

# Unit 1

## Lesson 9 : The Big M Method

### Learning outcomes

- **The Big M Method to solve a linear programming problem.**

In the previous discussions of the Simplex algorithm I have seen that the method must start with a **basic feasible solution**. In my examples so far, I have looked at problems that, when put into standard LP form, conveniently have an **all slack** starting solution. An all slack solution is only a possibility when all of the constraints in the problem have  $\leq$  inequalities. Today, we are going to look at methods for dealing with LPs having other constraint types.

Remember that simplex needs a place to start – it must start from a basic feasible solution then move to another basic feasible solution to improve the objective value.

With these assumptions, I can obtain an initial basic feasible solution /dictionary by letting all slack variables be basic, all original variables be non basic. Obviously, these assumptions do not hold for every LP. What do I do when they don't?

When a basic feasible solution is not readily apparent, the Big M method or the two-phase simplex method may be used to solve the problem.

### The Big M Method

If an LP has any  $\geq$  or  $=$  constraints, a starting basic feasible solution may not be readily apparent.

The Big M method is a version of the Simplex Algorithm that first finds a basic feasible solution by adding "artificial" variables to the problem. The objective function of the original LP must, of course, be modified to ensure that the artificial variables are all equal to 0 at the conclusion of the simplex algorithm.

### Steps

1. **Modify the constraints** so that the RHS of each constraint is nonnegative (This requires that each constraint with a negative RHS be multiplied by -1. Remember that if you multiply an inequality by any negative number, the direction of the inequality is reversed!). After modification, identify each constraint as a  $\leq$ ,  $\geq$ , or  $=$  constraint.
2. **Convert each inequality constraint to standard form** (If constraint  $i$  is a  $\leq$  constraint, we add a

slack variable  $s_i$ ; and if constraint  $i$  is a  $\geq$  constraint, we subtract an excess variable  $e_i$ ).

3. **Add an artificial variable  $a_i$  to the constraints identified as  $\geq$  or = constraints at the end of Step 1.** Also add the sign restriction  $a_i \geq 0$ .
4. **If the LP is a max problem, add (for each artificial variable)  $-Ma_i$  to the objective function** where  $M$  denote a very large positive number.
5. **If the LP is a min problem, add (for each artificial variable)  $Ma_i$  to the objective function.**
6. **Solve the transformed problem by the simplex.** Since each artificial variable will be in the starting basis, all artificial variables must be eliminated from row 0 before beginning the simplex. Now (In choosing the entering variable, remember that  $M$  is a very large positive number!).

**If all artificial variables are equal to zero in the optimal solution, we have found the optimal solution to the original problem.**

If any artificial variables are positive in the optimal solution, the original problem is **infeasible!!!**

Let's look at an example.

### Example 1

Minimize  $z = 4x_1 + x_2$

Subject to:

$$3x_1 + x_2 = 3$$

$$4x_1 + 3x_2 \geq 6$$

$$x_1 + 2x_2 \leq 4$$

$$x_1, x_2 \geq 0$$

By introducing a surplus in the second constraint and a slack in the third we get the following LP in standard form:

Minimize  $z = 4x_1 + x_2$

Subject to:

$$3x_1 + x_2 = 3$$

$$4x_1 + 3x_2 - s_2 = 6$$

$$x_1 + 2x_2 + s_3 = 4$$

$$x_1, x_2, s_2, s_3 \geq 0$$

Neither of the first two constraint equations has a slack variable or other variable that we can use to be basic in a feasible starting solution so we

**must use artificial variables.** If we introduce the artificial variables  $R_1$  and  $R_2$  into the first two constraints, respectively, and  $MR_1 + MR_2$  into the objective function, we obtain:

$$\begin{aligned} \text{Minimize } z &= 4x_1 + x_2 + MR_1 + MR_2 \\ \text{Subject to:} \\ 3x_1 + x_2 + R_1 &= 3 \\ 4x_1 + 3x_2 - S_2 + R_2 &= 6 \\ x_1 + 2x_2 + s_3 &= 4 \\ x_1, x_2, S_2, s_3, R_1, R_2 &\geq 0 \end{aligned}$$

