ENUMERATING 9 x 9 SUDOKU GRIDS

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ABSTRACT

Sudoku is a popular Japanese game which requires one to fill in a 9×9 game board entirely, so that the digits 1 through 9 occur exactly once in every row, column, and 3×3 sub-box of the grid. Some boxes have already been filled in by the setter to serve as clues. A lesser-known fact is that Sudoku is a special case of Latin squares, and hence the enumeration of the total number of possible grids proves to be an interesting combinatorial problem. Previous researchers have come up with an accurate answer to this question through various reduction methods as well as computer-based programming: they derived a way to place all sudoku grids into 44 different classes, after which each class was enumerated separately. Building upon their research, we realized that the answer can in fact be derived by another method, which allows one to come up with the 44 classes directly, through logical analysis of the structure of various sudoku grids. Thus, in this paper, we propose this alternative method of deriving the solution.

INTRODUCTION

Previous research by Felgenhauer & Jarvis (Mathematics of Sudoku I, 2006) puts the total number of possible sudoku grids as 6670903752021072936960. This is approximately 6.671×10^{21}. This number was calculated using various reduction methods and computer programmes.

The researchers first counted the total number of possibilities for the first three rows, which turned out to be 948109639680. To make enumeration easier, they used many reduction methods such as relabelling, putting the first grid into standard form, and lexicographical reduction, all of which are explained under the Method section. Firstly, they reduced 948109639680 by 9! in order to put the first grid in standard form, yielding the answer 2672136. This can be further reduced by lexicographical reduction. This cuts down the number of top 3 rows that we are required to count by a factor of 72, to 36288. This was eventually reduced to 1296, 71 and finally 44.

The final 44 classes can represent all the 2612736 different top 3 rows. The significance of the 44 classes is that every possible configuration of the top 3 blocks will have the same number of completions as one of the classes, and each member in every class has the same number of completions to a full grid. Thus, they can simply be placed into the same class for easier calculation.

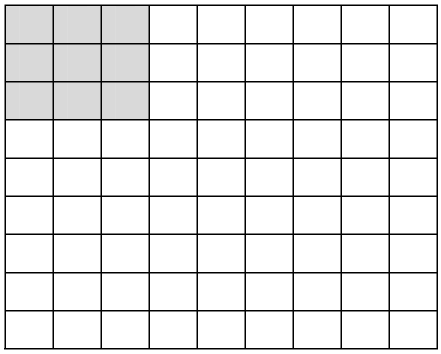
Each of the 44 classes is then enumerated, to find out the number of different completions to a full grid. Following that, the number is multiplied by the number of equivalent configurations to a full grid, totaled up and multiplied by 1881169920 ($9! \times 72 \times 72$) for the total number of valid Sudoku grids.

However, if we look closely at the 44 classes, we will realize that they can actually be derived by logical analysis. After deciding how we will order our classes, by making some simple assumptions, we will arrive at the first class. We will be able to derive the subsequent classes through a similar process of logical analysis and trial-and-error.

In this paper, we will explain our alternative method to coming up with these 44 classes by logical analysis instead of purely step-by-step reduction methods and computer programming.

METHOD

Firstly, we define a few terms:



Each sudoku grid is divided into 9 'blocks'. These 3 by 9 blocks are those which, according to the game's rules, must contain all digits from 1 to 9. Thus, rows 1-3, 4-6, 7-9 each have 3 blocks. An example of such has been shaded above. The block at the top left-hand corner will be referred to as the first block.

A 'class' refers to a large group of configurations of top three rows that are grouped together because they have the same number of completions to a full grid.

'Chronological order' is an ordering system which will play a large part in the organization of the enumeration later on. It is a way of arranging different sudoku classes. Suppose class A is smaller than class B, and the numbers in the columns of the two classes first begin to differ from each other in column X. Arrange the three numbers in column X of both classes in ascending order. Class A is smaller than class B if in column X, the smallest number of A which B does not possess is lesser than the smallest number of B which A does not possess. (The classes only take into consideration the top three rows of every sudoku grid.)

After defining the above terms, we then studied various possible reduction methods that we can perform on the first three rows which may help to reduce the total number of grids that we will have to enumerate. These reduction methods give us other possible top blocks that have the same number of possible completions to a full grid. These reduction methods are mainly (1) relabelling, (2) lexicographical reduction.

