failure trials with $p = 0.05$. By the binomial distribution, this probability is

$$\sum_{x=0}^{2} \binom{20}{x} 0.05^x 0.95^{20-x}$$

$$= 0.95^{20} + 20 \cdot 0.05 \cdot 0.95^{19} + \frac{20 \cdot 19}{2} \cdot 0.05^2 \cdot 0.95^{18} = 0.92.$$

8. [3-51]

The loss due to a fire in a commercial building is modeled by a random variable X with density function

$$f(x) = \begin{cases} 0.005(20-x) & \text{for } 0 < x < 20, \\ 0 & \text{otherwise.} \end{cases}$$

Given that a fire loss exceeds 8, what is the probability that it exceeds 16?

(A) 1/25 (B) 1/9 (C) 1/8 (D) 1/3 (E) 3/7

Answer: B: 1/9

Solution: This is an easy integration exercise. We need to compute $P(X \geq 16 | X \geq 8)$, which is the same as $P(X \geq 16)/P(X \geq 8)$. Integrating the given density function, we get

$$P(X \geq x) = \int_x^{20} 0.005(20-t)dt = 0.0025(20-x)^2, \quad 0 < x < 20.$$

Taking the ratio of these expressions with $x = 16$ and $x = 8$ gives the answer: $(20 - 16)^2/(20-8)^2 = 1/9$.

9. [1-101]

A large pool of adults earning their first driver's license includes 50% low-risk drivers, 30% moderate-risk drivers, and 20% high-risk drivers. Because these drivers have no prior driving record, an insurance company considers each driver to be randomly selected from the pool. This month, the insurance company writes 4 new policies for adults earning their first driver's license. What is the probability that these 4 will contain at least two more high-risk drivers than low-risk drivers?

(A) 0.006 (B) 0.012 (C) 0.018 (D) 0.049 (E) 0.073

Answer: D: 0.049

Solution: This requires a tedious case distinction, according to the types of the four drivers: Denoting high, moderate, and low risk drivers by the letters H, M, and L, we need to considers "words" of 4 of these letters containing at least two more H's than L's. The possible patterns and their probabilities are:

(a) 4 H's: HHHH: probability 0.2^4;

(b) 3 H's and 1 L: HHHL, HHLH, HLHH, LHHH: probability $0.2^3 0.5$ each, and $4 \cdot 0.2^3 0.5$ total;

(c) 3 H's and 1 M: HHHM, HHMH, HMHH, MHHH: probability $0.2^3 0.3$ each, and $4 \cdot 0.2^3 0.3$ total;

(d) 2 H's and 2 M: HHMM, HMHM, HMMH, MMHH, MHMH, MHHM; probability $0.2^3 0.3^2$ each, $6 \cdot 0.2^3 0.3^2$ total.

Adding up these probabilities, we get the answer:

$$0.2^4 + 4 \cdot 0.2^3 0.5 + 4 \cdot 0.2^3 0.3 + 6 \cdot 0.2^2 0.3^2 = 0.0489$$

10. [2-52]

An insurance company determines that N, the number of claims received in a week, is a random variable with $P(N = n) = 2^{-n-1}$, where $n \geq 0$. The company also determines that the number of claims received in a given week is independent of the number of claims received in any other week. Determine the probability that exactly seven claims will be received during a given two-week period.

(A) $\frac{1}{256}$ (B) $\frac{1}{128}$ (C) $\frac{7}{512}$ (D) $\frac{1}{64}$ (E) $\frac{1}{32}$

Answer: D: 1/64

Solution: Let N_1 and N_2, denote the number of claims during the first, respectively second, week. The total number of claims received during the two-week period then is $N_1 + N_2$, and so we need to compute $P(N_1 + N_2 = 7)$. By breaking this probability down into cases according to the values of (N_1, N_2), and using the independence of N_1 and N_2 and the given distribution, we get

$$\begin{aligned} P(N_1 + N_2 = 7) &= \sum_{n=0}^{7} P(N_1 = n, N_2 = 7 - n) = \sum_{n=0}^{7} 2^{-n-1} 2^{-(7-n)-1} \\ &= \sum_{n=0}^{7} 2^{-9} = 8 \cdot 2^{-9} = \frac{1}{64}. \end{aligned}$$

11. [4-9]

A device contains two components. The device fails if either component fails. The joint density function of the lifetimes of the components, measured in hours, is $f(s,t)$, where $0 < s < 1$ and $0 < t < 1$. What is the probability that the device fails during the first half hour of operation?