We can now set  $x_1$ ,  $x_2$  and  $S_2$  to zero and **use  $R_1$ ,  $R_2$  and  $s_3$  as the starting basic feasible solution.**

In tableau form we have:

Basic	z	$x_1$	$x_2$	$S_2$	$R_1$	$R_2$	$s_3$	Solution
z	1	-4	-1	0	-M	-M	0	0
$R_1$	0	3	1	0	1	0	0	3
$R_2$	0	4	3	-1	0	1	0	6
$s_3$	0	1	2	0	0	0	1	4

At this point, we have our starting solution in place but we **must adjust our z-row to reflect the fact that we have introduced the variables  $R_1$  and  $R_2$  with non-zero coefficients (M).**

**We can see that if we substitute 3 and 6 into the objective function for  $R_1$  and  $R_2$ , respectively, that  $z = 3M + 6M = 9M$ . In our tableau, however, z is shown to be equal to 0. We can eliminate this inconsistency by substituting out  $R_1$  and  $R_2$  in the z-row. Because each artificial variable's column contains exactly one 1, we can accomplish this by multiplying each of the first two constraint rows by M and adding them both to the current z-row.**

$$\text{New z-row} = \text{Old z-row} + M \cdot R_1\text{-row} + M \cdot R_2\text{-row}$$

$$\begin{aligned} \text{Old z-row:} & \quad (1 \quad -4 \quad -1 \quad 0 \quad -M \quad -M \quad 0 \quad 0) \\ + M \cdot R_1\text{-row:} & (0 \quad 3M \quad M \quad 0 \quad M \quad 0 \quad 0 \quad 3M) \\ + M \cdot R_2\text{-row:} & (0 \quad 4M \quad 3M \quad -M \quad 0 \quad M \quad 0 \quad 6M) \\ \text{New z-row:} & \quad (1 \quad -4+7M \quad -1+4M \quad -M \quad 0 \quad 0 \quad 0 \quad 9M) \end{aligned}$$

Our tableau now becomes

Basic	z	x <sub>1</sub>	x <sub>2</sub>	S <sub>2</sub>	R <sub>1</sub>	R <sub>2</sub>	s <sub>3</sub>	Solution
z	1	-4+7M	-1+4M	-M	0	0	0	9M
R <sub>1</sub>	0	3	1	0	1	0	0	3
R <sub>2</sub>	0	4	3	-1	0	1	0	6
s <sub>3</sub>	0	1	2	0	0	0	1	4

Now we have the expected form for our starting solution.

We now **apply the simplex method as before**. Since this is a **minimization problem** we **select the entering variable with the most positive objective row coefficient**. In this case, that is **x<sub>1</sub>**. Calculating the intercept ratios we get:

$$R_1 - 3/3 = 1$$

$$R_2 - 6/4 = 1.5$$

$$s_3 - 4/1 =$$

So we **select R<sub>1</sub> as our leaving variable**.

Performing the Gauss-Jordan row operations, we obtain the new tableau:

Basic	z	x <sub>1</sub>	x <sub>2</sub>	S <sub>2</sub>	R <sub>1</sub>	R <sub>2</sub>	s <sub>3</sub>	Solution
z	1	0	(1+5M)/3	-M	(4-7M)/3	0	0	4+2M
x <sub>1</sub>	0	1	1/3	0	1/3	0	0	1
R <sub>2</sub>	0	0	5/3	-1	-4/3	1	0	2
s <sub>3</sub>	0	0	5/3	0	-1/3	0	1	3

In this tableau, we can see that **x<sub>2</sub> will be our next entering variable and R<sub>2</sub> will leave**.

We can thus see that **the simplex algorithm will quickly remove both R<sub>1</sub> and R<sub>2</sub> from the solution just as we intended** when we assigned them the coefficient of M in the objective function. If we continue to apply the simplex algorithm, we will find that **the optimal solution is:**

$$x_1 = 2/5$$

$$x_2 = 9/5$$

$$S_2 = 1$$

**with z = 17/5**

Two important considerations accompany use of the M method.

