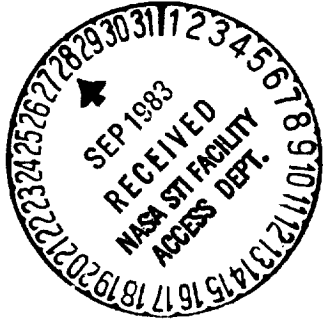


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A Simple Algorithm for Computing Canonical Forms

H. Ford, L.R. Hunt, Renjeng Su



Abstract

It is well known that all linear time-invariant controllable systems can be transformed to Brunovsky canonical form by a transformation consisting only of coordinate changes and linear feedback. However, the actual procedures for doing this have tended to be overly complex. The technique introduced here is envisioned as an on-line procedure and is inspired by George Meyer's tangent model for nonlinear systems. The process utilizes Meyer's block triangular form as an intermediate step in going to Brunovsky form. The method also involves orthogonal matrices, thus eliminating the need for the computation of matrix inverses. In addition, the Kronecker indices can be computed as a by-product of this transformation so it is not necessary to know them in advance.

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## A Simple Algorithm for Computing Canonical Forms

H. Ford\*, L.R. Hunt\*, and Renjeng Su\*\*

### I. Introduction

In his work at NASA Ames Research Center, George Meyer is applying the theory of transformations of nonlinear systems to linear systems in order to design automatic flight controllers for vertical and short take off aircraft [1], [2], [3], [4], [5], [6]. In these articles he introduces a linear system (called the tangent model) which approximates the nonlinear system, and he mentions the importance of taking the tangent model to Brunovsky [7] canonical form. Thus we feel that an on-line procedure for transforming a controllable linear system to Brunovsky form is highly desirable. Additionally in [8] the authors together with George Meyer present a modification of the tangent model in which the procedure introduced here plays a central role in the linearization

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process itself. This process involves a Taylor Series expansion using lie derivatives.

Our computation of the transformation proceeds in two steps:

- 1) An orthogonal coordinate change is used to move the linear system to Meyer's block triangular form (see [5]).
- 2) Once we are in block triangular form, the process becomes trivial and formally involves "Lie differentiation" of certain coordinate functions. The on-line procedure for doing this and its application to automatic flight control are given in [5].

In step 1) it is not necessary to calculate a matrix inverse and no systems of linear equations need be solved. As a by-product of step 1) we find the Kronecker indices of the system.

Understanding the fact that the block triangular form is a natural intermediate step in transforming to Brunovsky form is our main contribution in this paper. We have computer programs to carry out the entire process.

In section II, we describe how to transform a single control linear system to a string of integrators. In section III, we show how to generalize the results of section II to a multi-control system. Detailed algorithms for transforming linear systems to Brunovsky form are given in section IV, including pseudocode programs. Results achieved using a three control system and concluding comments are given in section V. We want to emphasize the simplicity and ease of implementation of the algorithm.

Though the explanation of why it works may seem cumbersome, the algorithm is very straightforward.

## II. Single Control Case

Definition 2.1. An  $n$  dimensional single control system  $\dot{x} = f(x,u)$  is called block triangular if  $\dot{x}_i$  is a function only of  $x_1, x_2, \dots, x_{i+1}$  for  $i = 1, \dots, n$  where  $x_{n+1} = u$  (see [5]).

For a linear single control system  $\dot{x} = Ax + bu$ , this is equivalent to saying that the square matrix  $H = \begin{bmatrix} \bar{A} & b \\ 0 & 0 \end{bmatrix}$  is a lower Hessenberg matrix [9], that is, all elements above the first super diagonal are zero. Notice that all elements on the first super diagonal must be nonzero if we are to have a controllable system. The outstanding characteristic of the block triangular system is that if we start with  $x_1$ , it is necessary to take  $n$  derivatives of  $x_1$  with respect to  $t$  before reaching the control.

Definition 2.2. An  $n$  dimensional single control system  $\dot{x} = f(x,u)$  is called a string of integrators if  $\dot{x}_i = x_{i+1}$  for  $i = 1, \dots, n$  where  $x_{n+1} = u$ .

Once we have a block triangular linear system  $\dot{x} = Ax + bu$ , we can transform it to a string of integrators  $\dot{y} = \tilde{A}y + \tilde{B}v$  by simply letting  $y_1 = x_1, y_2 = \dot{y}_1, \dots, y_n = \dot{y}_{n-1}$  and  $v = \dot{y}_n$ .

Theorem 2.1. The above transformation from block triangular form to a string of integrators consists of only coordinate changes and linear feedback.

Proof: We know that  $y_1 = x_1$  as mentioned above.