Simply put, relabelling refers to the exchanging of positions of two different digits throughout the entire grid. For example, for any sudoku grid, we can replace all the '1's with '2' and vice versa to get another valid grid. Thus, for any top three blocks, we can relabel the digits in order to get another set of top three blocks that complete to a valid grid in the same number of ways.

Lexicographical reduction is the name given to an operation that we can perform on the top three boxes of the sudoku grid. The columns in the second and third blocks of the first three rows will be permuted so that their top entries are in increasing order. After that, we may swap the two boxes so that the top left number of the second block is smaller than that of the third block. The first step reduces the number of sudoku grids we have to count by a factor of 36, because there are 6 ways to permute the columns in each block. The second step merely doubles this number. Thus, lexicographical reduction reduces the number of blocks we have to count by a factor of 72.

After defining and studying these reduction methods, we can begin the proper enumeration of sudoku grids.

To make enumeration easier, each grid will be relabeled so that the first block will be in 'standard form' as shown below:

1	2	3
4	5	6
7	8	9

This reduces the total number of top 3 rows we have to count by a factor of 9!. We just have to consider the number of sudoku grids whose top-left box is of this form. Then, we will also lexicographically reduce each grid that we are going to consider, which, as explained above, will reduce the number of top 3 rows we have to count by the factor 72.

Following that, we will be able to derive all 44 classes through logical analysis. Listing out the classes in chronological order, we derived the lexicographically reduced form of each class systematically through coming up with and rejecting possibilities. We list out columns 4-9 of each class, where the smallest entry for column 4 is smaller than that of column 5, which is lesser than that of column 6; and similarly for columns 6,7 and 8. Note that we can always use column permutations and relabelling on the root form to arrive at any sudoku grid that falls under the same class. It will then be apparent that all the grids can be divided into 44 classes.

Finally, we can enumerate the number of configurations of the top 3 rows that belong to each individual class, with the help of a computer programme, and multiply the answer by the number of completions to a full grid. The final sum for all the classes will then be multiplied by 1881169920, which is equal to $9!\times 72\times 72$, taking into account the reduction processes we have applied. This gives us the number of sudoku grids.

RESULTS

Before presenting the results, we would first give some examples as to how the classes were derived.

Let the first block be in standard form. We would list out the classes in chronological order. Shown below is how classes 1-4 can be logically thought out.

To derive the first class, which is the smallest chronologically, we observe that since the top row of the first block contains the digits 1,2,3, the leftmost box at the top row of the second block cannot contain any of these digits. Hence, the next smallest digit it can contain would be 4. Also, the numbers 1 and 2 would be able to fit in the other two boxes of column 4.

1	2	3	4					
4	5	6	1	3		2		
7	8	9	2			1		

Then, we continue by putting the numbers 3 and 5 into column 5.

1	2	3	4	7				
4	5	6	1	3		2		
7	8	9	2	5	6	1		

Next, we observe that 7,8,9 cannot be in the same column as this would result in a clash with block 1, where 7,8,9 are in the same row. Therefore, to obtain the smallest possible class chronologically, 7 is placed in column 5. As such, 6,8,9 are in column 6 and 3,5,7 are in column 5.

1	2	3	4	7	9			
4	5	6	1	3	8	2		
7	8	9	2	5	6	1		

For the third block, we observe that 1,2 can still be in column 7 (in the 2nd and 3rd row) but 3 and 4 cannot be in column 7 as it would result in a clash with blocks 1 and 2. Thus, 5, is in column 7, to achieve smallest possible class.

1	2	3	4	7	9	5		
4	5	6	1	3	8	2		
7	8	9	2	5	6	1		

Then, we notice that we can put 3 in column 8, but 4 cannot be in column 8. This is because 3 must be in the last row. (To avoid a clash with the 5 in column 7, the 5 in column 5 would have to be in the 3rd row. This leaves us with 3 in the middle row and 7 in the first row for column 5. Thus, 3 is in the last row for column 8. However, this would leave us with the choice of putting 4 in either the top or the second row, either of which would result in a clash.) Thus, the next logical thing would be to put 6 and 7 in column 8 and 4,8,9 in column 9.

1	2	3	4	7	9	5	6	8
4	5	6	1	3	8	2	7	9
7	8	9	2	5	6	1	3	4

[Class 1]

From the above, we can get the second class by making some minor changes. The numbers in the columns stay the same for columns 1 to 7. We simply have exchanged the positions of 7 and 8 in columns 8 and 9. Class 3 can be derived by exchanging the positions of 7 and 9. Case 4 is a simple case of swapping 6 and 7 in columns 8 and 9. (Note that the classes are in chronological order, so they gradually increase as we continue our derivation).