The use of the penalty **M** may not always force the artificial variable to zero level by the final iteration. This can occur in the case where the given LP has no feasible solution. If any artificial variable is positive in the final iteration than the LP has no feasible solution space.

**Theoretically**, the application of the M technique requires that **M approaches infinity** but to **computerize the solution algorithm, M must be finite while being “sufficiently large.”** The pitfall in this case is, however, **if M is too large it can lead to substantial round-off error yielding an incorrect optimal solution.** For this reason, **most commercial LP solvers do not apply the M-method** but use, rather, an artificial variable method called the two-phase method. For educational purposes, TORA, allows the implementation of the M-method with a user selected value for M where M is sufficiently large to allow solution of the problem. **The definition of the term “sufficiently large” is dependent upon the problem in question and requires some judgment for implementation.**

**Example 2** Minimise  $z = 2x_1 - 3x_2 + x_3$   
subject to

$$\begin{aligned} 3x_1 - 2x_2 + x_3 &\leq 5, \\ x_1 + 3x_2 - 4x_3 &\leq 9, \\ x_2 + 5x_3 &\geq 1, \\ x_1 + x_2 + x_3 &= 6, \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

*Solution* We obtain the linear programming problem: minimise  $x_8$  subject to

$$\begin{aligned} 3x_1 - 2x_2 + x_3 + x_4 &= 5, \\ x_1 + 3x_2 - 4x_3 + x_5 &= 9, \\ x_1 + x_2 + x_3 + x_6 &= 6, \\ -x_2 - 5x_3 + x_7 &= -1, \\ -2x_1 + 3x_2 - x_3 + x_8 &= 0, \\ x_1, \dots, x_7 &\geq 0, \end{aligned}$$

- where  $x_6$  is an artificial variable. In tableau  $T_1$  of Table 1, pivoting about  $y_{33}$  ( $= 1$ ) removes  $a_6$  from the basis. The rows of tableau  $T_2$  are then rearranged to give tableau  $T_3$  so that the bad row is below the others, and column  $a_6$  is ignored from here on. Pivoting in  $T_3$  about  $y_{12}$  ( $= 7$ ) gives tableau  $T_4$  which has the basic feasible solution  $(0, 33/7, 9/7, 92/7, 0, 0, 71/7, -90/7)$ . This has  $x_6 = 0$  and is an optimal solution, so  $x_1 = 0, x_2 = 33/7, x_3 = 9/7$  is an optimal solution of the original problem with optimal value  $-90/7$ .

$$\text{Min } z = 2x_1 + 3x_2$$

s.t.

$$1/2x_1 + 1/4x_2 \leq 4 \dots\dots\dots 1$$

$$x_1 + 3x_2 \geq 20 \dots\dots\dots 2$$

$$x_1 + x_2 = 10 \dots\dots\dots 3$$

$$x_1, x_2 \geq 0$$

Step1: Make the right hand side of all constraints positive.

We don't have any negative right hand side.

Step2: Identify each constraint which is  $\geq$  or  $=$ .

Constraints 2 and 3 apply the above conditions.

Step3: For each  $\leq$  constraint add a slack variable and for each  $=$  constraint subtract an excess variable to make them equalities.

$$1 \dots\dots\dots 1/2x_1 + 1/4x_2 + s_1 = 4$$

$$2 \dots\dots\dots x_1 + 3x_2 - e_1 = 20$$

Step4: For each  $\geq$  or  $=$  constraint add an artificial variable  $a_i$  ( $a_i$ 's  $> 0$ ), which is to be chosen in the starting bfs.

$$2 \dots\dots\dots x_1 + 3x_2 - e_1 + a_2 = 20$$

$$3 \dots\dots\dots x_1 + x_2 + a_3 = 10$$

Step5: If the LP is a min add  $+Ma_i$  to the objective function. If it is a max add  $-Ma_i$  to the objective function. Here M represents a very big number such that in the min problem  $+Ma_i$  is arbitrarily large so that  $a_i$  the artificial variable is best to be chosen as zero, which we require. Similar reasoning applies in the max problem.

$$\text{Min } z = 2x_1 + 3x_2 + Ma_2 + Ma_3$$

Step6: Choose those artificial variables in the starting bfs and proceed to find the Optimal Tableau. If in the end, artificial variables

are zero we find the solution, but if they are not equal to zero then we don't have a optimal solution. Thus, original LP is infeasible.