Assuming  $y_i = \sum_{j=1}^i \alpha_j x_j$  for some  $i \leq n-1$ , we have

$$y_{i+1} = \dot{y}_i = \sum_{j=1}^i \alpha_j \dot{x}_j = \sum_{j=1}^i \alpha_j \sum_{k=1}^{j+1} a_{j,k} x_k = \sum_{k=1}^{i+1} \beta_k x_k$$

Thus the state variables  $y$  are functions only of the state variables

$$x. \text{ The new control } v = \dot{y}_n = \sum_{k=1}^n \beta_k \dot{x}_k$$

$$\begin{aligned} &= \sum_{k=1}^{n-1} \beta_k \dot{x}_k + \beta_n \dot{x}_n = \sum_{k=1}^{n-1} \beta_k \sum_{\alpha=1}^{k+1} a_{k,\alpha} x_\alpha + \beta_n \left( \sum_{j=1}^n a_{n,j} x_j + bu \right) \\ &= \sum_{\ell=1}^n \gamma_\ell x_\ell + \gamma_{n+1} u, \end{aligned}$$

and our feedback is linear.  $\square$

If we start with a system  $\dot{x} = Ax + bu$  which is not block triangular, we first form the augmented matrix  $H = \begin{bmatrix} A & b \\ 0 & 0 \end{bmatrix}$ . From

Stewart [9] it is known that there exists an orthogonal transformation  $C = \begin{bmatrix} \tilde{C} & 0 \\ 0 & 1 \end{bmatrix}$  such that  $CHC^T$  is lower Hessenberg.

$$CHC^T = \begin{bmatrix} \tilde{C} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A & b \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{C}^T & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \tilde{C}A\tilde{C}^T & \tilde{C}b \\ 0 & 0 \end{bmatrix}$$

Thus the transformation  $y = \tilde{C}x$  yields a block triangular system  $\dot{y} = \tilde{C}A\tilde{C}^T y + \tilde{C}bu$ .

Notice  $\begin{bmatrix} y \\ u \end{bmatrix} = C \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} \tilde{C} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} \tilde{C}x \\ u \end{bmatrix}$  is a transformation involving coordinate changes only. Once we have a block triangular system, we go to a string of integrators as before.

### III. Multi-Control Case

For a multi-control system, the situation is slightly more complicated than for the single control case. Below is the usual definition of Kronecker indices [7].



and  $\hat{B}$  equals

$$\begin{array}{c}
 \left. \begin{array}{l} \kappa_1 \\ \kappa_2 \\ \kappa_H \end{array} \right\} \left[ \begin{array}{cccccc}
 0 & 0 & . & . & . & 0 \\
 0 & 0 & . & . & . & 0 \\
 . & . & . & . & . & . \\
 . & . & . & . & . & . \\
 . & . & . & . & . & . \\
 1 & 0 & . & . & . & 0 \\
 \hline
 0 & 0 & . & . & . & 0 \\
 0 & 0 & . & . & . & 0 \\
 . & . & . & . & . & . \\
 . & . & . & . & . & . \\
 . & . & . & . & . & . \\
 0 & 1 & 0 & . & . & 0 \\
 \hline
 & & . & & & \\
 & & . & & & \\
 \hline
 0 & 0 & . & . & . & 0 \\
 0 & 0 & & & & 0 \\
 . & . & . & . & . & . \\
 . & . & . & . & . & . \\
 . & . & . & . & . & . \\
 0 & 0 & . & . & . & 1
 \end{array} \right]
 \end{array}$$

That is, the Brunovsky form consists of  $m$  strings of integrators whose respective lengths are the Kronecker indices. For instance a ten dimensional system with three controls and Kronecker indices 5, 3, and 2 will have Brunovsky form

$$(3.1) \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ \dot{y}_4 \\ \dot{y}_5 \\ \dot{y}_6 \\ \dot{y}_7 \\ \dot{y}_8 \\ \dot{y}_9 \\ \dot{y}_{10} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \\ y_8 \\ y_9 \\ y_{10} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

Notice that for a system with only one control, the Brunovsky form is a string of integrators.

Though not necessary for the implementation of our algorithm, we shall for theoretical reasons want to be able to identify individual Kronecker indices with individual controls. For that reason we shall introduce the following alternate definition of Kronecker indices.

Definition 3.3. Let  $\dot{x} = Ax + Bu$  be an  $n$  dimensional linear system with  $m$  controls and with  $B = [b_1, \dots, b_m]$ . Consider the vectors  $b_m, b_{m-1}, \dots, b_1, Ab_m, \dots, Ab_1, A^2 b_m, \dots, A^2 b_1, \dots$  until we come to a vector dependent on the previous vectors, call it  $A^{k_i} b_i$ . Then  $k_i$  is the



Kronecker index associated with the control  $u_i$ . If we continue in this manner we will get  $m$  Kronecker indices whose sum is  $n$ . This definition will give us the same Kronecker indices as before. However, the subscripts will now associate each Kronecker index with a control rather than ordering the Kronecker indices (we can obviously renumber our controls so that  $\kappa_1 \geq \kappa_2 \geq \kappa_3 \geq \dots \geq \kappa_m$ ). This definition will be very useful in showing that the block triangular form of a system contains all information on the Kronecker indices.