(A) $\int_0^{0.5} \int_0^{0.5} f(s,t) ds dt$

(B) $\int_0^{1} \int_0^{0.5} f(s,t) ds dt$

(C) $\int_{0.5}^{1} \int_{0.5}^{1} f(s,t) ds dt$

(D) $\int_0^{0.5}\int_0^{1} f(s,t)dsdt + \int_0^{1}\int_0^{0.5} f(s,t)dsdt$

(E) $\int_0^{0.5}\int_{0.5}^{1} f(s,t)dsdt + \int_0^{1}\int_0^{0.5} f(s,t)dsdt$

Answer: E

Solution: The key to this problem is the correct interpretation of the event $(*)$ "the device fails within the first half hour" in terms of the variables s and t. Since failure occurs if *either* (i.e., the first, the second, or both) of the components fails, $(*)$ translates into $(**)$ "$s \le 1/2$ *or* $t \le 1/2$". The probability for failure is therefore given by double integral $\iint_R f(s,t)dsdt$, where R is the region inside the unit square $0 \le s \le 1, 0 \le t \le 1$ described by $(**)$. A sketch shows that R splits into two disjoint parts, the rectangle $0 \le s \le 0.5$, $0 \le t \le 1$, and the square $0.5 \le s \le 1$, $0 \le t \le 0.5$. Thus,

$$\iint_R f(s,t)dsdt = \int_0^{0.5}\int_{0.5}^{1} f(s,t)dsdt + \int_0^{1}\int_0^{0.5} f(s,t)dsdt,$$

which is answer (E).

12. [3-7]

The time to failure of a component in an electronic device has an exponential distribution with a median of four hours. Calculate the probability that the component will work without failing for at least five hours.

(A) 0.07 (B) 0.29 (C) 0.38 (D) 0.42 (E) 0.57

Answer: D: 0.42

Solution: The general form of the c.d.f. of an exponential distribution is $F(x) = 1 - e^{-x/\theta}$ for $x > 0$. Since the median is 4 hours, we have $F(4) = 1/2$. This allows us to determine the parameter θ: $1 - e^{-4/\theta} = 1/2$, so $\theta = 4/\ln 2$. The probability that the component will work for at least 5 hours is given by $1 - F(5) = e^{-5/\theta} = e^{-(5/4)\ln 2} = 0.42$.

13. [1-13]

A health study tracked a group of persons for five years. At the beginning of the study, 20% were classified as heavy smokers, 30% as light smokers, and 50% as nonsmokers. Results of the study showed that light smokers were twice as likely as nonsmokers to die during the five-year study, but only half as likely as heavy smokers. A randomly selected participant from the study died over the five-year period. Calculate the probability that the participant was a heavy smoker.

(A) 0.20 (B) 0.25 (C) 0.35 (D) 0.42 (E) 0.57

Answer: D: 0.42

Solution: A standard exercise in Bayes' Rule; the only non-routine part here is to properly interpret the phrase "light smokers were twice as likely as nonsmokers, and half as likely as heavy smokers to die ...". This boils down to relations between the conditional probabilities of dying given a light smoker, a nonsmoker, and a heavy smoker.

14. [4-108]

The stock prices of two companies at the end of any given year are modeled with random variables X and Y that follow a distribution with joint density function

$$f(x,y) = \begin{cases} 2x & \text{for } 0 < x < 1,\ x < y < x+1, \\ 0 & \text{otherwise.} \end{cases}$$

What is the conditional variance of Y given that $X = x$?