In the same vein, we can come up with all 44 classes.

Here are the 44 classes that can be derived by logical analysis in their lexicographically reduced form, and their number of completions to a full grid (which can be calculated by a computer programme). The numbers in the 8th column stand for the number of configurations of top three rows that have the same completions to a full grid (where the first block is in standard form), and the numbers in the last column are the number of possible completions to a full grid for each of these configurations.

Number	**Column 4**	**Column 5**	**Column 6**	**Column 7**	**Column 8**	**Column 9**	**Number of equivalent configurations**	**Number of completions to a full grid**
1	1,2,4	3,5,7	6,8,9	1,2,5	3,6,7	4,8,9	**2484**	**97961464**
2	1,2,4	3,5,7	6,8,9	1,2,5	3,6,8	4,7,9	**2592**	**97539392**
3	1,2,4	3,5,7	6,8,9	1,2,5	3,6,9	4,7,8	**1296**	**98369440**
4	1,2,4	3,5,7	6,8,9	1,2,5	3,7,8	4,6,9	**1512**	**97910032**
5	1,2,4	3,5,7	6,8,9	1,2,6	3,4,8	5,7,9	**2808**	**96482296**
6	1,2,4	3,5,7	6,8,9	1,2,6	3,4,9	5,7,8	**684**	**97549160**
7	1,2,4	3,5,7	6,8,9	1,2,6	3,5,7	4,8,9	**1512**	**97287008**
8	1,2,4	3,5,7	6,8,9	1,2,6	3,5,8	4,7,9	**1944**	**97416016**
9	1,2,4	3,5,7	6,8,9	1,2,6	3,5,9	4,7,8	**2052**	**97477096**
10	1,2,4	3,5,7	6,8,9	1,2,7	3,4,8	5,6,9	**288**	**96807424**
11	1,2,4	3,5,7	6,8,9	1,2,7	3,5,8	4,6,9	**864**	**98119872**
12	1,2,4	3,5,7	6,8,9	1,2,8	3,4,7	5,6,9	**1188**	**98371664**
13	1,2,4	3,5,7	6,8,9	1,2,8	3,5,7	4,6,9	**648**	**98128064**
14	1,2,4	3,5,7	6,8,9	1,2,8	3,6,9	4,5,7	**2592**	**98733568**
15	1,2,4	3,5,7	6,8,9	1,3,5	2,6,9	4,7,8	**648**	**97455648**
16	1,2,4	3,5,7	6,8,9	1,3,5	2,7,8	4,6,9	**360**	**97372400**
17	1,2,4	3,5,7	6,8,9	1,3,6	2,5,9	4,7,8	**3240**	**97116296**
18	1,2,4	3,5,7	6,8,9	1,3,8	2,6,7	4,5,9	**540**	**95596592**
19	1,2,4	3,5,7	6,8,9	1,3,8	2,6,9	4,5,7	**756**	**97346960**
20	1,2,4	3,5,7	6,8,9	1,4,5	2,6,9	3,7,8	**324**	**97714592**
21	1,2,4	3,5,7	6,8,9	1,4,5	2,7,8	3,6,9	**432**	**97992064**
22	1,2,4	3,5,7	6,8,9	1,4,6	2,3,9	5,7,8	**756**	**98153104**
23	1,2,4	3,5,7	6,8,9	1,4,7	2,6,9	3,5,8	**864**	**98733184**
24	1,2,4	3,5,7	6,8,9	1,4,8	2,6,9	3,5,7	**108**	**98048704**
25	1,2,4	3,5,7	6,8,9	1,5,6	2,3,9	4,7,8	**756**	**96702240**
26	1,2,4	3,5,8	6,7,9	1,2,5	3,6,8	4,7,9	**516**	**98950072**
27	1,2,4	3,5,8	6,7,9	1,2,6	3,4,8	5,7,9	**576**	**97685328**
28	1,2,4	3,5,8	6,7,9	1,2,7	3,5,8	4,6,9	**432**	**98784768**
29	1,2,4	3,5,8	6,7,9	1,3,7	2,6,9	4,5,8	**324**	**98493856**
30	1,2,4	3,5,8	6,7,9	1,4,7	2,5,8	3,6,9	**72**	**100231616**
31	1,2,4	3,5,8	6,7,9	1,4,7	2,6,9	3,7,8	**216**	**99525184**