What do we mean by the block triangular form of a linear system with  $m$  controls? The most obvious choice would be a system consisting of  $m$  block triangular systems, each in one control. For instance, a ten dimensional block triangular system with three controls and Kronecker indices 5, 3, and 2 might be of the form

$$(3.2) \quad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \\ \dot{x}_7 \\ \dot{x}_8 \\ \dot{x}_9 \\ \dot{x}_{10} \end{bmatrix} = \begin{bmatrix} X & X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ X & X & X & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ X & X & X & X & 0 & 0 & 0 & 0 & 0 & 0 \\ X & X & X & X & X & 0 & 0 & 0 & 0 & 0 \\ X & X & X & X & X & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & X & X & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & X & X & X & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & X & X & X & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & X & X \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & X & X \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ X & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & X & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & X \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

Here each of the 0's represents zero. Each of the rightmost X's represents a nonzero element. The other X's may or may not

be zero. We emphasize that 5 derivatives of  $x_1$  must be taken to reach a control, 3 derivatives of  $x_6$ , and 2 derivatives of  $x_9$ , precisely the Kronecker indices.

We found it convenient to rename the state variables, ordering them in terms of their distance from the controls. By distance, we mean the number of derivatives of a variable we must take to reach the controls. Renaming  $x_1 \rightarrow x_1$ ,  $x_2 \rightarrow x_2$ ,  $x_3 \rightarrow x_3$ ,  $x_6 \rightarrow x_4$ ,  $x_4 \rightarrow x_5$ ,  $x_7 \rightarrow x_6$ ,  $x_9 \rightarrow x_7$ ,  $x_5 \rightarrow x_8$ ,  $x_8 \rightarrow x_9$ , and  $x_{10} \rightarrow x_{10}$ . We then get

$$(3.3) \quad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \\ \dot{x}_7 \\ \dot{x}_8 \\ \dot{x}_9 \\ \dot{x}_{10} \end{bmatrix} = \begin{bmatrix} X & X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ X & X & X & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ X & X & X & 0 & X & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & X & 0 & X & 0 & 0 & 0 & 0 \\ X & X & X & 0 & X & 0 & 0 & X & 0 & 0 \\ 0 & 0 & 0 & X & 0 & X & 0 & 0 & X & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & X & 0 & 0 & X \\ X & X & X & 0 & X & 0 & 0 & X & 0 & 0 \\ 0 & 0 & 0 & X & 0 & X & 0 & 0 & X & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & X & 0 & 0 & X \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ X & 0 & 0 \\ 0 & X & 0 \\ 0 & 0 & X \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

We must take 5 derivatives of  $x_1$  to reach the controls, 3 derivatives of  $x_4$ , and 2 derivatives of  $x_7$ . Remarkably, the only thing which affects the number of derivatives necessary to reach the controls is the rightmost nonzero element in each row.

For instance,  $\dot{x}_4$  is a function of  $x_6$  but not a function of  $x_j$  for  $j > 6$  nor  $u_i$  for  $1 \leq i \leq 3$ .  $\dot{x}_6$  is a function of  $x_9$  but not a function of  $x_{10}$  nor  $u_i$  for  $1 \leq i \leq 3$ . This implies  $\ddot{x}_4$  is a function of  $x_9$

but not a function of  $x_{10}$  nor  $u_i$  for  $1 \leq i \leq 3$ .  $\dot{x}_9$  is a function of  $u_2$  but not  $u_3$ . This implies  $\ddot{x}_4$  is a function of  $u_2$  but not  $u_3$ . Notice in determining that it takes three derivatives of  $x_4$  before we reach the controls, all we used was knowledge of the rightmost nonzero elements. Thus a block triangular system with ten dimensions, three controls, and Kronecker indices 5,3, and 2 could have the form

$$(3.4.) \quad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \\ \dot{x}_7 \\ \dot{x}_8 \\ \dot{x}_9 \\ \dot{x}_{10} \end{bmatrix} = \begin{bmatrix} X & X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ X & X & X & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ X & X & X & X & X & 0 & 0 & 0 & 0 & 0 \\ X & X & X & X & X & X & 0 & 0 & 0 & 0 \\ X & X & X & X & X & X & X & 0 & 0 & 0 \\ X & X & X & X & X & X & X & X & 0 & 0 \\ X & X & X & X & X & X & X & X & X & 0 \\ X & X & X & X & X & X & X & X & X & X \\ X & X & X & X & X & X & X & X & X & X \\ X & X & X & X & X & X & X & X & X & X \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ x & 0 & 0 \\ X & X & 0 \\ X & X & X \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

In each row, the X furthest to the right is nonzero. The other X's may or may not be zero. Notice it still takes 5 derivatives of  $x_1$  to reach the controls, 3 derivatives of  $x_4$ , and 2 derivatives of  $x_7$ .