(A) $1/12$ (B) $7/6$ (C) $x + 1/2$ (D) $x^2 - 1/6$ (E) $x^2 + x + 1/3$

Answer: A: $1/12$

Solution: The conditional density of Y given $X = x$ is $h(y|x) = f(x,y)/f_X(x)$. Here $f(x,y)$ is given, and the marginal density $f_X(x)$ can be computed by integrating $f(x,y)$ with respect to y:

$$f_X(x) = \int_{y=x}^{y=x+1} 2x dy = 2x, \quad 0 < x < 1.$$

It follows that $h(y|x) = (2x)/(2x) = 1$ for $0 < x < 1$, $x \le y \le x+1$. This means that the conditional density of Y given $X = x$ is uniform on the interval $[x, x+1]$. By the formula for the variance of a uniform distribution, it follows that $\mathrm{Var}(Y|x) = 1/12$, so the answer is $1/12$.

Alternatively, one can compute $\mathrm{Var}(Y|x)$ directly, without resorting to variance formulas for the uniform distribution, by computing the appropriate integrals over the conditional density $h(y|x)$:

$$\begin{aligned} E(Y|x) &= \int_{y=x}^{x+1} y h(y|x) dy = \frac{1}{2}\left((x+1)^2 - x^2\right), \\ E(Y^2|x) &= \int_{y=x}^{x+1} y^2 h(y|x) dy. = \frac{1}{3}\left((x+1)^3 - x^3\right), \\ \mathrm{Var}(X|x) &= E(Y^2|x) - E(Y|x)^2 \\ &= \frac{1}{3}\left(3x^2 + 3x + 1\right) - \frac{1}{4}(2x+1)^2 = \frac{1}{12}. \end{aligned}$$

15. [3-17]

An insurance company sells an auto insurance policy that covers losses incurred by a policyholder, subject to a deductible of 100. Losses incurred follow an exponential distribution with mean 300. What is the 95th percentile of actual losses that exceed the deductible?

(A) 600 (B) 700 (C) 800 (D) 900 (E) 1000

Answer: E: 1000

Solution: The main difficulty here is the correct interpretation of the "95th percentile of actual losses that exceed the deductible". The proper interpretation involves a conditional probability: we seek the value x such that the conditional probability that the loss is at most x, given that it exceeds the deductible, is 0.95, i.e., that $P(X \le x | X \ge 100) = 0.95$, where X denotes the loss. By the complement formula for conditional probabilities, this is equivalent to $P(X \ge x | X \ge 100) = 0.05$. Since X is exponentially distributed with mean 300, we have $P(X \ge x) = e^{-x/300}$, so for $x > 100$,

$$P(X \ge x | X \ge 100) = \frac{P(X \ge x)}{P(X \ge 100} = \frac{e^{-x/300}}{e^{-100/300}} = e^{-(x-100)/300}.$$

Setting this equal to 0.05 and solving for x, we get $(x - 100)/300 = -\ln(0.05)$, so $x = -300\ln(0.05) + 100 = 1000$.

16. [4-107]

An insurance company insures a large number of drivers. Let X be the random variable representing the company's losses under collision insurance, and let Y represent the company's losses under liability insurance. X and Y have joint density function

$$f(x,y) = \begin{cases} \frac{1}{4}(2x+2-y) & \text{for } 0 < x < 1 \text{ and } 0 < y < 2, \\ 0 & \text{otherwise.} \end{cases}$$

What is the probability that the total loss is at least 1?

(A) 0.33 (B) 0.38 (C) 0.41 (D) 0.71 (E) 0.75

Answer: D: 0.71

Solution: This is conceptually not a difficult problem, but it can lead to somewhat lengthy integral computations.

The probability to compute is $P(X+Y \geq 1)$. This probability is given by the double integral $\iint_R f(x,y)dxdy$, where $f(x,y)$ is the given density and R is the part of the rectangle $0 < x < 1, 0 < y < 2$ on which $x + y \geq 1$. A direct computation of this double integral is feasible, but rather lengthy. It is easier to consider first the complementary probability, $P(X+Y < 1)$. The corresponding region of integration is the triangle $0 \leq x \leq 1$, $0 \leq y \leq 1-x$. Thus,

$$\begin{aligned} P(X+Y<1) &= \int_{x=0}^{1}\int_{y=0}^{1-x} \frac{1}{4}(2x+2-y)dydx \\ &= \frac{1}{4}\int_{x=0}^{1}\left((2x+2)(1-x) - \frac{(1-x)^2}{2}\right)dx \\ &= \frac{1}{4}\int_{x=0}^{1}\left(-\frac{5}{2}x^2 + x + \frac{3}{2}\right)dx \\ &= \frac{1}{4}\left(-\frac{5}{2}\frac{1}{3} + \frac{1}{2} + \frac{3}{2}\right) = \frac{7}{24}. \end{aligned}$$

Hence $P(X+Y \geq 1) = 1 - 7/24 = 0.708$.