32	1,2,4	3,5,8	6,7,9	1,5,6	2,3,7	4,8,9	**252**	**96100688**
33	1,2,4	3,5,9	6,7,8	1,2,7	3,5,6	4,8,9	**288**	**96631520**
34	1,2,4	3,5,9	6,7,8	1,2,7	3,5,9	4,6,8	**864**	**97756224**
35	1,2,4	3,5,9	6,7,8	1,4,7	2,5,8	3,6,9	**216**	**99083712**
36	1,2,4	3,5,9	6,7,8	1,4,7	2,6,8	3,5,9	**432**	**98875264**
37	1,2,4	3,6,9	5,7,8	1,2,5	3,6,9	4,7,8	**216**	**102047904**
38	1,2,4	3,6,9	5,7,8	1,2,7	3,6,9	4,5,8	**144**	**101131392**
39	1,2,4	3,6,9	5,7,8	1,3,5	2,6,7	4,8,9	**324**	**96380896**
40	1,2,4	3,6,9	5,7,8	1,4,7	2,5,8	3,6,9	**108**	**102543168**
41	1,2,4	3,7,9	5,6,8	1,4,6	2,3,9	5,7,8	**12**	**99258880**
42	1,2,6	3,4,8	5,7,9	1,3,5	2,4,9	6,7,8	**20**	**94888576**
43	1,2,6	3,7,8	4,5,9	1,4,7	2,5,8	3,6,9	**24**	**97282720**
44	1,4,7	2,5,8	3,6,9	1,4,7	2,5,8	3,6,9	**4**	**108374976**

To get the final answer, multiply the number in the 8th column in each row by the number in the last column and total all 44 products. Finally, multiply this by 1881169920 and we will get the answer 6709037520210729396960.

DISCUSSION

It is fairly obvious from the above table that some classes which are apparently correct are not included in the table. An example will be (1,2,4), (3,5,7), (6,8,9), (1,2,6), (3,4,7), (5,8,9) for columns 4,5,6,7,8,9, which seems to fit in between classes 4 and 5 but are not in the table. There are also many similar cases as we go down the results table.

The reason why these classes have not been included is because they already fit into a previous class. After some analysis, it will be apparent that these un-included classes share common characteristics with the standard element of the class that they belong in.

For example, the above-highlighted example actually belongs to class 1. We will analyze why that is so.

The top three rows of the un-included example can be written in the form

1	2	3	4	5	8	6	7	9
4	5	6	1	7	9	1	3	8
7	8	9	2	3	6	2	4	5

While the top three rows of class 1 has a basic structure of

1	2	3	4	7	8	5	6	9
4	5	6	2	3	9	1	7	8
7	8	9	1	5	6	2	3	4

Note that the placements of 1,2 and 8,9 in both cases are similar. Thus, we can attempt to relabel and do some modifications on the un-included case so that it becomes a part of class 1. This can be accomplished in a few steps: (1), swapping the 1st and 3rd row; (2)

interchanging the 1st and 3rd columns of every block (3) relabelling so that the first box is back to standard form.

Now, we will give another slightly different example of an un-included class and which class it actually belongs to. After the previous example, as one continues to generate classes, the next apparently correct, but un-included, class will be:

1	2	3	4	5	9	6	7	8
4	5	6	1	7	8	3	2	9
7	8	9	2	3	6	4	1	5

This class seems to come between classes 9 and 10. However, further research shows that the above example actually belongs to class 2. Let us take a look at how it is so.

It is easy to see that the above converts easily to

9	8	7	6	3	2	5	1	4
6	5	4	8	7	1	9	2	3
3	2	1	9	5	4	8	7	6

which relabels to 1,2,3; 3,8,9; 4,6,7. Then, to interchange 1 and 4, we can exchange the 2 by 2 subrectangle (3,6) with (2,3) . This shifts 1 and 4 around in the first two boxes, which after relabeling means that 1 and 4 moves in the last box.

Similar operations can be carried out on the rest of the un-included, but apparently correct, classes to ensure that they fit into one of the classes in the table. Finally, we can see that no two classes are the same, since all 44 classes have a different number of completions to a full grid.

Thus, it can be seen that we are able to confirm the 44 classes through the process elaborated above; the cases derived by logical analysis which do not fit into the table in fact belong to other, already included, classes.

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