The above discussion motivates our definition of block triangular for a linear system with several controls.

Definition 3.4. Let  $\dot{x} = Ax + Bu$  be an  $n$  dimensional system with  $m$  controls. We say the system is block triangular if the square matrix  $H = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}$  is a generalization of a lower Hessenberg matrix that

we simply call generalized lower Hessenberg and which we now define. First, all elements of H above the mth super diagonal are zero. Second, if  $H(I,J)$ , with  $I < n$ , is a zero element with no nonzero element to its right in row I, then all elements above  $H(I,J)$  in the same super diagonal must also be zero. For instance,

$$(3.5.) \quad H = \begin{bmatrix} X & X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ X & X & X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ X & X & X & X & X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ X & X & X & X & X & X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ X & X & X & X & X & X & X & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ X & X & X & X & X & X & X & X & 0 & 0 & 0 & 0 & 0 & 0 \\ X & X & X & X & X & X & X & X & X & 0 & 0 & 0 & 0 & 0 \\ X & X & X & X & X & X & X & X & X & X & 0 & 0 & 0 & 0 \\ X & X & X & X & X & X & X & X & X & X & X & 0 & 0 & 0 \\ X & X & X & X & X & X & X & X & X & X & X & X & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where the rightmost X's are nonzero, represents a ten dimensional block triangular system with three controls and Kronecker indices 5, 3, and 2. Notice that since  $H(4,7) = 0$ , we have  $H(3,6) = H(2,5) = H(1,4) = 0$ . Similarly,  $H(2,4) = 0$  implies  $H(1,3) = 0$ .

If  $\dot{x} = Ax + Bu$  is an n dimensional system with m controls which is not in block triangular form, we can easily make it block triangular in the following way. Form the augmented matrix  $H = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}$ ,

and find an orthogonal matrix  $C = \begin{bmatrix} \tilde{C} & 0 \\ 0 & I \end{bmatrix}$  (where  $I$  is the  $m$  dimensional identity) such that  $CHC^T$  is generalized lower Hessenberg. We will explain in the next section precisely how to find  $C$ . This matrix  $C$  satisfies  $CHC^T = \begin{bmatrix} \tilde{C} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{C}^T & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} \tilde{C}A\tilde{C}^T & CB \\ 0 & 0 \end{bmatrix}$

Thus the transformation  $y = \tilde{C}x$  yields a block triangular system  $\dot{y} = \tilde{C}A\tilde{C}^T y + \tilde{C}Bu$ . As in the single control case, this transformation is just a change of coordinates on our state space.

Definition 3.5. By zero pattern we shall mean the pattern of the rightmost nonzero elements of the matrix  $H = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}$  of a system  $\dot{x} = Ax + Bu$  in block triangular form. That is, for each block triangular form there is an  $n$  tuple of integers  $(\ell_1, \ell_2, \dots, \ell_n)$  so that  $\ell_i$  is the column number of rightmost nonzero element in the  $i^{\text{th}}$  row.

Theorem 3.1. Let  $\dot{x} = Ax + Bu$  be an  $n$  dimensional system with  $m$  controls. There is a one to one correspondence between the possible ordered sets of Kronecker indices and the possible zero patterns of the block triangular form. That is, a given set of Kronecker indices with given associations with the controls (see def. 3.3) results in a distinct and unique zero pattern.

Comment: This implies we can retrieve the Kronecker indices of a system knowing only the zero pattern of the block triangular form.

Proof: Consider the original system  $\dot{x} = Ax + Bu$ .

Let  $\tilde{H} = \begin{bmatrix} \tilde{C} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{C}^T & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} \tilde{A} & \tilde{B} \\ 0 & 0 \end{bmatrix}$  be the block triangular represent-

ation. Notice that  $\dot{y} = \tilde{A}y + \tilde{B}u$  has the same Kronecker indices as the original system  $\dot{x} = Ax + Bu$ . To see this consider that

$$\begin{aligned} [\tilde{b}_m, \dots, \tilde{b}_1, \tilde{A}\tilde{b}_m, \dots, \tilde{A}\tilde{b}_1, \dots] &= [\tilde{C}b_m, \dots, \tilde{C}b_1, \tilde{C}A\tilde{C}^T\tilde{C}b_m, \dots, \tilde{C}A\tilde{C}^T\tilde{C}b_1, \dots] \\ &= [\tilde{C}b_m, \dots, \tilde{C}b_1, \tilde{C}Ab_m, \dots, \tilde{C}Ab_1, \dots] = \tilde{C}[b_m, \dots, b_1, Ab_m, \dots, Ab_1, \dots]. \end{aligned}$$

Since  $\tilde{C}$  is nonsingular, the rank of any selection of columns from  $[\tilde{b}_m, \dots, \tilde{b}_1, \tilde{A}\tilde{b}_m, \dots, \tilde{A}\tilde{b}_1, \dots]$  is the same as the rank of the corresponding selection of columns from  $[b_m, \dots, b_1, Ab_m, \dots, Ab_1, \dots]$ . Therefore, we can assume without loss of generality that the original system is already in block triangular form.