After these steps we have the LP:

$$\text{Min } z=2x_1+3x_2+Ma_2+Ma_3$$

st

$$1/2x_1+1/4x_2+s_1 = 4$$

$$x_1+3x_2 -e_1+a_2=20$$

$$x_1+x_2 +a_3=10$$

After all, we have the table:

		↓ Entering variable							
Basis	z	x <sub>1</sub>	x <sub>2</sub>	s <sub>1</sub>	e <sub>2</sub>	a <sub>2</sub>	a <sub>3</sub>	RHS	
z	1	2M-2	4M-3	0	-M	0	0	30M	
s <sub>1</sub>	0	1/4	1/4	1	0	0	0	4	
← a <sub>2</sub>	0	1	3	0	-1	1	0	20	
a <sub>3</sub>	0	1	1	0	0	0	1	10	
		↓							
z	1	(2M-3)/3	0	0	(M-3)/3	(3-4M)/3	0	60+10M	
s <sub>1</sub>	0	5/12	0	1	1/12	-1/12	0	7/3	
x <sub>2</sub>	0	1/3	1	0	-1/3	1/3	0	20/3	
← a <sub>3</sub>	0	2/3	0	0	1/3	-1/3	1	10/3	
		↓							
z	1	0	0	0	-1/2	(1-2M)/2	(3-2M)/2	25	
s <sub>1</sub>	0	0	0	1	-1/8	1/8	-5/8	1/4	
x <sub>2</sub>	0	0	1	0	-1/2	1/2	-1/2	5	
x <sub>1</sub>	0	1	0	0	1/2	-1/2	3/2	5	

In the optimal solution, we have:  $\{z, s_1, x_2, x_1\} = \{25, 1/4, 5, 5\}$ . Since we don't have any of artificial variables  $a_1$  and  $a_2$  in the optimal solution, the solution is feasible. If any of the artificial variables  $a_1$ ,  $a_2$  were not equal to zero then we would have infeasibility as described below:

$$\text{Min } z=x_1+3x_2$$

s.t.

$$1/2x_1 + 1/4x_2 \leq 4$$

$$x_1 + 3x_2 \geq 10$$

$$x_1 + x_2 = 10$$

After going through all the steps described above we end with the optimal tableau:

Basis	z	$x_1$	$s_2$	$s_1$	$e_2$	$a_2$	$a_3$	RHS
z	1	$1-2M$	0	0	$-M$	0	$3-4M$	$30+6M$
$s_1$	0	$1/4$	0	1	0	0	$-1/4$	$3/2$
$a_2$	0	-2	0	0	-1	1	-3	6
$s_2$	0	1	1	0	0	0	1	10

In the above example, we have the optimal tableau since reduced costs of all non basic variables are non positive. However note that the optimal solution contains the very big number M, which should not have been the case for a min problem, thus we say that our original LP was infeasible. We also say that from the fact that we have the artificial variable  $a_2$  in the basic variables, which shouldn't have been the case for a feasible LP.

### Example 3

**Maximize  $Z = x_1 + 5x_2$**

Subject to:

$$3x_1 + 4x_2 \leq 6$$

$$x_1 + 3x_2 \geq 2$$

Where  $x_1, x_2 \geq 0$

### Solution

Introducing slack and surplus variables

$$3x_1 + 4x_2 + x_3 = 6$$

$$x_1 + 3x_2 - x_4 = 2$$

Where:

$x_3$  is a slack variable

$x_4$  is a surplus variable.