17. [2-55]

An insurance company sells a one-year automobile policy with a deductible of 2. The probability that the insured will incur a loss is 0.05. If there is a loss, the probability of a loss of amount N is K/N, for $N = 1, \dots, 5$ and K a constant. These are the only possible loss amounts and no more than one loss can occur. Determine the net premium for this policy.

(A) 0.031 (B) 0.066 (C) 0.072 (D) 0.110 (E) 0.150

Answer: A: 0.031

Solution: The "net premium" is the premium (per policy) at which the company breaks even. This is equal to the expected insurance payout (per policy). Taking into account the deductible, the payout is 1 if $N = 3$; 2 if $N = 4$; 3 if $N = 5$; and 0 in all other cases. Thus, the expected payout is

$$1P(N=3) + 2P(N=4) + 3P(N=5) = 1\frac{K}{3} + 2\frac{K}{4} + 3\frac{K}{5} = \frac{43}{30}K.$$

To determine the constant K, we use the fact that the (overall) probability of a loss is 0.05. Thus, the sum of the probabilities for the possible loss amounts has to equal 0.05:

$$0.05 = \sum_{N=1}^{5} \frac{K}{N} = \frac{137}{60}K.$$

Hence $K = 3/137$, and the above expectation betcomes $(43/30)(3/137) = 0.03138$.

18. [5-10]

An insurance policy pays a total medical benefit consisting of a part paid to the surgeon, X, and a part paid to the hospital, Y, so that the total benefit is $X + Y$. It is known that $\text{Var}(X) = 5,000$, $\text{Var}(Y) = 10,000$, and $\text{Var}(X + Y) = 17,000$. Due to increasing medical costs, the company that issues the policy decides to increase X by a flat amount of 100 per claim and to increase Y by 10% per claim. Calculate the variance of the total benefit after these revisions have been made.

(A) $18,200$ (B) $18,800$ (C) $19,300$ (D) $19,520$ (E) $20,670$

Answer: C: 19,300

Solution: We need to compute $\text{Var}(X + 100 + 1.1Y)$. Since adding constants does not change the variance, this is the same as $\text{Var}(X + 1.1Y)$, which expands as follows:

$$\begin{aligned}\text{Var}(X + 1.1Y) &= \text{Var}(X) + \text{Var}(1.1Y) + 2\,\text{Cov}(X, 1.1Y)\\ &= \text{Var}(X) + 1.1^2\,\text{Var}(Y) + 2 \cdot 1.1\,\text{Cov}(X, Y).\end{aligned}$$

We are given that $\text{Var}(X) = 5,000$, $\text{Var}(Y) = 10,000$, so the only remaining unknown quantity is $\text{Cov}(X, Y)$, which can be computed via the general formula for $\text{Var}(X + Y)$:

$$\begin{aligned}\text{Cov}(X, Y) &= \frac{1}{2}\left(\text{Var}(X + Y) - \text{Var}(X) - \text{Var}(Y)\right)\\ &= \frac{1}{2}(17,000 - 5,000 - 10,000) = 1,000.\end{aligned}$$

Substituting this into the above formula, we get the answer:

$$\text{Var}(X + 1.1Y) = 5,000 + 1.1^2 \cdot 10,000 + 2 \cdot 1.1 \cdot 1,000 = 19,520$$

19. [5-54]

Let T_1 be the time between a car accident and reporting a claim to the insurance company. Let T_2 be the time between the report of the claim and payment of the claim. The joint density function of T_1 and T_2, $f(t_1, t_2)$, is constant over the region $0 < t_1 < 6, 0 < t_2 < 6, 0 < t_1 + t_2 < 10$, and zero otherwise. Determine $E(T_1 + T_2)$, the expected time between a car accident and payment of the claim.