The rest of the proof depends on two basic principles from linear algebra:

1. When we multiply a matrix times a vector, the product is a linear combination of the columns of the matrix.
2. If we have a collection of  $n$  nonzero vectors  $v_1, \dots, v_n$ , then the linear combination  $\sum_{i=1}^n \alpha_i v_i$ , with  $\alpha_n \neq 0$ , is linearly independent of  $v_1, \dots, v_{n-1}$  if and only if  $v_n$  is linearly independent of  $v_1, \dots, v_{n-1}$ .

Using the first principle, we see that  $Ab_m = a_m b_{n,m}$  where  $a_n$  is the  $n^{\text{th}}$  column of  $A$ , and  $b_{n,m}$  is the  $n^{\text{th}}$  entry in  $b_m$ . For an integer  $i_0$ ,  $0 \leq i_0 \leq m-1$ ,  $Ab_{m-i_0} = \sum_{i_1=0}^{i_0} a_{n-i_1} b_{n-i_1, m-i_0}$

$$A^k b_{m-i_0} = \sum_{i_1=0}^{i_0} a_{n-i_1} b_{n-i_1, m-i_0}$$

$$A^k b_{m-i_0} = \sum_{i_1=0}^{i_0} \sum_{i_2=0}^{m+i_1} \dots \sum_{i_k=0}^{m+i_{k-1}} a_{n-i_k} a_{n-i_k, n-i_{k-1}} \dots a_{n-i_2, n-i_1} b_{n-i_1, m-i_0}$$

Thus  $A^k b_{m-i_0}$  is a linear combination of columns of  $A$ .

Using the second principle,  $Ab_m$  is linearly independent of  $b_m, \dots, b_1$  if and only if  $a_n$  is linearly independent of  $b_m, \dots, b_1$ .

Thus for purposes of checking independence  $Ab_m$  can be represented by  $a_n$ . Similarly  $A^k b_{m-i_0}$  can be represented by the leftmost column of  $A$  in the linear combination which has a nonzero coefficient. Zero coefficients are caused by previous vectors being dependent. Thus  $A^k b_{m-i_0}$  can be represented by column  $a_{n-[(k-1)m + i_0] + h}$  where  $h$  is the number of vectors already found to be dependent on previous vectors. To see this, consider  $A^k b_{m-i_0} = A(A^{k-1} b_{m-i_0})$ . The index of the column representing  $A^k b_{m-i_0}$  will be the first (from the top) nonzero element of  $A^{k-1} b_{m-i_0}$ . Since we move up a row for every independent vector, the first nonzero element of  $A^{k-1} b_{m-i_0}$  has index  $n-[(k-1)m + i_0] + h$ .

Thus  $A^k b_{m-i_0}$  can be dependent on the previous vectors in two ways. One way is that  $A^{k-1} b_{m-i_0}$  was dependent on vectors previous to it. The second way is that the column  $a_{n-[(k-1)m + i_0] + h}$  is dependent on columns to the right of it. In this second way  $A^k b_{m-i_0}$  is the lowest power of  $A$  times  $b_{m-i_0}$  which is dependent on previous vectors. Thus there is a one to one correspondence between the ordered sets of Kronecker indices (ordered by def 3.3) and the zero patterns of the block triangular form.  $\square$

Once we have the block triangular matrix  $H$ , how do we retrieve the lead variables (to be defined) for the Brunovsky form? To see how this is done and to better understand block triangular systems, it is useful to think in terms of derivative levels.

Definition 3.6. Let  $\dot{x} = Ax + Bu$  be a control system in block

triangular form. A state variable  $x_i$  is said to be on the  $j^{\text{th}}$  derivative level if it takes  $j$  derivatives of  $x_i$  to reach the controls. For instance in the block triangular system illustrated in equations (3.4) and (3.5),  $x_8, x_9$ , and  $x_{10}$  are on the first derivative level.  $x_5, x_6$ , and  $x_7$  are on the second derivative level,  $x_3$  and  $x_4$  are on the third level, and  $x_2$  is on the fourth level. Lastly the variable  $x_1$  is on the fifth derivative level.