The surplus variable  $x_4$  represents the extra units.



Now if we let  $x_1$  and  $x_2$  equal to zero in the initial solution, we will have  $x_3 = 6$  and  $x_4 = -2$ , which is not possible because a surplus variable cannot be negative. Therefore, we need **artificial variables**.

$$\text{Maximize } x_1 + 5x_2 - MA_1$$

Subject to:

$$3x_1 + 4x_2 + x_3 = 6$$

$$x_1 + 3x_2 - x_4 + A_1 = 2$$

Where:

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, A_1 \geq 0$$

	$C_j$	1	5	0	0	-M	
$C_B$	Basic variables B	$x_1$	$x_2$	$x_3$	$x_4$	$A_1$	Solution values b (= $X_B$ )
0	$x_3$	3	4	1	0	0	6
-M	$A_1$	1	3	0	-1	1	2
$Z_j - C_j$		$-M - 1$	$-3M - 5$	0	M	0	

Here,  $a_{11} = 3, a_{12} = 4, a_{13} = 1, a_{14} = 0, a_{15} = 0, b_1 = 6$   
 $a_{21} = 1, a_{22} = -3, a_{23} = 0, a_{24} = -1, a_{25} = 1, b_2 = 2$

### Calculating $Z_j - C_j$

$$\text{First column} = 0 * 3 + (-M) * 1 - 1 = -M - 1$$

$$\text{Second column} = 0 * 4 + (-M) * 3 - 5 = -3M - 5$$

$$\text{Third column} = 0 * 1 + (-M) * 0 - 0 = 0$$

$$\text{Fourth column} = 0 * 0 + (-M) * (-1) - 0 = M$$

$$\text{Fifth column} = 0 * 0 + (-M) * 1 - (-M) = 0$$

**Choose the smallest negative value from  $Z_j - C_j$ .**

**Substitute  $M = 0$**

**Smallest negative value is -5. So second column is the element column.**

**Now find out the minimum positive value.**

Minimum  $(6 / 4, 2 / 3) = 2 / 3$   
 So second row is the element row.  
 Here, the pivot (key) element = 3.  
 Therefore,  $A_1$  departs and  $x_2$  enters.

**Calculating values for table 2**

**Calculating values for first row**

$$a_{11} = 3 - 1 * 4 / 3 = 5 / 3$$

$$a_{12} = 4 - 3 * 4 / 3 = 0$$

$$a_{13} = 1 - 0 * 4 / 3 = 1$$

$$a_{14} = 0 - (-1) * 4 / 3 = 4 / 3$$

$$b_1 = 6 - 2 * 4 / 3 = 10 / 3$$

**Calculating values for key row**

$$a_{21} = 1 / 3$$

$$a_{22} = 3 / 3 = 1$$

$$a_{23} = 0 / 3 = 0$$

$$a_{24} = -1 / 3$$

$$b_2 = 2 / 3$$

**Table 2**

	$C_j$	1	5	0	0	
$C_B$	Basic variables B	$x_1$	$x_2$	$x_3$	$x_4$	Solution values $b (= X_B)$
0	$x_3$	$5 / 3$	0	1	$4 / 3$	$10 / 3$
5	$x_2$	$1 / 3$	1	0	$-1 / 3$	$2 / 3$
$Z_j - C_j$		$2 / 3$	0	0	$-5 / 3$	

**Table 3**

	$C_j$	1	5	0	0	
$C_B$	Basic variables B	$x_1$	$x_2$	$x_3$	$x_4$	Solution values $b (= X_B)$
0	$x_4$	$5 / 4$	0	$3 / 4$	1	$5 / 2$
5	$x_2$	$3 / 4$	1	$1 / 4$	0	$3 / 2$

$Z_j - C_j$		$11/4$	$0$	$5/4$	$0$	
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Since all the values of  $Z_j - C_j$  are positive, *this is the optimal solution*.

$$x_1 = 0, x_2 = 3/2$$

$$Z = 0 + 5 * 3/2 = 15/2$$