(A) 4.9 (B) 5.0 (C) 5.7 (D) 6.0 (E) 6.7

Answer: C: 5.7

Solution: We work in the t_1t_2-plane. A sketch shows that the given region is the square $[0, 6] \times [0, 6]$ with a right triangle of side length 2 deleted in the upper right hand corner of the square. Thus, the total area of this region is $6^2 - (1/2)2^2 = 34$, and because of the uniform distribution the joint density function of T_1 and T_2 is constant and equal to $f(t_1, t_2) = 1/34$ in this region. Now, $E(T_1 + T_2) = E(T_1) + E(T_2)$. By symmetry, $E(T_2)$

is the same as $E(T_1)$, so it suffices to compute the latter expectation and then double the result.

We use the double integral formula $E(T_1) = \iint_R t_1 f(t_1,t_2)dt_2dt_1$, where R is the above region and $f(t_1,t_2) = 1/34$ in this region. A sketch shows that R can be broken into two disjoint pieces, described by the inequalities $0 \le t_1 \le 4$, $0 \le t_2 \le 6$, and $4 \le t_1 \le 6$, $0 \le t_2 \le 10 - t_1$, respectively. It follows that

$$\begin{aligned} E(T_1) &= \int_{t_1=0}^{4}\int_{t_2=0}^{6} t_1 f(t_1,t_2)dt_2dt_1 + \int_{t_1=4}^{6}\int_{t_2=0}^{10-t_1} t_1 f(t_1,t_2)dt_2dt_1 \\ &= \frac{1}{34}\int_{t_1=0}^{4} 6t_1 dt_1 + \frac{1}{34}\int_{t_1=4}^{6} t_1(10-t_1)dt_1 = 2.86 \end{aligned}$$

Hence $E(T_1 + T_2) = 2 \cdot 2.86 = 5.72$.

20. [5-61]

A company agrees to accept the highest of four sealed bids on a property. The four bids are regarded as four independent random variables with common cumulative distribution function

$$F(x) = \frac{1}{2}(1 + \sin \pi x) \quad \text{for } 3/2 \le x \le 5/2.$$

Which of the following represents the expected value of the accepted bid?

(A) $\pi \displaystyle\int_{3/2}^{5/2} x \cos \pi x dx$

(B) $\dfrac{1}{16} \displaystyle\int_{3/2}^{5/2} (1 + \sin \pi x)^4 dx$

(C) $\dfrac{1}{16} \displaystyle\int_{3/2}^{5/2} x(1 + \sin \pi x)^4 dx$

(D) $\dfrac{\pi}{4} \displaystyle\int_{3/2}^{5/2} (\cos \pi x)(1 + \sin \pi x)^3 dx$

(E) $\dfrac{\pi}{4} \displaystyle\int_{3/2}^{5/2} x(\cos \pi x)(1 + \sin \pi x)^3 dx$

Answer: E

Solution: Let $X_1, \dots, X_4$ denote the four bids, and let $X^* = \max(X_1, \dots, X_4)$ denote the largest of these bids, i.e., the bid that is accepted. The c.d.f. of X^* then is given by (for $3/2 \le x \le 5/2$) (note the "maximum trick" here!)

$$\begin{aligned} F^*(x) &= P(X^* \le x) = P(\max(X_1, \dots, X_4) \le x) \\ &= P(X_1 \le x, \dots, X_4 \le x) = P(X_1 \le x) \dots P(X_4 \le x) = F(x)^4. \end{aligned}$$

Differentiating, we get the density of X^*:

$$\begin{aligned} f^*(x) &= \frac{d}{dx}F(x)^4 = 4F(x)^3F'(x) \\ &= 4\frac{1}{2^3}(1 + \sin \pi x)^3 \frac{1}{2}(\cos \pi x)\pi \\ &= \frac{\pi}{4}(1 + \sin \pi x)^3(\cos \pi x) \quad (3/2 \le x \le 5/2). \end{aligned}$$

Hence,

$$E(X^*) = \int_{3/2}^{5/2} x f^*(x) dx = \frac{\pi}{4} \int_{3/2}^{5/2} x(1 + \sin \pi x)^3 (\cos \pi x) dx.$$