Definition 3.7. Let  $\dot{x} = Ax + Bu$  be a control system in block triangular form. Let  $x_i$  be a state variable on the  $j^{\text{th}}$  derivative level. Then  $\dot{x}_i$  is said to be a lead variable if it cannot be reached by taking the derivative of a state variable on the  $(j+1)^{\text{th}}$  level.

For all block triangular systems,  $x_1$  is a lead variable. For the particular system in equations 3.4,  $x_4$  and  $x_7$  are also lead variables. Notice that  $x_4$  is on the third level;  $x_2$  is the only variable on the fourth level but its derivative is not a function of  $x_4$ . Also  $x_7$  is on the second level;  $x_3$  and  $x_4$  are on the third level, but their derivatives do not involve  $x_7$ .

Theorem 3.2. Let  $\dot{x} = Ax + Bu$  be a control system in block triangular form. The derivative levels of the lead variables are precisely the Kronecker indices.

Proof: This theorem is really just a restatement of Theorem 3.1. Notice that a lead variable occurs because of the inability to reach the variable from a higher level which in turn is caused by a column dependency in  $A$ .  $\square$

Once we have the lead variables in the block triangular form, we simply let them be the lead variables in the Brunovsky form.



For instance for the block triangular system  $\dot{x} = Ax + Bu$  of equation 3.4, we let

$$\begin{array}{lll}
 y_1 = x_1 & y_6 = x_4 & y_9 = x_7 \\
 y_2 = \dot{y}_1 & y_7 = \dot{y}_6 & y_{10} = \dot{y}_9 \\
 y_3 = \dot{y}_2 & y_8 = \dot{y}_7 & v_3 = \dot{y}_{10} \\
 y_4 = \dot{y}_3 & v_2 = \dot{y}_8 & \\
 y_5 = \dot{y}_4 & & \\
 v_1 = \dot{y}_5 & & 
 \end{array}$$

We know the respective lengths of the integral strings because we know the derivative levels of  $x_1, x_4,$  and  $x_7$ .

Again, the essential characteristic of the block triangular system is that the controls do not appear "too soon". Thus the state variables in Brunovsky form are functions only of the state variables in the original system (that is, not functions of the controls).

Theorem 3.3. The above transformation from block triangular form to Brunovsky canonical form consists of only coordinate changes and linear feedback.

Proof: As in the single control case, the derivative of  $x_i$  is a linear combination of  $x_j$ 's except when  $x_i$  is on the first derivative level. In that case  $\dot{x}_i$  is a linear combination of the  $x_j$ 's and  $u_\ell$ 's. Subsequent derivatives of the state variables are also linear combinations of the appropriate variables. Let  $x_i$  be a lead variable and let  $k$  be the derivative level of  $x_i$ . For  $j < k$ , the  $j^{\text{th}}$  derivative of  $x_i$  with respect to  $t, x_i^{(j)}$ , is a linear

combination of  $x_\ell$ 's and  $x_i^{(k)}$  is a linear combination of the  $x_\ell$ 's and  $u_\ell$ 's. Since the lengths of the strings in the Brunovsky form are determined by the derivative levels of the lead variables, our proof is complete.  $\square$

#### IV. Algorithms

The actual algorithm for finding a matrix  $C$  to transform a matrix  $H$  to block triangular form is a generalization of the method found in Stewart [9] for transforming a matrix to Hessenberg form.

We do this by placing the appropriate zeros in  $H$  one column at a time starting with the rightmost column. For each column, we multiply on the left and right by what is called an elementary reflector or Householder transformation  $u_k$  (see Stewart [9]). Each of the  $u_k$  is of the form  $u_k = I - \frac{2vv^T}{v^T v}$  where  $I$  is the identity matrix and  $v$  is a vector. To illustrate this process, suppose we want to zero out the first  $k-1$  elements in some column and at the same time insure that the  $k^{\text{th}}$  element is nonzero, suppose

$\begin{bmatrix} v_1 \\ \vdots \\ v_k \\ ? \end{bmatrix}$  is the column in question then we multiply

$$\left\{ I - \frac{1}{\|v\| (v_k + \|v\|)} \begin{bmatrix} v_1 \\ \vdots \\ v_{k-1} \\ v_{k+} \|v\| \\ 0 \end{bmatrix} [v_1, \dots, v_{k-1}, v_{k+} \|v\|, 0] \right\} \begin{bmatrix} v \\ ? \end{bmatrix} =$$

$$= \begin{bmatrix} v \\ ? \end{bmatrix} - \begin{bmatrix} v_1 \\ \vdots \\ v_{k-1} \\ v_{k+} \quad ||v|| \\ 0 \end{bmatrix} \frac{||v||^2 + v_k ||v||}{||v|| (v_k + ||v||)} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -||v|| \\ ? \end{bmatrix}$$

where  $||v||$  is the Euclidean norm of  $\begin{bmatrix} v_1 \\ \vdots \\ v_k \end{bmatrix}$ .

Starting with an index = m and k = n+m we zero out the first k-index -1 in elements column k. Then reduce k by 1 (or move one column to the left) and continue. If we find that the first k-index elements of column k are already zero, we reduce the index by 1, reduce k by 1 and continue. All of the work is done by multiplication on the left, but it is easy to see that multiplication on the right does not undo the work. For instance  $u_k$ , the matrix which produces zeros in the  $k^{\text{th}}$  column is of the form  $u_k = \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}$  where C is at most  $(k-1) \times (k-1)$ . Thus multiplication of H by  $u_k$  on the right cannot affect columns k thru n+m, precisely the columns that have already been transformed.

We give the essential part of the algorithm below. In so doing we use the pseudo code INFL of Stewart [4]. Here N is the dimension of the system and M is the number of controls.

- 1.) INDEX = M
- 2.) For K = N+M, ..., 1
  - 1.) If  $H(K+1 - \text{INDEX}, K+1) = 0$ 
    - 1) INDEX = INDEX - 1
  - 2.) If K-Index  $\leq$  1 Exit Loop
  - 3.)  $\text{ETA} = \max \{ |H(I, K)| : I = 1, \dots, K - \text{INDEX} \}$

- 4.) If  $\text{ETA} = 0$  Step  $K$
- 5.)  $V(I, K) = H(I, K) / \text{ETA}$   $I = 1, \dots, K - \text{INDEX}$
- 6.)  $\text{SIGMA} = \text{sign} (V(K - \text{INDEX}, K)) \sqrt{V^2(1, K) + \dots + V^2(K - \text{INDEX}, K)}$
- 7.)  $V(K - \text{INDEX}, K) = V(K - \text{INDEX}, K) + \text{SIGMA}$
- 8.)  $\text{PI}(K) = \text{SIGMA} * V(K - \text{INDEX}, K)$

After finding the vector  $V$  and the scalar  $\pi$ , the elementary reflector  $U_k$  consists of  $U_k = I - \frac{VV^T}{\pi}$  and the transformation matrix  $C$  is  $C = U_1 U_2 \dots U_{n+m}$ . Of course some of the  $U_k$  may be identities.

A very similar procedure was used by Minimis and Paige [5], as a first step, for the purpose of placing eigenvalues. They use a generalized upper Hessenberg matrix rather than a generalized lower Hessenberg matrix. With renaming of coordinates they would have obtained the block triangular form. Once in block triangular form, as we have shown in this paper, the transformation to Brunovsky canonical form and hence the placing of eigen values is extremely easy. In Brunovsky form, the system is decoupled into several single control systems, each represented by a string of integrators.

Once we have a block triangular system, we can utilize the zero pattern of  $H$  to select the lead variables for the Brunovsky form. We do this by working our way up the super diagonals of  $H$  until encountering zeros. The column number of the first zero in each of the first  $m$  superdiagonal working upwards tells us which variables have been "skipped over" and must be lead variables. Recall also that  $x_1$  is always a lead variable. The following algorithm shows precisely how this is done.

- 1.)  $K = M$
- 2.)  $I = N$
- 3.) For  $L = 1, N+M$ 
  - 1.) If  $|H(I, I+K)| \neq 0$ 
    - 1.) If  $I = 1$  Exit Loop
    - 2.)  $I = I-1$
    - 3.) Step L
  - 2.)  $IOTA(K) = I+K$
  - 3.)  $Z(K) = \sum_{J=1}^N C(I+K, J) X(J)$
  - 4.)  $K = K-1$
  - 5.) Step L
- 4.) If  $|H(1, J+K)| \neq 0$ 
  - 1.) For  $L = 1, K$ 
    - 1.)  $IOTA(L) = L$
    - 2.)  $Z(L) = \sum_{J=1}^N C(L, J) X(J)$

Notice the lead variables  $Z$  of the Brunovsky form are computed in terms of the original  $x$ , not in terms of the intermediate block triangular system. The variable  $Z(K) = Y(I+K) = \sum_{J=1}^N C(I+K, J) X(J)$  would be the  $I+K$ th variable in the block triangular system.

Once we know the lead variables in the Brunovsky form, all we need to know are the Kronecker indices. These can easily be found by counting the number of derivatives necessary to go from the lead variables in block triangular form to the controls. The following algorithm does this, using the value of  $IOTA$  from the previous algorithm.

- 1.) For K=1, M
  - 1.) I = IOTA(K)
  - 2.) KAPPA(K) = 1
  - 3.) For J = N+M, ..., 1
    - 1.) If |H(I,J)| ≠ 0
      - 1.) If J > N Step K
      - 2.) I = J
      - 3.) KAPPA(K) = KAPPA(K) + 1
      - 4.) Begin J Loop again
    - 2.) Step J
  - 4.) Step K

#### V. RESULTS

We apply our theory and algorithms to a linear system on  $\mathbb{R}^7$  with 3 controls.

Example: The following results were achieved using single precision Fortran on a VAX 11/780 machine. For purposes of distinguishing zeros, numbers having absolute value less than 0.00001 were considered zero.

#### Original System

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \\ \dot{x}_7 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 5 \\ 3 & 2 & 0 & 0 & 0 & 1 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 3 & 0 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

## Transformation to Brunovsky form

$$z_1 = 0.16273 x_2 + 0.16781 x_3 + 0.75770 x_4 + 0.50853 x_5 - 0.33563 x_6$$

$$\kappa_1 = 3$$

$$z_4 = -0.85120 x_2 - 0.09259 x_3 - 0.03736 x_4 + 0.48083 x_5 + 0.18518 x_6$$

$$\kappa_2 = 3$$

$$z_7 = -x_1$$

$$\kappa_3 = 1$$

Here  $\kappa_1, \kappa_2,$  and  $\kappa_3$  are in order as in definition 3.1 and are not necessarily the Kronecker indices associated with the original controls  $u_1, u_2,$  and  $u_3$  respectively. That is, the subscript 1 does not identify  $\kappa_1$  with  $u_1$ . In a Brunovsky canonical form, the controls appear only at the end of integral strings; that is, after three derivatives of  $z_1$ , three derivatives of  $z_4$ , and one derivative of  $z_7$ . All state variables  $z$  should be functions only of the state variables  $x$ .

To get some idea of the accuracy of our numerical method we will look at the actual results in the first string.

We have

$$\begin{aligned} z_2 = \dot{z}_1 &= 0.16273 x_3 + 0.49327 x_4 + 0.75770 x_5 - 0.32546 x_6 \\ &\quad - 0.00002 x_7 - 0.00003 u_2 \end{aligned}$$

Notice that the control  $u_2$  does appear in  $z_2$ , but only negligibly.

Computing, we find

$$\begin{aligned} z_3 = \dot{z}_2 = \ddot{z}_1 &= -0.00060 x_1 - 0.00004 x_2 + 0.16273 x_4 + 0.49327 x_5 \\ &\quad + 0.01354 x_6 - 0.64098 x_7 - 0.00004 u_1 - 0.00002 u_2 \\ &\quad - 0.00003 \dot{u}_2 \end{aligned}$$

We again have in  $Z_3$  that the appearance of the controls  $u_1$  and  $u_2$  and the derivative  $\dot{u}_2$  is negligible.

$$\begin{aligned} \text{The new control } v_1 = \dot{z}_3 = \ddot{z}_2 = \dddot{z}_1 = \\ - 1.92294 x_1 - 1.28202 x_2 - 0.00004 x_3 - 0.00014 x_4 \\ + 0.16273 x_5 + 0.39972 x_6 - 3.45276 x_7 \\ - 1.28196 u_1 - 0.60036 u_2 - 0.00024 u_3 - 0.00004 \dot{u}_1 \\ - 0.00002 \dot{u}_2 - 0.00003 \ddot{u}_2 \end{aligned}$$

In the new control  $v_1$ , the original controls appear substantially but derivatives of controls appear only negligibly. These results can be improved by using more precise arithmetic, but we want to show how well it works even in single precision.

The weakest link in the algorithm is of course the recognition of zeros. Because the computer uses discrete arithmetic, there are no absolute zeros and nonzeros. We must have some criteria for deciding which numbers are zero. We chose an arbitrary cutoff point of 0.00001 in our example. Perhaps a more objective approach would be to compare an element with other elements in the matrix by using a matrix norm. We expect however that in an actual application, there will always be some "tailoring" of this cutoff point to suit the particular situation.

As mentioned previously, George Meyer is using transformations to Brunovsky canonical forms to design automatic flight controllers for vertical and short take off aircraft. In his scheme, the weak dependencies of the new state variables on the controls in our example are ignored, and errors are treated by a regulator which resides in the aircraft's on board computer.



In practice, we may be receiving noisy measurements for state and control variables. For our process, once we reach block triangular form, time derivatives are required to complete the transformations to Brunovsky form. Meyer [2] has a beautiful technique for moving from block triangular to Brunovsky form which involves "smoothing integration" using the inflight computer before the differentiation process takes place.

The theory of transformations of nonlinear systems to linear systems is developed in [11],[12],[13],[14],[15], [16]. Recent applications of this theory to automatic flight control are found in [17],[18],[19]. The techniques of this paper are employed in [8] to build approximate transformations of nonlinear systems to Brunovsky form.

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